A variational approach to parabolic equations under general and p, q-growth conditions

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ABSTRACT

We consider *variational solutions* to the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u = \operatorname{div} D_\xi f(x, u, Du) - D_u f(x, u, Du) & \text{in } \Omega_T \\ u = u_0 & \text{on } \partial_{\mathrm{par}} \Omega_T \end{cases}$$

where the function $f = f(x, u, \xi)$, $f : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to [0, \infty)$, is convex with respect to (u, ξ) and coercive in $\xi \in \mathbb{R}^{N \times n}$, but it not necessarily satisfies a growth condition from above. A motivation to consider a class of such energy functions f can be also easily found in the stationary case, where a large literature in the *calculus of variations* is devoted to the minimization of *p, q-growth* problems [45] and to *double phase* problems [23], [24], [4], [5], [6]. In the parabolic context the notion of variational solution (see the references from [8] to [15]) is compatible with the lack of *the same polynomial growth* from below and from above.

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1. General and $p, q-$growth conditions in the stationary elliptic case

Main motivations to consider general and $p, q-$growth conditions in *evolution problems* appear similarly in the *stationary context* too, where the differential problems under consideration are of *elliptic type*, and where the mathematical literature nowadays is wider; thus we start to briefly consider in this section the stationary elliptic case.

We describe a mathematical model which does not fit in the classical framework for the existence and the regularity theory and which is motivated by a class of integrals of the *calculus of variations* introduced in *nonlinear elasticity*. Following the ideas introduced by Morrey [48,49] and Ball [2], a model energy-integral considered in nonlinear elasticity for a generic map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is

$$u \to \int_\Omega f(Du(x)) \, dx, \tag{1.1}$$

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where Ω is a bounded open set of $\mathbb{R}^n$ and f is a *quasiconvex* and coercive function, $f : \mathbb{R}^{n \times n} \to \mathbb{R}$ such as, for instance,

$$f\left(Du\left(x\right)\right) = \left|Du\left(x\right)\right|^p + g\left(\det Du\left(x\right)\right) \tag{1.2}$$

for some $p > 1$. Here $g : (0, +\infty) \to (0, +\infty)$ is a real function which grows at $+\infty$ as a power, i.e., $g(t) \leq c(1 + t^r)$ for some $r, t_0 \geq 1$ and every $t \geq t_0$. Since $\left|\det Du\left(x\right)\right| \leq n^{-\frac{n}{2}} \left|Du\left(x\right)\right|^n$, then, if $\left|Du\left(x\right)\right| \geq t_0$,

$$\left|Du\left(x\right)\right|^p \leq f\left(Du\left(x\right)\right) \leq \text{const} \left(1 + \left|Du\left(x\right)\right|^p + \left|Du\left(x\right)\right|^{rn}\right). \tag{1.3}$$

Therefore the integral in (1.1) is coercive in $W^{1,p}\left(\Omega, \mathbb{R}^n\right)$ and, being a model in nonlinear elasticity where *discontinuous* solutions are admitted, then reasonably $p < n$. Thus, if we denote by $q := rn$, then the $p-$growth from below is different from the $q-$growth from above. We are here in presence of $p, q-$growth *conditions* and $p < q$.

The energy integral (1.1), (1.2), which has been used to modelize the phenomenon of *cavitation* by Ball [3], see also Marcellini [40,42,43] and Celada–Perrotta [20], is also associated to some other mathematical difficulties to be studied, for instance it is related to *vector valued maps* $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and thus the first variation is a *system*; the dependence on the Jacobian determinant $\det Du\left(x\right)$, which produces a not convex function f in (1.2); the possible singularity of $g\left(\det Du\left(x\right)\right)$ as $\det Du\left(x\right) \to 0^+$; that is, we could expect that $g\left(\det Du\left(x\right)\right) \to +\infty$ as $\det Du\left(x\right) \to 0^+$; it may also happen the lack of convexity (for some experimental elastic materials) of $g : (0, +\infty) \to (0, +\infty)$; the singularity of $f\left(Du\left(x\right)\right)$ as $\left|Du\left(x\right)\right| \to +\infty$; i.e., the $p-$growth from below, different from the $q-$growth from above, as we emphasized above, which gives the genuine $p, q-$growth in (1.3), with p strictly less than q.

Other motivations are well described in the well known article published in 2006 by Rosario Mingione [47]. Some more reasons to consider general and $p, q-$growth conditions, again stimulated by the work of Rosario Mingione, are considered here in the following.

1.1. Double phase integrals

For instance, we can minimize, with fixed boundary values, the integral

$$F\left(u\right) = \int_\Omega \left\{a\left(x\right)\left|Du\left(x\right)\right|^p + b\left(x\right)\left|Du\left(x\right)\right|^q\right\} dx, \tag{1.4}$$

with $q \neq p$, let us say $q > p > 1$, and

$$\begin{cases} a\left(x\right), b\left(x\right) \geq 0 \\ a\left(x\right) + b\left(x\right) > 0. \end{cases}$$

This class of energy functionals enters in the context of general $p, q-$ growth conditions; it is also named *double phase integrals* and has been recently (starting from 2015) explored in a series of interesting papers by M.Colombo–Mingione [23,24], Baroni–M.Colombo–Mingione [4–6]; from a different point of view see Eleuteri–Marcellini–Mascolo [34,35]. See also Rădulescu–Zhang [50], Cencelja–Rădulescu–Repovš [21] and De Filippis [29]. For related recent references we quote [45] and Cupini–Giannetti–Giova–Passarelli [25], Carozza–Giannetti–Leonetti–Passarelli [18], Cupini–Marcellini–Mascolo [26,27], Harjulehto–Hästö–Toivanen [36], Hästö–Ok [37], Bousquet–Brasco [16].

Independently of the continuity of the coefficients $a\left(x\right), b\left(x\right)$, a local boundedness result for minimizers of the energy integral in (1.4) can be deduced from Theorem 1.1 in [28]:

Theorem 1.1. *Let* $q \geq p > 1$, $a^{-1} \in L^r_{loc}\left(\Omega\right)$ *and* $b \in L^s_{loc}\left(\Omega\right)$ *for some exponents* $r \in \left(\frac{1}{p-1}, +\infty\right]$, $s \in (1, +\infty]$, *with*

$$\frac{1}{pr} + \frac{1}{qs} + \frac{1}{p} - \frac{1}{q} < \frac{1}{n}. \tag{1.5}$$

Then every local minimizer of the energy integral (1.4) *is locally bounded in* Ω.

Note that in the special relevant case $r = s = +\infty$ then condition (1.5) reduces to $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, that is

$$\frac{q}{p} < 1 + \frac{q}{n}. \tag{1.6}$$

More regularity of minimizers, in fact the local Hölder continuity of their gradients, has been obtained in the quoted papers by M.Colombo–Mingione [23,24] and Baroni–M.Colombo–Mingione [4–6] (see also [34,35]). The following results have been proved in [23]. Of course in the first of the two ones we need a more strict assumption than (1.5), (1.6).

Theorem 1.2. *Let $q \geq p > 1$, $a^{-1} \in L^\infty_{loc}(\Omega)$ and $a, b \in C^\alpha_{loc}(\Omega)$ for some $\alpha \in (0, 1]$, with*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}. \tag{1.7}$$

Then every local minimizer of the energy integral (1.4) is of class $C^{1,\beta}_{loc}(\Omega)$ for some $\beta \in (0.1)$.

Theorem 1.3. *Let $q \geq p$ and $1 < p \leq n$, $a^{-1} \in L^\infty_{loc}(\Omega)$ and $a, b \in C^\alpha_{loc}(\Omega)$ for some $\alpha \in (0, 1]$, with*

$$\frac{q}{p} < 1 + \frac{\alpha}{p} \tag{1.8}$$

Let us also assume that a local minimizer of the energy integral (1.4) is locally bounded. Then it is of class $C^{1,\beta}_{loc}(\Omega)$ for some $\beta \in (0.1)$.

The following is a related regularity result in [35], valid for a generalized class of double (or *multi*) phase energy integrands, whose prototype is given by

$$f(x, \xi) = a(x)|\xi|^p + b(x)|\xi|^s + |\xi_n|^q, \tag{1.9}$$

ξ_n being the last (or any other) component of the vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ and $s \leq \frac{p+q}{2}$. Here we can also consider the more general case with $f = f(x, \xi)$ without a structure, i.e. not necessarily depending on the modulus of ξ. We assume that $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a convex function with respect to the gradient variable and it is strictly convex only at infinity; more precisely, $f_{\xi\xi}$, $f_{\xi x}$ are Carathéodory functions satisfying

$$\begin{cases} M_1 |\xi|^{p-2}|\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi)\lambda_i \lambda_j \\ |f_{\xi_i \xi_j}(x, \xi)| \leq M_2 |\xi|^{q-2} \\ |f_{\xi x}(x, \xi)| \leq h(x)|\xi|^{\frac{p+q-2}{2}} \quad \text{or} \quad |f_{\xi x}(x, \xi)| \leq h(x)|\xi|^{q-1} \end{cases} \tag{1.10}$$

for some constants $M_0, M_1, M_2 > 0$, for almost every $x \in \Omega$ and for all $\lambda, \xi \in \mathbb{R}^n$ with $|\xi| \geq M_0$. Here $1 < p \leq q$ and $h \in L^r(\Omega)$ for some $r > n$.

Theorem 1.4. *Under the given growth assumptions (1.10) with exponents p, q satisfying*

$$\frac{q}{p} < 1 + 2\left(\frac{1}{n} - \frac{1}{r}\right), \tag{1.11}$$

any local minimizer of the energy integral $\int_\Omega f(x, Du(x))\, dx$ is locally Lipschitz continuous in Ω.

If we specialize the above theorem with integrand $f(x, \xi)$ as in (1.9) with $a(x) = 1$ and $b(x) = |x|^\alpha$ for some $\alpha \in (0, 1)$ and $0 \in \Omega$, then $b \in C^{0,\alpha} \cap W^{1,r}$ with $\frac{1}{r} = \frac{1-\alpha}{n}$. Also the function h in (1.10) belongs to L^r for the same $r = \frac{n}{1-\alpha}$ and condition (1.11) can be written in terms of the parameter α in the equivalent form

$$\frac{q}{p} < 1 + \frac{2\alpha}{n}. \tag{1.12}$$

Differently, if we take under consideration the double phase integral (1.4) with the same coefficients $a(x) = 1$ and $b(x) = |x|^\alpha$, then a computation gives $\frac{q}{p} < 1 + \frac{\alpha}{n}$, as in the Colombo–Mingione Theorem 1.2.

1.2. Minimization versus solving a PDE equation

A problem in the calculus of variation is mainly formulated in terms of the *minimization* of a functional; while an evolution problem is usually described by a *differential equation* or a *system*. Let us see a *first known difficulty*, even in the definition, and even in the stationary (elliptic) case.

In the context of the double phase energy integral (1.4), if we assume the *continuity* of the coefficients $a(x), b(x)$ in the compact set $\overline{\Omega}$, since $q \geq p$ then

$$a(x)\,|Du(x)|^p + b(x)\,|Du(x)|^q \geq \text{const} \cdot |Du(x)|^p, \quad \text{if} \quad |Du(x)| \geq 1.$$

By its coercivity, the functional in (1.4) has *local minimizers* in the Sobolev class $W^{1,p}_{\text{loc}}(\Omega)$. We expect (however it is not always true!) that any local minimizer u is also a *weak solution* to the corresponding Euler's first variation, i.e., the PDE in divergence form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, Du) = 0, \quad x \in \Omega,$$

related to the vector-field $a(x, Du) = (a_i(x, Du))_{i=1,2,\dots,n}$ given by

$$a(x, \xi) =: \left\{ p\,a(x)\,|Du(x)|^{p-2} + q\,b(x)\,|Du(x)|^{q-2} \right\} Du(x),$$

which satisfies the growth condition

$$|a(x, \xi)| \leq M \left(1 + |\xi|^{q-1} \right).$$

for some constant $M > 0$ and for every $\xi \in \mathbb{R}^n$. Is it enough the summability property that comes from the coercivity condition $u \in W^{1,p}_{\text{loc}}(\Omega)$? We have in fact to require that $u \in W^{1,q}_{\text{loc}}(\Omega)$. Let us recall here the reason. If $u \in W^{1,q}_{\text{loc}}(\Omega)$ we obtain

$$|a(x, Du)| \leq M \left(1 + |Du|^{q-1} \right) \in L^{\frac{q}{q-1}}_{\text{loc}}(\Omega) = L^{q'}_{\text{loc}}(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

and $u \in W^{1,q}_{\text{loc}}(\Omega)$ would satisfy the (correct) weak form of the equation

$$\int_{\Omega} \sum_{i=1}^{n} a_i(x, Du) \frac{\partial \varphi}{\partial x_i}\, dx = 0, \quad \forall \varphi \in W^{1,q}_0(\Omega),\ \text{supp } \varphi \subset \Omega.$$

However a *minimizer* is only a function of class $W^{1,p}_{\text{loc}}(\Omega)$! While, for the validity of the *weak equation* we have to impose *a priori* that $u \in W^{1,q}_{\text{loc}}(\Omega)$. This is a difference (and a difficulty) with respect to the case of the so-called *natural growth conditions* with $q = p$.

We refer to Carozza–Kristensen–Passarelli [19] for general conditions for the validity of the Euler–Lagrange equation in the weak sense.

We emphasize that in the *parabolic context* below we do not assume growth conditions from above in the existence result; while in the regularity results – when possible – we consider general growth assumptions, of the type of the $p, q-$growth conditions above.

2. Evolution problems

As shown in the previous section, even in the elliptic case there is a gap between minimization and solution of equations and systems. Of course we have this difficulty in the evolution problems too, emphasized by the fact that an evolution problem is usually formulated by a differential equation and not as a minimization.

We adopt here a different point of view and we pose this "philosophical" question: does it exist in *evolution problems* a counterpart of *the minimization property*? I.e., may *a solution of an evolution problem* be *a variational minimizer*?

Let us recall a parabolic (in this case) *equation* (later a *system*), for instance, of the type

$$\partial_t u = \operatorname{div} a\left(x, t, Du\right),$$

on a parabolic space–time cylinder $\Omega_T = \Omega \times (0, T)$. A *weak solution* to this parabolic equation, by definition, should satisfy the *weak form* of the equation

$$\int_\Omega \left\{ u\varphi_t - \left(a\left(x, t, Du\right), D\varphi\right) \right\} dx \, dt = 0, \quad \forall \varphi \in C_0^1\left(\Omega_T\right).$$

In general this definition, which corresponds to the *distributional definition*, does not give existence of an associated *Cauchy–Dirichlet problem*. To get existence of weak solutions we have to impose growth conditions on the vector field $a\left(x, t, Du\right)$. For instance the natural growth conditions $(q = p)$ or, more generally, the so-called p, q−growth conditions, where a weak solution to an associated *Cauchy–Dirichlet problem*

$$\begin{cases} \partial_t u = \operatorname{div} a\left(x, t, Du\right) & \text{in } \Omega_T \\ u = u_0 & \text{on } \partial_{\mathrm{par}} \Omega_T \end{cases} \tag{2.1}$$

is a function u in the class $u \in L^p\left(0, T; W^{1,p}\left(\Omega\right)\right) \cap L_{\mathrm{loc}}^q\left(0, T; W_{\mathrm{loc}}^{1,q}\left(\Omega\right)\right)$. Here the Cauchy–Dirichlet problem needs an initial and boundary datum u_0 in the class

$$\begin{cases} u_0 \in L^{\frac{p(q-1)}{p-1}}\left(0, T; W^{1, \frac{p(q-1)}{p-1}}\left(\Omega\right)\right) \cap C^0\left([0, T]; L^2\left(\Omega\right)\right) \\ \partial_t u_0 \in L^{p'}\left(0, T; W^{-1,p'}\left(\Omega\right)\right) \end{cases};$$

on $\partial\Omega \times (0, T)$ the lateral boundary condition holds in the usual sense

$$u\left(\cdot, t\right) \in u_0\left(\cdot, t\right) + W_0^p\left(\Omega\right), \quad \text{a.e. } t \in (0, T)$$

and the initial datum $u\left(x, 0\right) = u_0\left(x, 0\right)$ in the L^2−sense, i.e.

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h dt \int_\Omega \left| u\left(x, t\right) - u_0\left(x, 0\right) \right|^2 dx = 0.$$

Classical existence theory for parabolic equations with the basic standard assumptions is well established and goes back to the book in 1969 by J.L.Lions [39] and that one in 1973 by H.Brézis [17]. About regularity we could quote much more, included the more recent book, published in 2012 by DiBenedetto–Gianazza–Vespri [32]. Specific for the minimal surfaces operator, and mainly related to the subject considered here, is the article appeared in 1978 by Lichnewsky–Temam [38]. A similar definition has been given in 1987 by Wieser [54]. The theory has been also developed in the context of metric spaces by the use of a *discretization method in time*, which is nowadays called *minimal movements*; we refer in particular to the monograph published in 2008 by Ambrosio–Gigli–Savaré [1].

2.1. The variational approach for parabolic equations and systems

We are concerned with the existence for evolutionary problems possessing a variational structure, in the sense that we construct solutions which inherit a certain minimizing property. A systematic approach started in 2013 in a joint research project by Bögelein–Duzaar–Marcellini (see the references from [8–15]), also in collaboration with S.Signoriello and C.Scheven.

We consider a parabolic system; with respect to the previous notations (2.1), the vector field $a\,(x, u, \xi)$ is now given by the $N \times n$ matrix

$$a\,(x, u, \xi) = D_\xi f\,(x, u, \xi)\,,$$

associated with a *convex integrand* $f : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to [0, \infty)$, convex with respect to (u, ξ), only satisfying a *coercivity condition* from below, but not satisfying any growth condition a-priori fixed from above!

To establish the existence of solutions we introduce the concept of *variational solution*, also named *parabolic minimizer*. Variational solutions exist under weaker assumptions than weak solutions. The advantage of these variational solutions also comes from the fact that they might exist even in situations where the associated parabolic system makes no sense. This is the main point: we construct variational solutions to evolutionary problems under general assumptions on the integrand, where *a-priori* it is not clear at all that these minimizers also solve the associated evolutionary system. At the same time we propose further sufficient growth conditions to ensure that the variational solution are also weak solutions to the differential system.

To explain the main ideas and the results in more detail, we start with a variational integrand $f \colon \Omega \times \mathbb{R}^{N \times n} \to (-\infty, +\infty]$, a given map $h \colon \Omega \to \mathbb{R}^N$ and an initial datum $u_o \colon \Omega \to \mathbb{R}^N$. Here, Ω denotes a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and $\Omega_\infty := \Omega \times (0, \infty)$ stands for the infinite space–time cylinder over Ω. Again $N > 1$ and we are dealing with parabolic systems. As before, the Cauchy–Dirichlet problem − formally − takes the form

$$\begin{cases} \partial_t u = \mathrm{div} D_\xi f(x, Du) + h(x) & \text{in } \Omega_\infty, \\ u = u_o & \text{on } \partial_{\mathcal{P}} \Omega_\infty, \end{cases}$$

where $u \colon \Omega_\infty \subset \mathbb{R}^{n+1} \to \mathbb{R}^N$ and $\partial_{\mathcal{P}} \Omega_\infty := [\partial \Omega \times (0, \infty)] \cup [\overline{\Omega} \times \{0\}]$ denotes the parabolic boundary of Ω_∞. Here the assumptions on the Carathéodory function $f \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$: it is a *convex* function with respect to (u, ξ) which satisfies the coercivity condition

$$f(x, u, \xi) \geq \nu |\xi|^p - g(x)(1 + |u|), \quad \forall (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n}, \tag{2.2}$$

for some $\nu > 0$ and $p > 1$, with $g \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $g \geq 0$ a.e. in Ω.

The assumptions of our theorem cover a large variety of interesting variational functionals already considered in the literature. Among them there are variational integrands fulfilling a standard growth condition from below and above, functionals of non-standard $p, q-$growth and double phase (as in [4–6,23,24] and [21,29,34,35]), functionals with exponential growth (see [7,44,46]), variable exponents [21,26,33,36,37] and Orlicz-type functionals (as in [22]):

$$\begin{aligned} f_1(x, Du) &= \alpha(x)|Du|^p + \beta(x)|Du|^q \quad (p, q- \text{ growth, double phase}), \\ f_2(x, Du) &= \alpha_1(x)|u_{x_1}|^{p_1} + \cdots + \alpha_n(x)|u_{x_n}|^{p_n} \quad (\text{anisotropic}), \\ f_3(x, Du) &= |Du|^{p(x)} \quad (\text{variable exponents}), \\ f_4(Du) &= |Du|^p \log(1 + |Du|) \quad (\text{Orlicz-type}), \\ f_5(Du) &= e^{|Du|^r} \quad (\text{exponential growth}). \end{aligned} \tag{2.3}$$

Note that all functions f_i in (2.3), with the only exception of the last exponential case f_5, are examples of $p, q-$*growth*. For instance, for f_2 we have

$$p = \min \{p_i, \ i = 1, 2, \ldots, n\}\,, \qquad q = \max \{p_i, \ i = 1, 2, \ldots, n\}\,;$$

while for f_3, with a bounded exponent $p\,(x)$, we can consider p, q as the infimum and the supremum of $p\,(x)$; i.e., $p \leq p\,(x) \leq q$. In the relevant case when the exponent $p\,(x)$ is continuous, in the context of the regularity

theory for every $x_0 \in \Omega$ it is sufficient to consider a ball $B_r(x_0) \subset \Omega$ with center in x_0 and with small radius r, and define

$$p = \inf\{p(x): \ x \in B_r(x_0)\}, \qquad q = \sup\{p(x): \ x \in B_r(x_0)\},$$

so that p, q can be chosen as close as necessary (in dependence of r small) in order to apply the regularity theory valid under $p, q-$growth conditions.

For the initial and boundary datum $u_o \in W^{1,p}(\Omega, \mathbb{R}^N)$ we assume that

$$u_o \in L^2(\Omega, \mathbb{R}^N) \quad \text{and} \quad \int_\Omega f(x, u_o, Du_o)\, dx \in \mathbb{R}. \tag{2.4}$$

The associated Cauchy–Dirichlet problem on Ω_∞ is

$$\begin{cases} \partial_t u = \operatorname{div} D_\xi f(x, u, Du) - D_u f(x, u, Du) & \text{in } \Omega_\infty, \\ u = u_o & \text{on } \partial_\mathcal{P} \Omega_\infty. \end{cases} \tag{2.5}$$

In the following definition we describe the concept of *variational solutions* to Cauchy–Dirichlet problems. We follow an idea by Lichnewsky and Temam [38] which was first used in the context of the evolutionary parametric minimal surface equation. Variational solutions are sometimes also called *parabolic minimizers*.

How to treat the time-derivative $\partial_t u$ in the equation $\partial_t u = \operatorname{div} D_\xi f - D_u f$? As usually, we multiply by a test function φ which has the zero value at the *lateral boundary* , say $\varphi = v - u$, with $v \in L^p(0, T; W_{u_o}^{1,p}(\Omega, \mathbb{R}^N))$. Formally we obtain

$$\int_0^T \int_\Omega \partial_t u \cdot (v - u)\, dxdt$$

$$= \int_0^T \int_\Omega \partial_t (u - v) \cdot (v - u)\, dxdt + \int_0^T \int_\Omega \partial_t v \cdot (v - u)\, dxdt$$

$$= -\frac{1}{2} \int_0^T \int_\Omega \partial_t (|v - u|^2)\, dxdt + \int_0^T \int_\Omega \partial_t v \cdot (v - u)\, dxdt$$

$$= \frac{1}{2}\left\{ \|v(\cdot, 0) - u_o\|_{L^2(\Omega)}^2 - \|(v - u)(\cdot, T)\|_{L^2(\Omega)}^2 \right\} + \int_0^T \int_\Omega \partial_t v \cdot (v - u)\, dxdt .$$

Definition 2.1 (*Variational Solution*). Let $f: \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \overline{\mathbb{R}} = (-\infty, +\infty]$ be a convex function satisfying the coercivity assumption (2.2). A map $u: \Omega_\infty = \Omega \times (0, \infty) \to \mathbb{R}^N$ in the class

$$u \in L^p\left(0, T; W_{u_o}^{1,p}(\Omega, \mathbb{R}^N)\right) \cap C^0\left([0, T]; L^2(\Omega, \mathbb{R}^N)\right), \quad \text{for any } T > 0$$

is a *variational solution* (in Ω_∞) to the stated Cauchy–Dirichlet problem (2.5) if

$$\int_0^T \int_\Omega f(x, u, Du)\, dxdt \leq \int_0^T \int_\Omega f(x, v, Dv)\, dxdt$$

$$+ \int_0^T \int_\Omega \partial_t v \cdot (v - u)\, dxdt + \tfrac{1}{2}\|v(\cdot, 0) - u_o\|_{L^2(\Omega)}^2 - \tfrac{1}{2}\|(v - u)(\cdot, T)\|_{L^2(\Omega)}^2$$

for every $T > 0$ and $v \in L^p(0, T; W_{u_o}^{1,p}(\Omega, \mathbb{R}^N))$ with $\partial_t v \in L^2(\Omega_T, \mathbb{R}^N)$.

The following two theorems give existence and uniqueness of the variational solution. For their proofs see Section 3 (see also Bögelein–Duzaar–Marcellini and the references from [8–13]).

Theorem 2.2. _Let_ $f \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to [0, +\infty]$ _(for simplicity we state the result in the case_ $f \geq 0$_) be a convex function satisfying the coercivity assumption_ (2.2) _and let_ $u_o \in L^2\left(\Omega, \mathbb{R}^N\right)$ _be a Cauchy–Dirichlet datum with finite energy; i.e.,_ $\int_\Omega f\left(x, u_o, Du_o\right) dx < \infty$. _Then there exists a variational solution according to_ Definition 2.1.

Theorem 2.3. _Under the assumptions of the previous_ Theorem 2.2 _the variational solution_ u _is unique, it belongs to the functional space_ $C^{0,1/2}\left([0, T]; L^2(\Omega, \mathbb{R}^N)\right)$ _and_ $\partial_t u \in L^2\left(\Omega_T, \mathbb{R}^N\right)$.

2.2. From variational solutions to weak solutions

The passage from the minimality condition satisfied by a parabolic variational solution to the validity of the associated parabolic system in weak form is possible under certain additional assumptions on the convex function f. Precisely − for instance − we can restrict ourselves to a classical case, where the integrand $f \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is a Carathéodory-function, coercive and convex with respect to (u, ξ) and in addition the following growth condition from above

$$f\left(x, u, \xi\right) \leq M\left(1 + |u|^p + |\xi|^p\right).$$

Or we can assume that f satisfies a non-standard and $p, q-$growth condition of the type

$$\nu|\xi|^p \leq f\left(x, u, \xi\right) \leq M(1 + |u|^q + |\xi|^q).$$

Let us first proceed formally, then we will give a complete proof. We consider the minimality condition for the variational solution

$$\int_0^T \int_\Omega f(x, u, Du)\, dxdt \leq \int_0^T \int_\Omega f(x, v, Dv)\, dxdt$$
$$+ \int_0^T \int_\Omega \partial_t v \cdot (v - u)\, dxdt + \tfrac{1}{2}\|v(\cdot, 0) - u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(v - u)(\cdot, T)\|^2_{L^2(\Omega)},$$

valid for every $T > 0$ and $v \in L^p(0, T; W^{1,p}_{u_o}\left(\Omega, \mathbb{R}^N\right))$ with $\partial_t v \in L^2(\Omega_T, \mathbb{R}^N)$. By Theorem 2.3 $\partial_t u \in L^2(\Omega_T, \mathbb{R}^N)$, thus we can use the test function $v \equiv u + \varepsilon\varphi$, with $\varepsilon \in (0, 1)$ and $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$ and we obtain

$$\int_0^T \int_\Omega \partial_t \left(u + \varepsilon\varphi\right) \cdot \varepsilon\varphi\, dxdt \qquad\qquad (2.6)$$
$$+ \int_0^T \int_\Omega \left\{f(x, u + \varepsilon\varphi, Du + \varepsilon D\varphi) - f(x, u, Du)\right\}\, dxdt \geq 0.$$

We divide both sides by $\varepsilon > 0$ and we let $\varepsilon \to 0^+$; we get

$$\int_0^T \int_\Omega \left\{\partial_t u \cdot \varphi + D_\xi f(x, u, Du) \cdot D\varphi + D_u f(x, u, Du) \cdot \varphi\right\}\, dxdt \geq 0$$

for every $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$. Here, we can replace φ by $-\varphi$ to obtain the reversed inequality, so that the variational solution solves the associated parabolic system and therefore is a weak solution to the Cauchy–Dirichlet problem (2.5), since also $u = u_o$ on $\partial_\mathcal{P} \Omega_\infty$.

As we said, this formal computation becomes a real proof of the existence of a weak solution to the Cauchy–Dirichlet problem if we assume some growth conditions from above for the convex function f. Precisely − for instance − we can restrict ourselves to a classical case, where the integrand f is a Carathéodory-function, coercive and convex with respect to (u, ξ) and satisfying the growth condition from above

$$f(x, u, \xi) \leq M\left(1 + |u|^p + |\xi|^p\right). \qquad\qquad (2.7)$$

By its convexity f is almost everywhere differentiable, locally Lipschitz function with respect to (u, ξ) and, by a result proved in the Step 2 of [41], there exists a constant $c = c(M, p)$ such that

$$|D_\xi f(x, u, \xi)| + |D_u f(x, u, \xi)| \leq c\left(1 + |u|^{p-1} + |\xi|^{p-1}\right). \tag{2.8}$$

This growth allows us to pass to the limit above as $\varepsilon \to 0^+$ in (2.6) by the Lebesgue dominated convergence theorem.

More generally in [8,9] we obtained the derivation of the parabolic system if the integrand $f \in C^2$ satisfies a non-standard growth condition of the type

$$\nu|\xi|^p \leq f(\xi) \leq M(1 + |\xi|^q)$$

when $2 \leq p \leq q < p + \min\{1, \frac{4}{n}\}$.

2.3. The De Giorgi's conjecture

The proof of Theorem 2.2 goes in the spirit of a De Giorgi's conjecture stated in 1996 for the wave equation in [30] (it can be also found in De Giorgi's *Selected Papers* [31]). The conjecture concerns the existence of global weak solutions to the Cauchy problem for *non-linear hyperbolic wave equations* on $\mathbb{R}^n$. Nowadays it has been solved in 2012 by Serra and Tilli [51] only in the *linear context* with the Laplacian operator as the principal part in the hyperbolic wave equation. See also Stefanelli [52] and Tentarelli–Tilli [53] for similar results.

We refer to the original paper by Ennio De Giorgi in the hyperbolic context and we state here its *modified* conjecture, in the parabolic framework, for a given Carathéodory function $f(x, u, \xi)$ which is convex with respect to (u, ξ).

Conjecture 2.4 (*Parabolic Version of De Giorgi's Original One*). *As $\varepsilon \to 0^+$, up to a subsequence the minimizer $u_\varepsilon(x, t)$ of the De Giorgi's energy integral $F_\varepsilon(u)$ given by*

$$F_\varepsilon(u) := \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \left\{\frac{1}{2}|\partial_t u|^2 + \frac{1}{\varepsilon}f(x, u, Du)\right\} dx dt \tag{2.9}$$

converges in $L^2\left(\Omega_T, \mathbb{R}^N\right)$ to the variational solution $u(x, t)$ of the Cauchy–Dirichlet problem (2.1). The solution $u(x, t)$ is intended in the sense of Definition 2.1.

The proof of this conjecture in fact, at the same time, is the proof of Theorem 2.2 In the next section we see in detail the idea of the proof of this existence theorem. Before we give a heuristic explanation of this Ennio De Giorgi's conjecture.

To explain why the sequence u_ε is expected to converge to a solution of the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u = \operatorname{div} D_\xi f(x, u, Du) - D_u f(x, u, Du) & \text{in} \quad \Omega_T = \Omega \times (0, T) \\ u = u_o & \text{on } \partial_\mathcal{P} \Omega_T \end{cases}$$

we can compute the Euler–Lagrange system of F_ε in (2.9) in its classical form. From the classical form of the Euler first variation one easily deduces that, for every $\varepsilon > 0$, the minimizer u_ε of the integral F_ε formally solves the second order elliptic regularized system

$$-\varepsilon\,\partial_{tt}u_\varepsilon + \partial_t u_\varepsilon = \operatorname{div} D_\xi f(x, u_\varepsilon, Du_\varepsilon) - D_u f(x, u_\varepsilon, Du_\varepsilon), \tag{2.10}$$

and moreover fulfill the Cauchy–Dirichlet boundary condition $u_\varepsilon = u_o$ on $\partial_\mathcal{P}\Omega_\infty$. In fact, from the integral $F_\varepsilon(u)$ in (2.9) we formally get the Euler–Lagrange system

$$-\frac{\partial}{\partial t}\left(e^{-\frac{t}{\varepsilon}}\partial_t u\right) - \operatorname{div}_x\left(\frac{1}{\varepsilon}e^{-\frac{t}{\varepsilon}}D_\xi f\right) + \frac{1}{\varepsilon}e^{-\frac{t}{\varepsilon}}D_u f = 0\,,$$

that is, equivalently,

$$+ \tfrac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \partial_t u - e^{-\frac{t}{\varepsilon}} \partial_{tt} u - \tfrac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \{ \mathrm{div}_x D_\xi f - D_u f \} = 0 \,,$$
$$- \varepsilon \, \partial_{tt} u + \partial_t u - \mathrm{div}_x D_\xi f + D_u f = 0 \,.$$

Therefore it seems to be natural to consider in (2.10) the limit as $\varepsilon \to 0^+$. The term $\varepsilon \, \partial_{tt} u_\varepsilon$ should disappear as $\varepsilon \to 0^+$. Formally this would lead to a solution u of the Cauchy–Dirichlet problem, provided we could establish the convergence $u_\varepsilon \to u$ in an appropriate sense.

In general we cannot expect that minimizers satisfy the Euler–Lagrange system. Furthermore, even when the integrand is $f(\xi) = \tfrac{1}{p} |\xi|^p$ with $p \neq 2$, we would not be allowed to pass to the limit $\varepsilon \to 0^+$ in the Euler–Lagrange system, since this would require the a.e. pointwise convergence of $Du_\varepsilon \to Du$; we cannot expect this property, *unless we have a linear elliptic operator* which matches with the weak convergence of Du_ε.

The main idea to overcome this difficulty is to remain at the level of minimizers; i.e. not to pass to the Euler–Lagrange system. Instead of going from equality to equality, that is from the parabolic equation for an approximated problem to an equality represented by the parabolic equation for the final problem, we go from an inequality to another inequality. The minimality inequality for of u_ε is $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(u_\varepsilon + v)$; starting from here we arrive to the *variational parabolic solution*.

Note that the method of proof is *typically nonlinear*: here − for instance − the *inequalities* for convexity and for lower semicontinuity become *equalities* only if the elliptic operator is linear; i.e., if it is a second order linear elliptic operator − of the type of the Laplacian operator − and the parabolic equation is the heat equation. On the contrary below we treat the general nonlinear case.

3. Proof of the existence of parabolic variational solutions

We give here the proof of Theorem 2.2 . As we said above, the proof of the existence theorem is realized by the proof of the De Giorgi's Conjecture 2.4.

3.1. Step 1 (existence of the approximate solution $u_\varepsilon(x,t)$)

Let $T > 0$ be fixed. For a time independent datum $u_o : \Omega \to \mathbb{R}^N$ we consider mappings $u \colon \Omega_T = \Omega \times (0,T) \to \mathbb{R}^N$ satisfying the Cauchy–Dirichlet boundary condition $u = u_o$ on $\partial_{\mathcal{P}} \Omega_\infty$. For given $\varepsilon \in (0,1]$ we consider the *convex* energy integral $F_\varepsilon(u)$ defined in (2.9). For every $\varepsilon > 0$ the growth assumption from below on the integrand f imply that F_ε is coercive. The convexity of f ensures the lower semicontinuity of F_ε in L^1 and allows us the application of standard methods from the calculus of variations for the existence of minimizers u_ε in the class of maps with partial time-derivative $\partial_t u$ L^2−summable and its spatial-gradient Du L^p−summable with respect to $d\mu = e^{-\frac{t}{\varepsilon}} \, dxdt$.

3.2. Step 2 (first a priori easy bound of the functional)

The real sequence $F_\varepsilon(u_\varepsilon)$ is bounded. In fact

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon) &= \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \left\{ \tfrac{1}{2} |\partial_t u_\varepsilon|^2 + \tfrac{1}{\varepsilon} f(x, u_\varepsilon, Du_\varepsilon) \right\} dxdt \\
&\leq F_\varepsilon(u_0) = \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \left\{ \tfrac{1}{\varepsilon} f(x, u_o, Du_o) \right\} dxdt \\
&\leq \int_\Omega f(x, u_o, Du_o) \, dx \cdot \underbrace{\int_0^\infty \tfrac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \, dt}_{=1} = \\
&= \int_\Omega f(x, u_o, Du_o) \, dx \cdot \left[-e^{-\frac{t}{\varepsilon}} \right]_{t=0}^{t=+\infty} = \int_\Omega f(x, u_o, Du_o) \, dx \,.
\end{aligned}
\tag{3.1}
$$

3.3. Step 3 (use of the minimality and convexity to treat the exponential term)

We use here the minimality of u_ε

$$F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(u_\varepsilon + v),$$

valid for every test function v equal to zero on the parabolic boundary. That is

$$\int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \left\{ \frac{1}{2} |\partial_t u_\varepsilon|^2 + \frac{1}{\varepsilon} f(x, u_\varepsilon, Du_\varepsilon) \right\} dxdt$$
$$\leq \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \left\{ \frac{1}{2} |\partial_t(u_\varepsilon + v)|^2 + \frac{1}{\varepsilon} f(x, u_\varepsilon + v, D(u_\varepsilon + v)) \right\} dxdt.$$

Equivalently, we separate the x and the t variables

$$0 \leq \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \frac{1}{2} \left\{ |\partial_t(u_\varepsilon + v)|^2 - |\partial_t u_\varepsilon|^2 \right\} dxdt$$
$$+ \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \frac{1}{\varepsilon} \left\{ f(x, u_\varepsilon + v, D(u_\varepsilon + v)) - f(x, u_\varepsilon, Du_\varepsilon) \right\} dxdt.$$

We consider a test function of the form $v = \delta e^{+\frac{t}{\varepsilon}} \varphi(x, t)$:

$$0 \leq \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \frac{1}{2} \left\{ \left| \partial_t \left(u_\varepsilon + \delta e^{+\frac{t}{\varepsilon}} \varphi \right) \right|^2 - |\partial_t u_\varepsilon|^2 \right\} dxdt$$
$$+ \int_0^T \int_\Omega \frac{e^{-\frac{t}{\varepsilon}}}{\varepsilon} \left\{ f\left(x, u_\varepsilon + \delta e^{\frac{t}{\varepsilon}} \varphi, Du_\varepsilon + \delta e^{\frac{t}{\varepsilon}} D\varphi \right) - f(x, u_\varepsilon, Du_\varepsilon) \right\} dxdt.$$

By the convexity of $f(x, \cdot, \cdot)$, for δ small, precisely $\delta e^{+\frac{t}{\varepsilon}} \leq \delta e^{+\frac{T}{\varepsilon}} \leq 1$, we have

$$f\left(x, u_\varepsilon + \delta e^{+\frac{t}{\varepsilon}} \varphi, Du_\varepsilon + \delta e^{+\frac{t}{\varepsilon}} D\varphi \right)$$
$$= f\left(x, \left(1 - \delta e^{\frac{t}{\varepsilon}}\right) u_\varepsilon + \delta e^{\frac{t}{\varepsilon}} (u_\varepsilon + \varphi), \left(1 - \delta e^{\frac{t}{\varepsilon}}\right) Du_\varepsilon + \delta e^{\frac{t}{\varepsilon}} D(u_\varepsilon + \varphi) \right)$$
$$\leq \left(1 - \delta e^{+\frac{t}{\varepsilon}}\right) f(x, u_\varepsilon, Du_\varepsilon) + \delta e^{+\frac{t}{\varepsilon}} f(x, u_\varepsilon + \varphi, D(u_\varepsilon + \varphi))$$

or equivalently

$$f\left(x, u_\varepsilon + \delta e^{+\frac{t}{\varepsilon}} \varphi, Du_\varepsilon + \delta e^{+\frac{t}{\varepsilon}} D\varphi \right) - f(x, u_\varepsilon, Du_\varepsilon)$$
$$\leq \delta e^{+\frac{t}{\varepsilon}} \left\{ f(x, u_\varepsilon + \varphi, D(u_\varepsilon + \varphi)) - f(x, u_\varepsilon, Du_\varepsilon) \right\}$$

and thus we get

$$0 \leq \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \frac{1}{2} \left\{ \left| \partial_t \left(u_\varepsilon + \delta e^{+\frac{t}{\varepsilon}} \varphi \right) \right|^2 - |\partial_t u_\varepsilon|^2 \right\} dxdt$$
$$+ \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \frac{1}{\varepsilon} \left\{ f\left(x, u_\varepsilon + \delta e^{\frac{t}{\varepsilon}} \varphi, Du_\varepsilon + \delta e^{\frac{t}{\varepsilon}} D\varphi \right) - f(x, u_\varepsilon, Du_\varepsilon) \right\} dxdt$$
$$\leq \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \frac{1}{2} \left\{ \left| \partial_t \left(u_\varepsilon + \delta e^{+\frac{t}{\varepsilon}} \varphi \right) \right|^2 - |\partial_t u_\varepsilon|^2 \right\} dxdt$$
$$+ \int_0^T \int_\Omega \frac{e^{-\frac{t}{\varepsilon}}}{\varepsilon} \delta e^{+\frac{t}{\varepsilon}} \left\{ f(x, u_\varepsilon + \varphi, D(u_\varepsilon + \varphi)) - f(x, u_\varepsilon, Du_\varepsilon) \right\} dxdt.$$

Thus we obtained

$$
0 \leq \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \frac{1}{2} \left\{ \left| \partial_t \left(u_\varepsilon + \delta e^{+\frac{t}{\varepsilon}} \varphi \right) \right|^2 - |\partial_t u_\varepsilon|^2 \right\} \, dxdt
$$
$$
+ \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \frac{\delta}{\varepsilon} \cdot e^{+\frac{t}{\varepsilon}} \left\{ f\left(x, u_\varepsilon + \varphi, D\left(u_\varepsilon + \varphi \right) \right) - f\left(x, u_\varepsilon, Du_\varepsilon \right) \right\} \, dxdt
$$

On the right hand side the factors $e^{\frac{t}{\varepsilon}}$ and $e^{-\frac{t}{\varepsilon}}$ cancel. We divide by $\frac{\delta}{\varepsilon}$, we let $\delta \to 0^+$ and we get

$$
\int_0^T \int_\Omega f\left(x, u_\varepsilon, Du_\varepsilon \right) \, dxdt \leq \int_0^T \int_\Omega f\left(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi \right) \, dxdt
$$
$$
+ \int_0^T \int_\Omega \varepsilon e^{-\frac{t}{\varepsilon}} \partial_t u_\varepsilon \partial_t \left(e^{+\frac{t}{\varepsilon}} \varphi\left(x, t \right) \right) \, dxdt
$$

In order to simplify the second addendum on the right hand side in

$$
\int_0^T \int_\Omega f\left(x, u_\varepsilon, Du_\varepsilon \right) \, dxdt \leq \int_0^T \int_\Omega f\left(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi \right) \, dxdt
$$
$$
+ \int_0^T \int_\Omega \varepsilon e^{-\frac{t}{\varepsilon}} \partial_t u_\varepsilon \partial_t \left(e^{+\frac{t}{\varepsilon}} \varphi\left(x, t \right) \right) \, dxdt
$$

we compute $\partial_t \left(e^{+\frac{t}{\varepsilon}} \varphi\left(x, t \right) \right) = e^{+\frac{t}{\varepsilon}} \left(\frac{1}{\varepsilon} \varphi + \partial_t \varphi \right)$ and we get

$$
\int_0^T \int_\Omega f\left(x, u_\varepsilon, Du_\varepsilon \right) \, dxdt \leq \int_0^T \int_\Omega f\left(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi \right) \, dxdt \tag{3.2}
$$
$$
+ \int_0^T \int_\Omega \partial_t u_\varepsilon \left(\varphi + \varepsilon \partial_t \varphi \right) \, dxdt .
$$

3.4. Step 4 (a priori estimate for the spatial derivative Du_ε)

The minimization property

$$
\int_0^T \int_\Omega f\left(x, u_\varepsilon, Du_\varepsilon \right) \, dxdt \leq \int_0^T \int_\Omega f\left(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi \right) \, dxdt \tag{3.3}
$$
$$
+ \int_0^T \int_\Omega \partial_t u_\varepsilon \left(\varphi + \varepsilon \partial_t \varphi \right) \, dxdt
$$

is valid for a generic test function $\varphi = \varphi\left(x, t \right)$ equal to zero on the parabolic boundary $\partial_\mathcal{P} \Omega_T$. Since $u_\varepsilon = u_o$ on $\partial_\mathcal{P} \Omega_T$, we can choose $\varphi = u_o - u_\varepsilon$ and we obtain

$$
\int_0^T \int_\Omega f\left(x, u_\varepsilon, Du_\varepsilon \right) \, dxdt \leq \int_0^T \int_\Omega f\left(x, u_o, Du_o \right) \, dxdt
$$
$$
+ \int_0^T \int_\Omega \partial_t u_\varepsilon \left(u_o - u_\varepsilon - \varepsilon \partial_t u_\varepsilon \right) \, dxdt
$$

equivalently

$$
\int_0^T \int_\Omega f\left(x, u_\varepsilon, Du_\varepsilon \right) \, dxdt \leq T \int_\Omega f\left(x, u_o, Du_o \right) \, dx
$$
$$
- \frac{1}{2} \int_0^T \int_\Omega \partial_t \left| u_o - u_\varepsilon \right|^2 \, dxdt - \varepsilon \int_0^T \int_\Omega \left| \partial_t u_\varepsilon \right|^2 \, dxdt
$$

and therefore

$$\int_0^T \int_\Omega f(x, u_\varepsilon, Du_\varepsilon) \; dxdt + \frac{1}{2} \int_\Omega |u_o(x) - u_\varepsilon(x, T)|^2 \; dx$$

$$+ \varepsilon \int_0^T \int_\Omega |\partial_t u_\varepsilon|^2 \; dxdt \leq T \int_\Omega f(x, u_o, Du_o) \; dx \,.$$

Thus, by the coercivity condition (2.2), the spatial gradient-derivative Du_ε is bounded in $L^p(\Omega_T, \mathbb{R}^{N \times n})$ uniformly for $\varepsilon \in (0, 1]$ (while we did not jet get a bound for the time derivative $\partial_t u_\varepsilon$).

3.5. Step 5 (a priori estimate for the time derivative $\partial_t u_\varepsilon$)

Let us recall the previous estimate

$$\int_0^T \int_\Omega f(x, u_\varepsilon, Du_\varepsilon) \; dxdt \leq \int_0^T \int_\Omega f(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi) \; dxdt \qquad (3.4)$$

$$+ \int_0^T \int_\Omega \partial_t u_\varepsilon (\varphi + \varepsilon \partial_t \varphi) \; dxdt \,.$$

Instead to use the test function $\varphi = u_o - u_\varepsilon$, which is good for the term $f(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi) = f(x, u_o, Du_o)$ but it is not good for

$$\partial_t u_\varepsilon (\varphi + \varepsilon \partial_t \varphi) = \partial_t u_\varepsilon (u_o - u_\varepsilon - \varepsilon \partial_t u_\varepsilon) = \partial_t |u_o - u_\varepsilon|^2 - \varepsilon |\partial_t u_\varepsilon|^2 \,,$$

in principle we should insert $\varphi = -\partial_t u_\varepsilon$, which gives the nice term

$$\partial_t u_\varepsilon (\varphi + \varepsilon \partial_t \varphi) = \partial_t u_\varepsilon (-\partial_t u_\varepsilon - \varepsilon \partial_{tt} u_\varepsilon) = -|\partial_t u_\varepsilon|^2 - \varepsilon \dots$$

but however creates a problem in the expression

$$f(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi) = f(x, u_\varepsilon - \partial_t u_\varepsilon, Du_\varepsilon - D\partial_t u_\varepsilon) \,.$$

A better idea is to use a test function φ of the type (h is a small positive parameter)

$$\varphi = -h\partial_t [u_\varepsilon]_h = [u_\varepsilon]_h - u_\varepsilon \,, \qquad (3.5)$$

where $[u_\varepsilon]_h$ is a kind of "substitute" of u_ε. More precisely, $[u_\varepsilon]_h(x, t)$ is the solution of the above ODE in time (3.5), with initial value at $t = 0$ given by u_o (x is a parameter here). By solving the one-dimensional Cauchy problem we get the precise representation for $[u_\varepsilon]_h$

$$[u_\varepsilon]_h(x, t) = e^{-\frac{t}{h}} u_o(x) + \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} u_\varepsilon(x, s) \; ds \,, \qquad (3.6)$$

and thus

$$\partial_t [u_\varepsilon]_h = -\frac{1}{h} e^{-\frac{t}{h}} u_o(x) + \frac{1}{h} u_\varepsilon(x, t) - \frac{1}{h^2} \int_0^t e^{\frac{s-t}{h}} u_\varepsilon(x, s) \; ds = \frac{1}{h}(u_\varepsilon - [u_\varepsilon]_h) \,.$$

Note that the test function $\varphi = \varphi(x, t) = [u_\varepsilon]_h - u_\varepsilon$ is equal to zero on the parabolic boundary $\partial_\mathcal{P} \Omega_\infty$. In fact $[u_\varepsilon]_h(x, 0) = u_o(x)$. Moreover $u_\varepsilon(x, t) = u_o(x)$ if $x \in \partial\Omega$; thus

$$x \in \partial\Omega \quad \Rightarrow \quad [u_\varepsilon]_h(x, t) = e^{-\frac{t}{h}} u_o(x) + u_o(x) \underbrace{\frac{1}{h} \int_0^t e^{\frac{s-t}{h}} ds}_{=1-e^{-\frac{t}{h}}} = u_o(x) \,.$$

Now a computation gives the desired bound for the time derivative. In fact in the previous estimate

$$\int_0^T \int_\Omega f(x, u_\varepsilon, Du_\varepsilon)\, dxdt \leq \int_0^T \int_\Omega f(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi)\, dxdt$$
$$+ \int_0^T \int_\Omega \partial_t u_\varepsilon (\varphi + \varepsilon \partial_t \varphi)\, dxdt$$

we insert the test function $\varphi = [u_\varepsilon]_h - u_\varepsilon = -h\partial_t [u_\varepsilon]_h$ and we obtain

$$\int_0^T \int_\Omega f(x, u_\varepsilon, Du_\varepsilon)\, dxdt \leq \int_0^T \int_\Omega f(x, [u_\varepsilon]_h, D[u_\varepsilon]_h)\, dxdt \tag{3.7}$$
$$- h \int_0^T \int_\Omega \partial_t u_\varepsilon (\partial_t [u_\varepsilon]_h + \varepsilon \partial_{tt} [u_\varepsilon]_h)\, dxdt .$$

We rewrite (3.7) by reordering the addenda end we use Jensen's inequality

$$h \int_0^T \int_\Omega \partial_t u_\varepsilon \partial_t [u_\varepsilon]_h\, dxdt + \varepsilon h \int_0^T \int_\Omega \partial_t u_\varepsilon \partial_{tt} [u_\varepsilon]_h\, dxdt \tag{3.8}$$
$$\leq \int_0^T \int_\Omega \{f(x, [u_\varepsilon]_h, [u_\varepsilon]_h) - f(x, u_\varepsilon, Du_\varepsilon)\}\, dxdt$$
$$\leq \int_0^T \int_\Omega \{[f(x, u_\varepsilon, Du_\varepsilon)]_h - f(x, u_\varepsilon, Du_\varepsilon)\}\, dxdt$$
$$= -h \int_0^T \int_\Omega \partial_t [f(x, u_\varepsilon, Du_\varepsilon)]_h\, dxdt ,$$

since, as before, $[w]_h - w = -h\, \partial_t [w]_h$. Separately in (3.8) we estimate the addendum with the second derivative in time. Since $[u_\varepsilon]_h - u_\varepsilon = -h\partial_t [u_\varepsilon]_h$, we have

$$\partial_t u_\varepsilon \partial_{tt} [u_\varepsilon]_h = \partial_t [u_\varepsilon]_h \partial_{tt} [u_\varepsilon]_h + (\partial_t u_\varepsilon - \partial_t [u_\varepsilon]_h)\partial_{tt} [u_\varepsilon]_h$$
$$= \partial_t [u_\varepsilon]_h \partial_{tt} [u_\varepsilon]_h + \partial_t (h\partial_t [u_\varepsilon]_h)\partial_{tt} [u_\varepsilon]_h$$
$$= \frac{1}{2}\partial_t |\partial_t [u_\varepsilon]_h|^2 + h|\partial_{tt} [u_\varepsilon]_h|^2 \geq \frac{1}{2}\partial_t |\partial_t [u_\varepsilon]_h|^2 .$$

By integrating both sides of the above inequality we get

$$\int_0^T \int_\Omega \partial_t u_\varepsilon \partial_{tt} [u_\varepsilon]_h\, dxdt \geq \int_0^T \int_\Omega \frac{1}{2}\partial_t |\partial_t [u_\varepsilon]_h|^2\, dxdt \tag{3.9}$$
$$= \frac{1}{2} \int_\Omega \left\{ |\partial_t [u_\varepsilon]_h|^2 \Big|_{t=T} - |\partial_t [u_\varepsilon]_h|^2 \Big|_{t=0} \right\} dx$$
$$= \int_\Omega \left\{ |\partial_t [u_\varepsilon]_h|^2 \Big|_{t=T} \right\} dx \geq 0$$

the last equality being possible since $|\partial_t [u_\varepsilon]_h|^2\Big|_{t=0} = 0$. Therefore, from (3.8), (3.9) we deduce

$$h \int_0^T \int_\Omega \partial_t u_\varepsilon \partial_t [u_\varepsilon]_h\, dxdt \leq -h \int_0^T \int_\Omega \partial_t [f(x, u_\varepsilon, Du_\varepsilon)]_h\, dxdt$$

and also

$$\int_0^T \int_\Omega \partial_t u_\varepsilon \partial_t [u_\varepsilon]_h\, dxdt \leq - \int_0^T \int_\Omega \partial_t [f(x, u_\varepsilon, Du_\varepsilon)]_h\, dxdt \tag{3.10}$$
$$= \int_\Omega \left\{ f(x, u_\varepsilon, Du_\varepsilon)_h|_{t=0} - f(x, u_\varepsilon, Du_\varepsilon)_h|_{t=T} \right\} dx .$$
$$\leq \int_\Omega f(x, u_o, Du_o)\, dx .$$

In the limit in (3.10) as $h \to 0^+$ we finally obtain

$$\|\partial_t u_\varepsilon\|_{L^2(\Omega \times (0,T), \mathbb{R}^N)}^2 = \int_0^T \int_\Omega |\partial_t u_\varepsilon|^2 \ dxdt \le \int_\Omega f(x, u_o, Du_o) \ dx. \tag{3.11}$$

Thus also the time-derivative $\partial_t u_\varepsilon$ is bounded in $L^2(\Omega \times (0, +\infty), \mathbb{R}^N)$ uniformly for $\varepsilon \in (0,1]$.

3.6. Step 6 (passing to the limit as $\varepsilon \to 0^+$)

By the *a priori* bounds (uniform with respect to $\varepsilon \in (0,1]$), up to a subsequence

$$\begin{cases} u_\epsilon \rightharpoonup u & \text{weakly in } L^p(\Omega_T, \mathbb{R}^N) \text{ and in } L^2(\Omega_T, \mathbb{R}^N) \\ Du_\epsilon \rightharpoonup Du & \text{weakly in } L^p(\Omega_T, \mathbb{R}^{N \times n}) \\ \partial_t u_\epsilon \rightharpoonup \partial_t u & \text{weakly in } L^2(\Omega_T, \mathbb{R}^N) \end{cases}, \tag{3.12}$$

where $u \colon \Omega_\infty \to \mathbb{R}^N$ is a map in the class $u \in L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$, $\partial_t u \in L^2(\Omega_T, \mathbb{R}^N)$. Further, $u = u_o$ on the parabolic boundary $\partial_{\mathcal{P}} \Omega_\infty$ in the sense of traces. Moreover u_ϵ converges to u also strongly in $L^p(\Omega_T, \mathbb{R}^N)$ if $p > 2_* = \frac{2n}{n+2}$ (such that $(2_*)^* = 2$).

We go back to the minimality condition $F_\varepsilon(u_\varepsilon) \le F_\varepsilon(u_\varepsilon + v)$ for a generic test function v equal to zero on the parabolic boundary. By using the test function $v = \delta e^{+\frac{t}{\varepsilon}} \varphi(x, t)$, after some simplifications we get

$$\int_0^T \int_\Omega f(x, u_\varepsilon, Du_\varepsilon) \ dxdt \le \int_0^T \int_\Omega f(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi) \ dxdt \tag{3.13}$$
$$+ \int_0^T \int_\Omega \partial_t u_\varepsilon (\varphi + \varepsilon \partial_t \varphi) \ dxdt.$$

We need a modification with the new test function $v = \delta \eta(t) e^{+\frac{t}{\varepsilon}} \varphi(x, t)$

$$\int_0^T \int_\Omega \eta(t) f(x, u_\varepsilon, Du_\varepsilon) \ dxdt \le \int_0^T \int_\Omega \eta(t) f(x, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi) \ dxdt$$
$$+ \int_0^T \int_\Omega \eta(t) \partial_t u_\varepsilon (\varphi + \varepsilon \partial_t \varphi) \ dxdt + \int_0^T \int_\Omega \varepsilon \eta'(t) \partial_t u_\varepsilon \varphi \ dxdt,$$

and then we change the test function, by posing $\varphi = v - u_\varepsilon$

$$\int_0^T \int_\Omega \eta f(x, u_\varepsilon, Du_\varepsilon) \ dxdt \le \int_0^T \int_\Omega \eta f(x, v, Dv) \ dxdt \tag{3.14}$$
$$+ \int_0^T \int_\Omega \eta \partial_t u_\varepsilon (v - u_\varepsilon + \varepsilon \partial_t (v - u_\varepsilon)) \ dxdt + \int_0^T \int_\Omega \varepsilon \eta' \partial_t u_\varepsilon (v - u_\varepsilon) \ dxdt.$$

Under some analytic and algebraic computation (in particular $\eta(t) \to 1$ in $(0, T)$, uniqueness of the variational parabolic solution, which implies the independence of $u(x, t)$ of T, then $T \to +\infty$), by the lower semicontinuity of the integral we can go to the limit as $\epsilon \to 0^+$ and we finally get the existence of the variational solution.

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