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APPROXIMATION OF QUASICONVEX FUNCTIONS, AND LOWER SEMICONTINUITY OF MULTIPLE INTEGRALS

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We study semicontinuity of multiple integrals $\int_\Omega f(x,u,Du)dx$, where the vector-valued function u is defined for $x \in \Omega \subset \mathbb{R}^n$ with values in $\mathbb{R}^N$. The function $f(x,s,\xi)$ is assumed to be Caratheodory and quasiconvex in Morrey's sense. We give conditions on the growth of f that guarantee the sequential lower semicontinuity of the given integral in the weak topology of the Sobolev space $H^{1,p}(\Omega;\mathbb{R}^N)$. The proofs are based on some approximation results for f. In particular we can approximate f by a nondecreasing sequence of quasiconvex functions, each of them being *convex* and *independent* of (x,s) for large values of ξ. In the special polyconvex case, for example if n=N and f(Du) is equal to a convex function of the Jacobian detDu, then we obtain semicontinuity in the weak topology of $H^{1p}(\Omega;\mathbb{R}^n)$ for *low* p, i.e. $p > n^2/(n+1)$, in particular for some p lower than n.

1. Introduction

Let us consider a function $f(x,s,\xi)$ defined for x in a bounded open set Ω of $\mathbb{R}^n$, $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$. We assume that f is a Caratheodory function, i.e., it is measurable with respect to x and continuous with respect to (s,ξ), and satisfies the growth conditions

$$(1.1) \qquad -C_1|\xi|^r - C_2|s|^t - C_3(x) \leq f(x,s,\xi) \leq g(x,s)(1+|\xi|^p).$$

Here $C_1, C_2 \geq 0$; $C_3 \in L^1(\Omega)$; $g \geq 0$ is a Caratheodory function (no growth conditions are required for g). For the exponents we assume: $p \geq 1$; $1 \leq r < p$ ($r = 1$ if $p = 1$) and $1 \leq t < np/(n-p)$ ($t \geq 1$ if $p \geq n$).

Moreover, if $C_2 \neq 0$, we assume also that the boundary $\partial\Omega$ is Lipschitz continuous.

Finally we assume that f is *quasiconvex* with respect to ξ in Morrey's sense ([25]; [26], section 4.4):

$$(1.2)\qquad \int_\Omega f(x,s,\xi + D\phi(y))dy \geq |\Omega| f(x,s,\xi) \qquad \forall\, \phi \in H_0^{1p}(\Omega;\mathbb{R}^N).$$

In section 4 we prove the following result:

THEOREM 1.1 - *Let* $f(x,s,\xi)$ *be a Caratheodory function, quasiconvex with respect to* ξ*, and satisfying the growth conditions* (1.1). *Then the integral*

$$(1.3)\qquad \int_\Omega f(x,u(x),Du(x))\,dx$$

is sequentially lower semicontinuous in the weak topology of $H^{1p}(\Omega;\mathbb{R}^N)$.

Theorem 1.1 improves the analogous results by Morrey [25], [26] and by Meyers [24], who assume a type of uniform continuity of f with respect to its arguments, and a recent result by Acerbi and Fusco [2], obtained assuming slightly more restrictive growth conditions.

We recall that either if $N = 1$ or $n = 1$, then f is quasiconvex if and only if f is convex; while, if both n and N are greater than one, then quasiconvexity is a more general condition than convexity (for properties of quasiconvex functions we refer to [26], [5], [8], [20]; see also our section 5). Therefore it seems not possible to reduce theorem 1.1 to the semicontinuity results known in convex case (see, i.e., Serrin [31], De Giorgi [10], and more recently Ekeland and Temam [15], chap. 8, theo. 2.1, and Eisen [13]).

The proof of theorem 1.1, different from that of [25], [24], [2], but similarly to other classical semicontinuity results, is based on the possibility to approximate f by a nondecreasing sequence of functions f_k, each of them being easier to handle. To quote the main approximation theorem, proved in section 3, we assume also that $p > 1$ and that f satisfies the coercivity condition ($C_0 > 0$):

$$(1.4) \qquad C_0|\xi|^p \leq f(x,s,\xi) \leq g(x,s)(1 + |\xi|^p) \ .$$

THEOREM 1.2 – *Let* $f(x,s,\xi)$ *be a Caratheodory function, quasiconvex with respect to* ξ, *and satisfying the growth condition* (1.4) *with* $p > 1$. *Then there exists a sequence* $f_k(x,s,\xi)$ *of Caratheodory functions, quasiconvex with respect to* ξ, *and such that:*

$$(1.5) \qquad C_0|\xi|^p \leq f_k(x,s,\xi) \leq k(1 + |\xi|^p) \ ;$$

$$(1.6) \qquad f_k(x,s,\xi) = C_0|\xi|^p, \ \textit{either for } |s| \geq k \textit{ or } |\xi| \geq k;$$

$$(1.7) \qquad f_k \leq f_{k+1} \ , \ \sup_k f_k = f \ .$$

It is clear that this approximation result is usefull to prove theorem 1.1; in fact in the zone where $|Du| \geq k$, that is critical for the integral (1.3), f_k reduces to a convex function, independent of x and s. One of the difficulties in the proof of theorem 1.2 is that the definition of quasiconvexity involves an integral inequality, instead that a pointwise inequality, like convexity. In our proof we follow a procedure introduced in a similar context by Marcellini and Sbordone [22]. and we use a representation formula by Dacorogna [7], a variational principle by Ekeland [14], and a regularity result by Giaquinta and Giusti [17].

In section 2 we prove a semicontinuity result for $f = f(\xi)$, independent of x and s. The proof is particularly simple and self contained. Although this is a special case, it is a crucial step to obtain theorem 1.1.

In sections 3 and 4 we prove respectively theorems 1.2 and 1.1.

In section 5 we specialize (1.2). On assuming that f is *polyconvex* in Ball's sense [5] (an example is given by (6.8)), we can prove a semicontinuity result that, with respect to theorem 1.1, roughly speaking, allow us to consider semicontinuity in the weak topology of H^{1p}, for p strictly (but slightly) smaller than in theorem 1.1. Let us mention that, in the same context of polyconvex integrals, Acerbi, Buttazzo and Fusco [1] proved a semicontinuity theorem in the strong topology of L^∞, while they have shown a counterexample to semicontinuity in the strong topology of L^p, if p is finite.

In section 6 we give some contraexamples to the semicontinuity theorems 1.1 and 5.5, when some of the assumptions are not satisfied.

2. The case $f = f(\xi)$

THEOREM 2.1 - *Let* $f = f(\xi)$ *be a quasiconvex function such that*

(2.1) $$0 \le f(\xi) \le C_4(1 + |\xi|^p) ,$$

for $C_4>0$ *and* $p \ge 1$. *Then the integral*

(2.2) $$\int_\Omega f(Du(x))\, dx$$

is sequentially lower semicontinuous in the weak topology of $H^{1p}(\Omega;\mathbb{R}^N)$.

We divide the proof of this theorem into 3 steps:

Step 1 - We assume first that u is affine, i.e., $Du = \xi$ in Ω for some $\xi \in \mathbb{R}^{nN}$.

Let u_h be a sequence in $H^{1p}(\Omega ; \mathbb{R}^N)$ that converges to u in the weak topology. If u_h had the same boundary value as u, then the semicontinuity result would trivially follow from quasiconvexity of f. To change the boundary datum of u_h, we use a method introduced by De Giorgi [11], and well known in the context of Γ-convergence theory (see, i.e. Sbordone [30] and Dal Maso - Modica [9], theorem 6.1).

Let Ω_0 be a fixed open set compactly contained in Ω, let $R = 1/2 \operatorname{dist}(\overline{\Omega}_0, \partial\Omega)$, let ν be a positive integer, and for $i = 1,2,\dots,\nu$ let us define

$$(2.3) \qquad \Omega_i = \{ x \in \Omega : \operatorname{dist}(x, \Omega_0) < \frac{i}{\nu} R \} .$$

Let us choose smooth functions $\phi_i \in C_0^1(\Omega_i)$ such that

$$(2.4) \quad \begin{cases} 0 \leq \phi_i \leq 1; \ \phi_i = 1 \text{ on } \Omega_{i-1}, \ \phi_i = 0 \text{ out of } \Omega_i; \\ |D\phi_i| \leq (\nu+1)/R. \end{cases}$$

Let us define $v_{hi} = u + \phi_i(u_h - u)$. The support of v_{hi} is contained in Ω; thus by quasiconvexity of f we have

$$(2.5) \quad \begin{aligned} \int_\Omega f(Du)dx &= f(\xi)|\Omega| \leq \int_\Omega f(Dv_{hi})dx \\ &= \int_{\Omega \setminus \Omega_i} f(Du)dx + \int_{\Omega_i \setminus \Omega_{i-1}} f(Dv_{hi})dx + \int_{\Omega_{i-1}} f(Du_h)dx . \end{aligned}$$

We sum up with respect to $i = 1,2,\dots,\nu$, and we divide by ν. We obtain

(2.6) $\int_\Omega f(Du)dx \leq \int_{\Omega\setminus\Omega_0} f(Du)dx + \frac{1}{\nu}\int_{\Omega_\nu} f(Dv_{hi})dx + \int_\Omega f(Du_h)dx$.

Since $Dv_{hi} = (1-\phi_i)Du + \phi_i Du_h + (u_h-u)D\phi_i$, we have

$$\int_{\Omega_\nu} f(Dv_{hi})dx \leq C_4|\Omega| + C_5\{\int_\Omega |Du|^p dx +$$

(2.7)

$$+ \int_\Omega |Du_h|^p dx + (\frac{\nu+1}{R})^p \int_{\Omega_\nu} |u_h-u|^p dx\} .$$

Let us go to the limit as $h \to +\infty$ in (2.6), (2.7). The sequence Du_h is bounded in $L^p(\Omega;\mathbb{R}^{nN})$, and u_h converges strongly to u in $L^p(\Omega_\nu,\mathbb{R}^N)$. Thus we have

(2.8) $\int_\Omega f(Du)dx \leq \int_{\Omega\setminus\Omega_0} f(Du)dx + \frac{C_6}{\nu} + \liminf_h \int_\Omega f(Du_h)dx$.

As $\nu \to +\infty$ and $\Omega_0 \nearrow \Omega$ we obtain our result.

<u>Step 2</u> - f is continuous in the following way:

(2.9) $|f(\xi) - f(\eta)| \leq C_7(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|$

(this step is similar to theorem 4.4.1 of Morrey [26]). The function $\phi: \mathbb{R} \to \mathbb{R}^+$, defined through $f(\xi)$ when only one component (say ξ_i) of ξ vary, is convex. Thus it is definite almost everywhere the derivative $\phi'(\xi_i)$, and we have

(2.10) $\phi'(\xi_i) \gtrless (\phi(\xi_i+h) - \phi(\xi_i))/h$ if $h \gtrless 0$.

For $h = \pm(|\xi|+1)$ we obtain

$$|f_{\xi_i}| = |\phi'| \leq \frac{\phi(\xi_i \pm |\xi| \pm 1) - \phi(\xi_i)}{|\xi| + 1} \leq \frac{\phi(\xi_i \pm |\xi| \pm 1)}{|\xi| + 1}$$

(2.11)

$$\leq C_8 (1 + |\xi|^{p-1}).$$

Of course this inequality implies (2.9).

Step 3 - We prove the semicontinuity result for general $u \in H^{1p}(\Omega;\mathbb{R}^N)$. Let us consider a partition of Ω into open cubes Ω_i ($\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, $\bar{\Omega} = \cup \bar{\Omega}_i$) and let us define vectors $\xi_i \in \mathbb{R}^{n\,N}$ by

$$(2.12) \qquad \xi_i = \frac{1}{|\Omega_i|} \int_{\Omega_i} Du \, dx .$$

For every $\varepsilon > 0$ we can choose the partition of Ω in such a way that

$$(2.13) \qquad \sum_i \int_{\Omega_i} |Du - \xi_i|^p \, dx < \varepsilon.$$

Let u_h be a sequence that converges to u on $w - H^{1p}(\Omega;\mathbb{R}^N)$. For every i, let us define in Ω_i the sequence $v_h(x) = u_h(x) - u(x) + \langle \xi_i, x \rangle$. As $h \to +\infty$, v_h converges to $v(x) = \langle \xi_i, x \rangle$ in $w\text{-}H^{1P}(\Omega_i;\mathbb{R}^N)$. Thus, by step 1, we have

$$(2.14) \qquad \liminf_h \sum_i \int_{\Omega_i} f(Dv_h)dx \geq \sum_i \int_{\Omega_i} f(\xi_i)dx .$$

By step 2, and by Hölder's inequality with exponents p/(p-1) and p, we have

$$\left|\int_\Omega f(Du_h)\, dx - \sum_i \int_{\Omega_i} f(Dv_h)dx\right| \leq$$

(2.15) $$\leq C_7 \sum_i \int_{\Omega_i} (1 + |Du_h|^{p-1} + |Dv_h|^{p-1}) |Du_h - \xi_i| dx$$

$$\leq C_9 \left(\int_\Omega (1 + |Du_h| + |Dv_h|)^p dx \right)^{p-1/p} \left(\sum_i \int_{\Omega_i} |Du - \xi_i|^p \right)^{1/p} < C_{10} \varepsilon^{1/p} .$$

For the same reason

(2.16) $$\left| \int_\Omega f(Du) dx - \sum_i \int_{\Omega_i} f(\xi_i) dx \right| < C_{10} \varepsilon^{1/p} .$$

Our semicontinuity result is a consequence of (2.14), (2.15), (2.16).

3. Approximation of quasiconvex functions

In this section we assume, like in (1.4), that $f(x,s,\xi)$ is a Caratheodory function, quasiconvex with respect to ξ, and satisfying the growth conditions

(3.1) $$C_0 |\xi|^p \leq f(x,s,\xi) \leq g(x,s)(1 + |\xi|^p),$$

where $p \geq 1$, $C_0 > 0$, and g Caratheodory function.

For every integer i, let $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $\phi_i(t) = 1$ for $0 \leq t \leq i-1$, and $\phi_i(t) = 0$ for $t \geq i$. Let us define

(3.2) $$g_i(x,s,\xi) = \phi_i(|s|) f(x,s,\xi) + (1 - \phi_i(|s|)) C_0 |\xi|^p .$$

Let A be a subset of Ω, with zero measure, such that $g(x,s)$ is continuous with respect to s for every $x \in \Omega \setminus A$. For i,j integers ($j \geq C_0$), we define

(3.3) $$A_{ij} = \{ x \in \Omega \setminus A : \max \{ g(x,s) : |s| \leq i \} < j \} .$$

A_{ij} is a measurable set. We define $\psi_{ij}(x) = 1$ if $x \in A_{ij}$, and $\psi_{ij}(x) = 0$ otherwise. We define also

$$(3.4) \qquad g_{ij}(x,s,\xi) = \psi_{ij}(x) g_i(x,s,\xi) + (1 - \psi_{ij}(x)) C_0 |\xi|^p .$$

LEMMA 3.1 - *For every* i,j,g_{ij} *is a Caratheodory function, quasiconvex with respect to* ξ, *satisfying:*

$$(3.5) \qquad C_0 |\xi|^p \leq g_{ij}(x,s,\xi) \leq j(1 + |\xi|^p) \ ;$$

$$(3.6) \qquad g_{ij}(x,s,\xi) = C_0 |\xi|^p \quad \text{for} \quad |s| \geq i;$$

$$(3.7) \qquad \sup_{ij} g_{ij}(x,s,\xi) = f(x,s,\xi) \quad , \quad \forall\, x \in \Omega \setminus A, \ \forall\, s, \ \forall\, \xi .$$

Proof. g_{ij} is a Caratheodory function, since $\phi_i(|s|)$ is continuous and $\psi_{ij}(x)$ is measurable. With respect to ξ, g_i and g_{ij} are quasiconvex functions; in fact they are convex combination of quasiconvex functions. If $|s| \geq i$ then $g_{ij} = C_0 |\xi|^p$; $g_{ij} = C_0 |\xi|^p$ also if $x \notin A_{ij}$; while, if $|s| < i$ and $x \in A_{ij}$ we have

$$(3.8) \qquad g_{ij} \leq g_i \leq f \leq g(x,s)(1+|\xi|^p) \leq j(1 + |\xi|^p).$$

Thus (3.5), (3.6) are proved. To obtain (3.7) we observe that g_{ij} is nondecreasing with respect to i and j separately, since $f \geq g_i \geq C_0 |\xi|^p$; moreover $\lim_i \lim_j g_{ij} = f$.

For every integer $m \geq 1$ let us define in $\Omega \setminus A \times \mathbb{R}^N \times \mathbb{R}^{nN}$:

$$(3.9) \quad G_{ijm}(x,s,\xi) = \begin{cases} g_{ij}(x,s,\xi) & \text{for } |\xi| \leq m , \\ C_0 |\xi|^p & \text{for } |\xi| > m ; \end{cases}$$

$$g_{ijm}(x,s,\xi) = \sup\{G(x,s,\xi):\ G \text{ is quasiconvex with respect to } \xi \text{ and } G \le G_{ijm}\}. \tag{3.10}$$

LEMMA 3.2 - g_{ijm} *is quasiconvex with respect to* ξ *and satisfies:*

(3.11) $\quad C_0|\xi|^p \le g_{ijm}(x,s,\xi) \le j(1+|\xi|^p)$;

(3.12) $\quad g_{ijm}(x,s,\xi) = C_0|\xi|^p$ *either for* $|\xi| \ge m$ *or* $|s| \ge i$.

Proof. Since the supremum of a family of quasiconvex functions is quasiconvex, in (3.10) we have a maximum and g_{ijm} is quasiconvex. Since the convex function $G = C_0|\xi|^p$ is less than or equal to G_{ijm}, we have $C_0|\xi|^p \le g_{ijm}$, and thus (3.12).

Fixed $x \in \Omega\setminus A$ and $s \in \mathbb{R}^N$, we consider the infimum

(3.13) $\inf\{\frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,s,\xi + D\phi(y))dy:\ \phi \in H_0^{1p}(\Omega;\ \mathbb{R}^N)\}$.

LEMMA 3.3 - *The infimum in (3.13) is a continuous function of* (s,ξ).

Proof. Fixed $x \in \Omega\setminus A$, the function G_{ijm} is uniformly continuous for $s \in \mathbb{R}^N$ and $|\xi| \le m$. Thus for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $|s-t|+|\xi-\eta| < \delta$, we have (we decompose the integral over Ω into two integrals, and we use inequality (2.9) for the function $|\xi|^p$):

$$\left| \frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,s,\xi+D\phi(y))dy - \frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,t,\eta+D\phi(y))dy \right|$$

$$(3.14) \leq \varepsilon + \frac{C_0}{|\Omega|} \left| \int_\Omega (|\xi+D\phi(y)|^p - |\eta+D\phi(y)|^p)dy \right|$$

$$\leq \varepsilon + C_{11} \{ |\xi|^{p-1} + |\eta|^{p-1} + \frac{1}{|\Omega|}\int_\Omega |D\phi(y)|^{p-1} dy \} \delta .$$

Since $G_{ijm} \geq C_0 |\xi|^p$, in the infimum (3.13) we can limit ourself to consider test functions ϕ bounded in $H_0^{1p}(\Omega;\mathbb{R}^N)$, uniformly as ξ vary in a bounded set of $\mathbb{R}^{nN}$. For all such functions ϕ, if $|s-t|+ +|\xi-\eta| < \min\{\varepsilon, \delta\}$, we have

$$(3.15) \quad \left| \frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,s,\xi+D\phi(y))dy - \frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,t,\eta+D\phi(y))dy \right| < C_{12}\varepsilon.$$

Of course, this implies that the infimum in (3.13) is a continuous function of (s,ξ).

REMARK 3.4 - The previous result does not hold if the function inside the integral does not satisfy some properties of structure, such as, i.e., either coercivity with respect to ξ or continuity on s uniformly with respect to ξ (as suggested by corollary 3.12 of [21]). In fact, if we consider, like in [21], $G_{ijm} = (1+|\xi|)^{|s|}$, then the infimum in (3.13) is equal to G_{ijm} if $|s| \geq 1$, while is equal to 1 if $|s| < 1$. Thus, in this case, the infimum is not continuous (neither lower semicontinuous) with respect to s.

LEMMA 3.5 (Dacorogna) - $g_{ijm}(x,s,\xi)$ *is a Caratheodory function, and is equal to the infimum in* (3.13).

Proof. By lemma 3.3 the infimum in (3.13) is a continuous function of $\xi \in \mathbb{R}^{nN}$. It is necessary to use this fact as the first step in the argument of Dacorogna ([7], theorem 5; or [8], pag. 87). Then, like in steps 2,3,4 of [7]; [8] we obtain that g_{ijm} is the infimum in (3.13). Again by lemma 3.3 g_{ijm} is continuous in (s,ξ). g_{ijm} is measurable in x, since it is infimum of a family of measurable functions.

LEMMA 3.6 (Ekeland)- *There exists a sequence* u_m *that minimizes on* $H_0^{1p}(\Omega;\mathbb{R}^N)$ *for every m the functionals*

$$(3.16)\quad \phi \to \frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,s,\xi+D\phi(y))dy+\frac{1}{m}\int_\Omega |D\phi(y)-Du_m(y)|dy \quad ,$$

and satisfies

$$(3.17)\quad \frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,s,\xi+Du_m(y))dy < g_{ijm}(x,s,\xi) + \frac{1}{m} .$$

Proof. This lemma is a particular case of a variational principle given by Ekeland [14] in the general setting of a complete metric space V and a lower semicontinuous functional $F : V \to \mathbb{R} \cup \{+\infty\}$ ($F \not\equiv +\infty$). Here $V = H^{11}(\Omega;\mathbb{R}^N)$, and F is the integral of G_{ijm}, that is *strongly* lower semicontinuous in $H^{11}(\Omega;\mathbb{R}^N)$, by Fatou's lemma. In (3.17) we use the characterization of $g_{ij\,m}$ given in lemma 3.5.

The fact stated in (3.16), that u_m is a minimum function, allows us to get a Meyers' type result [23], introduced in the context of minimum problems for vector valued functions by Giaquinta and Giusti [17] (see also [4] for the convex case and [18] for quasi-minima). Like in lemma 3.2 of Marcellini and Sbordone [22], from (3.16) we deduce:

LEMMA 3.7 - *If* $p > 1$, *there exists* $\varepsilon > 0$ *such that the sequence* u_m *is bounded in* $H^{1,p+\varepsilon}_{loc}(\Omega;\mathbb{R}^N)$.

LEMMA 3.8 - *If* $p > 1$, *the nondecreasing sequence* g_{ijm} *converges, as* $m \to +\infty$, *to* g_{ij} .

Proof. Let Ω_0 be a fixed open set compactly contained in Ω. The sequence u_m of the previous lemma is bounded in $H^{1,p+\varepsilon}(\Omega_0;\mathbb{R}^N)$. Thus, if we denote by Ω_m the set $\Omega_m = \{ x\in \Omega_0 : |Du_m| \geq m\}$, we have

$$(3.18) \qquad \int_{\Omega_m} |Du_m|^p dx \leq |\Omega_m|^{\varepsilon/p+\varepsilon} \left(\int_{\Omega_0} |Du_m|^{p+\varepsilon} dx\right)^{p/p+\varepsilon}.$$

Therefore the left side in (3.18) converges to zero, as $m\to +\infty$. From (3.17) and (3.5) we obtain

$$g_{ijm} + \frac{1}{m} > \frac{1}{|\Omega|}\int_{\Omega_0} G_{ijm}(x,s,\xi+ Du_m(y))dy$$

$$(3.19) \qquad \geq \frac{1}{|\Omega|}\int_{\Omega_0\setminus\Omega_m} g_{ij}(x,s,\xi+Du_m(y))dy$$

$$\geq \frac{1}{|\Omega|}\int_{\Omega_0} g_{ij}(x,s,\xi+Du_m(y))dy - \frac{j}{|\Omega|}\int_{\Omega_m} (1+|Du_m(y)|^p)dy .$$

Since u_m is bounded in $H_0^{1p}(\Omega;\mathbb{R}^N)$, it has a subsequence that weakly converges. We still denote by u_m this subsequence, and we denote by $u\in H_0^{1p}(\Omega;\mathbb{R}^N)$ the weak limit. Let $m\to +\infty$; we use (3.18), (3.19), the semicontinuity result of section 2, and quasiconvexity of g_{ij}:

$$\lim_m g_{ijm} \geq \liminf_m \frac{1}{|\Omega|}\int_{\Omega_0} g_{ij}(x,s,\xi+Du_m(y))dy$$

$$(3.20) \qquad \geq \frac{1}{|\Omega|}\int_{\Omega_0} g_{ij}(x,s,\xi+Du(y))dy$$

$$\geq g_{ij}(x,s,\xi) - \frac{1}{|\Omega|}\int_{\Omega\setminus\Omega_0} g_{ij}(x,s,\xi+Du(y))dy .$$

As $\Omega_0 \nearrow \Omega$, we obtain our result, since $g_{ijm} \leq g_{ij}$.

REMARK 3.9 - Several lemmas, from 3.4 to 3.8, are devoted to the study of the convergence of g_{ijm} as $m\to +\infty$. Let us show that this study is much easier if we know that f is *convex* with respect to ξ:

for every m we can construct a function $G(x,s,\xi)$, convex with respect to ξ, that coincides with g_{ij} for $|\xi| \leq m$ and grows linearly at ∞ (the supremum of all iperplanes supporting g_{ij} where $|\xi| < m$). Since G grows linearly, there exists $m' > m$ such that $G \leq C_o |\xi|^p$ for $|\xi| \geq m'$. By the same definition (3.10), $G \leq g_{ijm'} \leq g_{ij}$, and thus $g_{ijm'} = g_{ij}$ for $|\xi| \leq m$. Note that, also in this simple argument for the convex case, we need $p > 1$.

Finally we obtain the approximation result stated in the introduction:

Proof of theorem 1.2 - For every integer k ($\geq 2 + C_o$) we define

(3.21) $$f_k(x,s,\xi) = \max\{ g_{ijm}(x,s,\xi) : i + j + m \leq k \}.$$

(1.5) is consequence of (3.11), while (1.6) follows from (3.12). Finally the supremum of f_k is f, by lemma 3.8 and formula (3.7).

REMARK 3.10 - Since each f_k is convex for large values of ξ, we can assume that it grows linearly at ∞. In fact it is enough to change f_k, where $|\xi| \geq k$, with the function $C_o p\, k^{p-1} |\xi|$.

4. Semicontinuity in the quasiconvex case

In this section we will prove theorem 1.1. It will be consequence of the approximation theorem 1.2., by proving a semicontinuity result for the functions f_k as in theorem 1.2. In the following lemmas we assume k is fixed and f_k satisfies (1.5), (1.6).

LEMMA 4.1 (Scorza-Dragoni) - *For every positive* ε *there exists a compact set* $C \subset \Omega$, *with* $|\Omega \setminus C| < \varepsilon$, *such that* $f_k(x,s,\xi)$ *is continuous in* $C \times \mathbb{R}^N \times \mathbb{R}^{nN}$.

Proof. See i.e. lemma 1 of pag. 37 of [10], or [15], chap. VIII, section 1.3.

LEMMA 4.2 - *There exists a continuous bounded function* $w:\mathbb{R}^+\to\mathbb{R}^+$, *with* $w(0)=0$, *such that, for* $x,y\in C$, $s,t\in\mathbb{R}^N$, $\xi\in\mathbb{R}^{nN}$ *we have*

$$(4.1)\qquad |f_k(x,s,\xi)-f_k(y,t,\xi)|\le w(|x-y|+|s-t|)\quad .$$

Proof. For $|\xi|\le k$, by (1.5), we have

$$(4.2)\qquad |f_k(x,s,\xi)-f_k(y,t,\xi)|\le 2k(1+k^p)\quad .$$

Inequality (4.2) holds also if $|\xi|>k$ since the left side is zero. The function f_k is continuous in the compact set $C\times\{|s|\le k+1\}\times\times\{|\xi|\le k+1\}$. Thus (4.1) holds on this set with w equal to the oscillation of f_k. By (4.2) the function w is bounded, and we can assume that $w(r)=2k(1+k^p)$ for $r\ge 1$. By (1.6), formula (4.1) holds also if either $|\xi|\ge k$ or $|s|$ and $|t|\ge k$. It remains to consider the case $|s|<k$, $|t|>k+1$ and $|\xi|<k$. In this case $w==2k(1+k^p)$, and thus (4.1) follows from (4.2).

LEMMA 4.3 - *The integral*

$$(4.3)\qquad \int_\Omega f_k(x,u(x),Du(x))dx$$

is sequentially lower semicontinuous in the weak topology of $H^{1p}(\Omega;\mathbb{R}^N)$.

Proof. Let $u\in H^{1p}(\Omega;\mathbb{R}^N)$. Let us consider a partition of Ω into open cubes Ω_i, with $\Omega_i\cap\Omega_j=\emptyset$. $\bigcup_i\bar\Omega_i=\bar\Omega$. Like in Morrey [25] we define the vector valued funtions (constant in each Ω_i):

(4.4) $$x_i = \frac{1}{|\Omega_i|}\int_{\Omega_i} x\,dx \quad ; \quad u_i = \frac{1}{|\Omega_i|}\int_{\Omega_i} u(x)dx .$$

By the dominate convergence theorem, for every $\varepsilon > 0$ we can choose the partition so that

(4.5) $$\int_\Omega w(|x-x_i| + |u(x)-u_i(x)|)\,dx < \varepsilon .$$

Let u_h be a sequence of $H^{1p}(\Omega;\mathbb{R}^N)$ that converges to u in the weak topology. We have

(4.6) $$\begin{aligned}\int_\Omega f_k(x,u_h,Du_h)dx &= \int_{\Omega\setminus C}\{f_k(x,u_h,Du_h) - f_k(x_i,u_i,Du_h)\}\,dx \\ &+ \int_C \{f_k(x,u_h,Du_h) - f_k(x,u,Du_h)\}\,dx \\ &+ \int_C \{f_k(x,u,Du_h) - f_k(x_i,u_i,Du_h)\}dx + \int_\Omega f_k(x_i,u_i,Du_h)dx .\end{aligned}$$

We use (4.2) , (4.1) and (4.5):

(4.7) $$\begin{aligned}\int_\Omega f_k(x,u_h,Du_h)dx &\geq - 2k(1+k^p)|\Omega\setminus C| \\ &- \int_C w(|u_h-u|)dx - \varepsilon + \sum_i \int_{\Omega_i} f_k(x_i,u_i,Du_h)\,dx .\end{aligned}$$

As $h \to +\infty$, by the semicontinuity theorem 2.1, we have

(4.8) $$\liminf_h \int_\Omega f_k(x,u_h,Du_h)dx \geq - C_{13}\,\varepsilon + \int_\Omega f_k(x_i,u_i,Du)dx .$$

We obtain the proof as the sides of the cubes Ω_i and ε go to zero.

Proof of theorem 1.1 - Let us assume first that $p > 1$. Similarly to Serrin [31], for $\varepsilon > 0$ let us define

(4.9) $$g_\varepsilon(x,s,\xi) = f(x,s,\xi) + C_2|s|^t + C_3(x) + \varepsilon|\xi|^p + C_\varepsilon \, .$$

Since $p > r$, we can choose the constant C_ε to obtain $g_\varepsilon(x,s,\xi) \geq \varepsilon/2\, |\xi|^p$. Let $f_{\varepsilon k}$ be the sequence of quasiconvex functions that converges, as $k \to +\infty$, to g_ε, according to theorem 1.2. If u_h weakly converges to u in $H^{1p}(\Omega;\mathbb{R}^N)$, by lemma 4.3. we have

$$\liminf_h \int_\Omega g_\varepsilon(x,u_h,Du_h)dx \geq \lim_k \liminf_h \int_\Omega f_{\varepsilon k}(x,u_h,Du_h)dx$$

(4.10)

$$\geq \lim_k \int_\Omega f_{\varepsilon k}(x,u,Du)\, dx = \int_\Omega g_\varepsilon(x,u,Du)\, dx \, .$$

Let C_{14} be an upper bound for the H^{1p} norm of u_h. Since u_h converges to u in the strong topology of $L^t(\Omega;\mathbb{R}^N)$ (here we use the assumption that $\partial\Omega$ is smooth if $C_2 \neq 0$), we obtain

$$\liminf_h \int_\Omega f(x,u_h,Du_h)\, dx \geq \liminf_h \int_\Omega g_\varepsilon(x,u_h,Du_h)dx$$

(4.11) $$- \lim_h \int_\Omega \{C_2|u_h|^t + C_3(x) + C_\varepsilon\}\, dx - \frac{\varepsilon}{2} C_{14}^p$$

$$\geq \int_\Omega f(x,u,Du)dx + \frac{\varepsilon}{2}\int_\Omega |Du|^p dx - \frac{\varepsilon}{2} C_{14}^p \, .$$

We complete the proof of the case $p > 1$ as $\varepsilon \to 0$. If $p = 1$, the proof is much simpler, since if u_h converges to u in the weak topology of $H^{11}(\Omega;\mathbb{R}^N)$ then the integrals of $|Du_h|$ are equiabsolutely continuous. We do not give the details; we can use the approximation lemma 3.1 and then the argument by Fusco [16], or the argument of section 2 of [22].

5. The polyconvex case

In this section we consider a particular case of quasiconvex functions. Following Ball [5], we say that a function $f(x,\xi)$ is *polyconvex* with respect to ξ if there exists a function $g(x,\eta)$, convex with respect to $\eta \in \mathbb{R}^m$, such that

$$(5.1) \qquad f(x,\xi) = g(x,\xi,\det_1\xi, \det_2\xi,\dots) \quad ,$$

where $\det_i \xi$ are subdeterminants (or adjoints) of the $n \times N$ matrix ξ. If $\xi = Du$, then each determinant is a divergence. For example, for $n = N = 2$, if $u \equiv (u^1,u^2) \in C^2(\Omega;\mathbb{R}^2)$, we have

$$(5.2) \qquad \det Du = u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1} = (u^1 u^2_{x_2})_{x_1} - (u^1 u^2_{x_1})_{x_2} \; .$$

Using (5.2) (and in general (5.4)), we can verify by Jensen's inequality that every polyconvex function is quasiconvex. By multiplying (5.2) by a test function $\phi \in C^\infty_0(\Omega)$ and by integrating by parts we have

$$(5.3) \qquad \int_\Omega \det Du \,\phi\, dx = - \int_\Omega \{ u^1 u^2_{x_2} \phi_{x_1} - u^1 u^2_{x_1} \phi_{x_2} \} dx \quad .$$

By continuity (5.3) holds for $u \in H^{12}(\Omega;\mathbb{R}^2)$. If $u \in H^{1p}(\Omega;\mathbb{R}^2)$ for $p < 2$, then $u \in L^{2p/2-p}_{loc}(\Omega;\mathbb{R}^2)$ and thus the product $u^1 u^2_{x_2}$ is summable if $1/p + (2-p)/2p \leq 1$, i.e., $p \geq 4/3$. Thus, if $u \in H^{1,4/3}(\Omega;\mathbb{R}^2)$, we can define by (5.3) the determinant of Du as a distribution. Moreover for the same reasons the map $u \to \det Du$ is continuous in the following sense: if u_h converges to u in $H^{1p}_{loc}(\Omega;\mathbb{R}^2)$ for $p > 4/3$ and if (5.3) holds for u_h and u, then $\det Du_h$ converges to $\det Du$ in the sense of distributions. In fact in this case u^1_h strongly converges to u in $L^4(\Omega)$ and we can go to the limit as $h \to +\infty$ in (5.3).

For general $n = N \geq 2$ we can still write det Du as a divergence (Morrey [26], pagg. 122-123)

$$(5.4)\quad \det Du = \frac{\partial(u^1,\ldots,u^n)}{\partial(x_1,\ldots,x_n)} = -\sum_{\alpha=1}^{n}(-1)^{\alpha}\frac{\partial}{\partial x_\alpha}\left(u^1\frac{\partial(u^2,\ldots,u^n)}{\partial(x_1,\ldots,x_{\alpha-1},x_{\alpha+1},\ldots,x_n)}\right).$$

This formula holds if $u \in H^{1,n}(\Omega;\mathbb{R}^n)$. Since if $u \in H^{1p}$ $(n-1 \leq p < n)$ then $u^1 \in L^{np/(n-p)}_{loc}$ and the Jacobian of order n-1 belongs to $L^{p/(n-1)}$, the right side of (5.4) is well defined in the sense of distributions if $(n-1)/p+(n-p)/np \leq 1$, i.e., $p \geq n^2/(n+1)$. Thus we have proved, as in Ball [5] and Ball, Currie and Olver [6], the following result:

LEMMA 5.1 ([5],[6]) - *Let* $n = N \geq 2$ *and* $u \in H^{1p}_{loc}(\Omega;\mathbb{R}^n)$. *If* $p \geq n^2/(n+1)$ *then* det Du *is defined by* (5.4) *as a distribution; while if* $p \geq n$, det Du *is defined as a* L^1_{loc} *-function and formula* (5.4) *holds. Moreover, if* u_h *converges to* u *in the weak topology of* H^{1p}_{loc} *for* $p > n^2/(n+1)$, *then* det Du_h *converges to* det Du *in the sense of distributions.*

To get a semicontinuity result for polyconvex functions, we consider a function $g(x,\eta)$ defined for $x \in \Omega \subset \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ with values in $[0,+\infty]$ (see in next section the reason to assume g continuous in x and independent of s) satisfying:

(5.5) *The set* $\{(x,\eta) : g(x,\eta) < +\infty\}$ *is open (and not empty) in* $\Omega \times \mathbb{R}^m$, *and* g *is continuous on this set.*

(5.6) $g(x,\eta)$ *is convex and lower semicontinuous with respect to* $\eta \in \mathbb{R}^m$.

REMARK 5.2 - We assume (5.5) to simplify the next lemma, but we could consider also other cases. On assuming (5.5), (5.6) we have in mind the situation described by Ball [5], of interest in nonlinear elasticity (see also the paper by Antman [3]), where are considered functions $f(x,Du) = g(x,\det Du)$ that are finite if and only if $\det Du > 0$, and go to $+\infty$ if $\det Du \to 0$.

Let us begin with two approximation lemmas.

LEMMA 5.3 (De Giorgi) - *There exists a nondecreasing sequence of real nonnegative functions* g_k *that converge, as* $k \to +\infty$, *to* g. *For every* $k, g_k(x,\eta)$ *is uniformly continuous in* $\Omega \times \mathbb{R}^m$, *it grows linearly with respect to* η, *it is convex with respect to* η *and it is equal to zero if* $\mathrm{dist}\,(x,\partial\Omega) \leq 1/k$.

Proof. Let $\Omega_i = \{x \in \Omega : \mathrm{dist}\,(x,\partial\Omega) > 1/i\}$, and let $\phi_i \in C_0^\infty(\Omega_i)$ be equal to one on Ω_{i-1}, and $\phi_i \geq 0$. For every $x \in \Omega$, the function g is lower semicontinuous on $\mathbb{R}^m$; thus it is the supremum of a sequence of affine functions $(a_j(x), \eta) + b_j(x)$. Like in pag. 31 of De Giorgi [10] (the argument of [10] can be applied, since g is finite in a neighbour of each supporting point), we can choose $a_j(x)$ and $b_j(x)$ to be continuous in Ω. Let us define a_0, $b_0 \equiv 0$, and

$$(5.7) \qquad g_k(x,\eta) = \max\{\phi_i(x)[(a_j(x),\eta) + b_j(x)] : i+j \leq k\}.$$

The sequence g_k has all the required properties.

LEMMA 5.4 - *There exists a sequence* $h_k(x,\eta)$ *of* C^∞*-functions satisfying all the properties stated in the previous lemma (except the fact that* $h_k \geq -1$*).*

Proof. Let α be a mollifier, i.e., $\alpha \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $\int \alpha\, dxdy = 1$, $\alpha \geq 0$. Let us define $h_{k\varepsilon} = h_k * \alpha_\varepsilon$, where $\alpha_\varepsilon(x,\eta) = \varepsilon^{-(n+m)}\alpha(x/\varepsilon, \eta/\varepsilon)$. If ε is sufficiently small, $h_{k\varepsilon}$ is a nonnegative C^∞-function, and is zero if dist $(x, \partial\Omega) \leq 1/k+1$. By the uniform continuity of g_k, as $\varepsilon \to 0$ $h_{k\varepsilon}$ converges to g_k uniformly in $\Omega \times \mathbb{R}^m$. Thus we can choose $\varepsilon = \varepsilon(k)$ such that

$$(5.8) \qquad |h_{k,\varepsilon(k)} - g_k| < \frac{1}{2(k+1)^2} \ .$$

For $k \geq 2$ let us define $h_k(x,\eta) = h_{k-1,\varepsilon(k-1)}(x,\eta) - (k-1)^{-1}$. The sequence h_k satisfies all the stated properties. For example, let us verify that h_k is increasing with respect to k:

$$h_k < g_{k-1} + \frac{1}{2k^2} - \frac{1}{k-1} \leq g_k + \frac{1}{2k^2} - \frac{1}{k-1}$$

$$(5.9) \qquad < h_{k,\varepsilon(k)} + \frac{1}{2(k+1)^2} + \frac{1}{2k^2} - \frac{1}{k-1}$$

$$< h_{k+1} + \frac{1}{k} + \frac{1}{k^2} - \frac{1}{k-1} < h_{k+1} \ .$$

Now we prove a semincontinuity result, that generalize an analogous result of Reshetnyak [28].

THEOREM 5.5 - *Let* $g: \Omega \times \mathbb{R}^n \to [0,+\infty]$ *be a function satisfying* (5.5), (5.6) . *Let* v_h *and* v *be functions of* $L^1_{loc}(\Omega;\mathbb{R}^m)$ *, and assume that* v_h *converges to* v *in the sense of distributions, i.e.:*

$$(5.10) \qquad \lim_h \int_\Omega (v_h,\phi)dx = \int_\Omega (v,\phi)dx \ , \qquad \forall\ \phi \in C_0^\infty(\Omega;\mathbb{R}^m).$$

Then we have

$$(5.11) \qquad \liminf_h \int_\Omega g(x,v_h(x))dx \geq \int_\Omega g(x,v(x))\, dx \ .$$

Proof. By using our approximation lemma 5.4 in the usual way like in (4.10), it is enough to prove the theorem assuming that g is $C^\infty(\Omega \times \mathbb{R}^m)$, and that there exists an open set Ω_0 compact_ly contained in Ω such that g is equal to zero for $x \notin \Omega_0$. Let α_ε be a mollifier and let $v_\varepsilon = v * \alpha_\varepsilon$. By the convexity of g, similarly to Serrin [31], we have

$$(5.12) \qquad g(x,v_h) \geq g(x,v_\varepsilon) + (D_\eta g(x,v_\varepsilon), \; v_h - v_\varepsilon) \quad .$$

Since $D_\eta g(x,v_\varepsilon(x)) \in C_0^\infty(\Omega;\mathbb{R}^m)$, by (5.10) we have

$$(5.13) \qquad \liminf_h \int_\Omega g(x,v_h)dx \geq \int_\Omega g(x,v_\varepsilon)dx + \int_\Omega (D_\eta g(x,v_\varepsilon), v - v_\varepsilon)dx \; .$$

We obtain the result for $\varepsilon \to 0$. In fact, in the first addendum of the right side we can use Fatou's lemma, while in the second addendum we can use the fact that $D_\eta g$ is bounded in $\Omega \times \mathbb{R}^m$ independently of ε.

By lemma 5.1 and theorem 5.5 we obtain two semicontinuity results for integrals of the type:

$$(5.14) \qquad F(u) = \int_\Omega f(x,Du)dx = \int_\Omega g(x,Du,\det_1 Du,\det_2 Du,\ldots)dx \; .$$

Here $f(x,\xi)$ is a polyconvex function like in (5.1), and $g(x,\eta)$ satifies (5.5), (5.6).

COROLLARY 5.6 - *Let* u_h *and* u *be functions of* $H^{1p}_{loc}(\Omega;\mathbb{R}^N)$, *for* $p > \min\{n^2/(n+1);N^2/(N+1)\}$. *Assume that the subdeterminants of the Jacobians* Du_h *and* Du *are defined as* L^1_{loc} *-functions and formula* (5.4) *holds for* u_h *and* u. *If* u_h *converges to* u *in the weak topology of* $H^{1p}_{loc}(\Omega;\mathbb{R}^N)$, *then* $\liminf_h F(u_h) \geq F(u)$.

COROLLARY 5.7 - *Let* u_h *and* u *be functions of* $H^{1r}_{loc}(\Omega;\mathbb{R}^N)$, *for* $r \geq \min\{n,N\}$. *If* u_h *converges to* u *in the weak topology of* $H^{1p}_{loc}(\Omega;\mathbb{R}^N)$

for $p > \min\{n^2/(n+1);\ N^2/(N+1)\}$ *then* $\liminf_h F(u_h) \geq F(u)$.

6. Some examples and remarks

Here we discuss the necessity of some assumptions of the semicontinuity theorems 1.1 and 5.5. Let us begin with theorem 5.5 and let us show that the result does not hold if $g(x,\eta)$ is only measurable with respect to x, or if $g = g(x,s,\eta)$ (with the usual meaning of s). Of course, to exhibit contraexamples, we must consider non coercive cases; in fact, if $g(x,\eta) \geq \text{cost}\ |\eta|^p$ for some $p > 1$, the semicontinuity theorem 5.5 reduces to the usual semicontinuity theorem in the weak topology of L^p.

EXAMPLE 6.1 - Let $n = m = 1$ and let $g(x,\eta) = a(x)\eta^2$, with $a(x)$ nonnegative, bounded and measurable in $(0,1)$. It has been proved in theorem 5 of [19] that for every $p \in [1,+\infty]$ and for every $u \in H^{12}$ there exists in H^{12} a sequence u_h that converges to u in L^p and satisfies

(6.1) $$\lim_h \int_0^1 a(x)\,(u_h')^2\,dx = \int_0^1 b(x)\,(u')^2\,dx\ ,$$

where

(6.2) $$b(x) = \lim_{\varepsilon \to 0^+} 2\varepsilon\left[\int_{x-\varepsilon}^{x+\varepsilon} a^{-1}(y)\,dy\right]^{-1} .$$

If we consider a function $a(x) \neq 0$ a.e., non locally summable a.e. in $(0,1)$, then $b(x)$ is zero a.e. Let us define $v_h = u_h'$ and $v = u'$. Then, for every $\phi \in C_0^\infty$, we have

(6.3) $$\lim_h \int_0^1 v_h\,\phi\,dx = -\lim_h \int_0^1 u_h\,\phi'\,dx = -\int_0^1 u\,\phi'\,dx = \int_0^1 v\,\phi\,dx\ .$$

Moreover v_h and v are in L^1, but they do not satisfy (5.11), since, if v is not identically zero,

$$(6.4) \qquad \lim_h \int_0^1 a(x)\, v_h^2(x)\, dx = 0 < \int_0^1 a(x) v^2(x) dx \ .$$

EXAMPLE 6.2 - We can adapt the counterexample of Eisen [12] to our situation. In fact Eisen showed that there exists a sequence of Lipschitz continuous functions $u_h \equiv (u_h^1, u_h^2)$ defined in (0,1), such that the product $u_h^1 (u_h^2)' = 0$ a.e., and u_h converges in L^1 to the function $u(x) \equiv (1,x)$. Let us define $v_h = (u_h^2)'$, $v = (u^2)' = 1$, $w_h = u_h^1$, $w = u^1 = 1$. Like in (6.3), v_h converges to v in the sense of distributions; v_h and v are in L^1, but

$$(6.5) \qquad \int_0^1 (w_h v_h)^2 dx = 0 \quad , \int_0^1 (wv)^2 dx = 1 \ .$$

This means that in general we cannot extend theorem 5.5 to integrals of the type $\int g(w(x), v(x))dx$, where $g(s,\eta)$ is continuous in (s,η), and convex with respect to η, and the topology considered is the product of the L^1 norm topology for w, and the topology of distributions for v.

EXAMPLE 6.3 - In theorem 1.1 the assumption $t < np/(n-p)$ if $p < n$ is necessary. We have a counterexample for $f(x,s,\xi) = |\xi|^p - \text{cost}|s|^{np/(n-p)}$, by choosing a sequence u_h that weakly converges in H^{1p}, but does not converge in the norm topology of $L^{np/(n-p)}$.

EXAMPLE 6.4 - If n and N are greater than or equal to 2, the assumption $r < p$ in theorem 1.1 is necessary. In fact there is a counterexample by Murat and Tartar (see the counterexample in section 4.1 of [27]), for $n = N = p = r = 2$, where it is shown that the integral

(6.6) $\int_\Omega a(x) \det Du \, dx$,

is not continuous in the weak topology of $H^{12}(\Omega;\mathbb{R}^2)$, even if a is a nonzero constant. It is a consequence of theorem 1.1, but it is also well known (see [29], [5], [6]; and for more general functionals [16], [22]), that, if $a \in L^\infty(\Omega)$, the integral in (6.6) is weakly sequentially continuous in the weak topology of $H^{1,2+\varepsilon}(\Omega;\mathbb{R}^2)$, for every positive ε.

EXAMPLE 6.5 - To discuss the necessity of the upper bound in (1.1) let us summarize our results in the special case $n = N \geq 2$ for the integral

(6.7) $\int_\Omega g(x,u(x))|\det Du(x)|^\alpha \, dx$;

we distinguish two cases: $\alpha \geq 1$ or $\alpha < 0$; in the second case we define $|\eta|^\alpha = +\infty$ if $\eta \leq 0$. If g is a nonnegative Caratheodory function and $\alpha \geq 1$, then the integral (6.7) is sequentially lower semicontinuous in the weak topology of $H^{1n}(\Omega;\mathbb{R}^n)$. This follows from theorem 1.1 in the general case (if $\alpha > 1$ we can approximate $|\eta|^\alpha$ with a nondecreasing sequence of convex functions on $\mathbb{R}$, each of them growing linearly at ∞), and from corollary 5.7, if g is independent of s and continuous. If $g = g(x)$ is a nonnegative continuous function, and either $\alpha \geq 1$ or $\alpha < 0$, then from corollary 5.7 it follows also that, if $p > n^2/(n+1)$:

(6.8) *If* u_h *and* u *are smooth (say* $u_h, u \in H^{1n}(\Omega;\mathbb{R}^n)$*) and* u_h *weakly converges to* u *in* $H^{1p}(\Omega;\mathbb{R}^n)$, *then* $\liminf_h \int_\Omega g(x)|\det Du_h|^\alpha dx \geq \geq \int_\Omega g(x)|\det Du|^\alpha dx$.

Let us mention explicitely that we have not proved that (6.8) is true if $1 \leq p \leq n^2/(n+1)$. Let us mention also that Acerbi, Buttazzo and Fusco [1] proved that (6.8) is not true if we replace the weak convergence in $H^{1p}(\Omega;\mathbb{R}^n)$ with the strong convergence in $L^p(\Omega;\mathbb{R}^n)$, whatever is $p \in [1,+\infty)$.

REMARK 6.6 - In (6.8) we make a distinction in between the space where the functional is well defined, and the space where the sequence u_h weakly converges. This is a natural point of view and it is not new. In this context of polyconvex functions we refer to theorem 9.2.1. by Morrey [26], and to [1]. We refer also to the well known semicontinuity theorems by Serrin [31] (see also [26], section 4.1), where the considered functions u_h are required to be in the space H^{11}, but the convergence is in the space L^1. We refer also to the theory of De Giorgi [11], related to this subject.

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7. Appendice

Traggo spunto dallo Step 2 di pag. 6 per fare due semplici osservazioni sulla Lipschitzianità delle funzioni quasi convesse.

Questa appendice separata, che non intendo pubblicare su una rivista matematica come il resto del lavoro, ma che diffondo in modo informale, è destinata soprattutto a quanti si interessano di problemi di Γ-convergenza e di omogeneizzazione, e che quindi ben conoscono l'interesse delle condizioni di Lipschitzianità espresse nelle proposizioni 1 e 2 che seguono. Circa la proposizione 1, faccio anche riferimento allo Step 2 di pag. 6, al paragrafo 4.4 del libro di Morrey, e al lemma 1.2 di Fusco (On the convergence of integral functionals depending on vector-valued functions, in corso di stampa su Ricerche di Mat.).

PROPOSIZIONE 7.1 - Sia $f = f(\xi)$ una funzione convessa rispetto ad ogni componente del vettore ξ (in particolare sia f convessa o quasi convessa), tale che

$$(7.1)\qquad |f(\xi)| \leq c_1 (1 + |\xi|^p) ,$$

con $c_1 > 0$ e $p \geq 1$. Allora esiste una costante positiva c_2, dipendente solo da c_1 e p ma non da f, tale che

$$(7.2)\qquad |f(\xi) - f(\eta)| \leq c_2 (1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta| .$$

PROPOSIZIONE 7.2 - Sia $f = f(\xi)$ una funzione convessa rispetto ad ogni componente del vettore ξ, e tale che

$$(7.3) \qquad c_0 (1 + |\xi|^p) \leqslant f(\xi) \leqslant c_1 (1 + |\xi|^p) ,$$

con $c_1 \geq c_0 > 0$ e $p \geqslant 1$. Allora esiste una costante positiva c_3, dipendente solo da c_0, c_1 e p, ma non da f, tale che

$$(7.4) \qquad |f^{1/p}(\xi) - f^{1/p}(\eta)| \leqslant c_3 |\xi - \eta| .$$

La dimostrazione della Proposizione 7.1 è come nello Step 2 di pag. 6. Per dimostrare la Proposizione 7.2 valutiamo la derivata

$$\left| \frac{\partial}{\partial \xi_i} f^{1/p} \right| = \frac{1}{p} \left| f^{(1-p)/p} \cdot f_{\xi_i} \right| \leqslant \frac{1}{p} \left[c_0 (1 + |\xi|^p) \right]^{(1-p)/p} |f_{\xi_i}| .$$

Utilizzando la (2.11), si vede che il secondo membro della disuguaglianza sopra scritta è limitato indipendentemente da ξ. Da ciò si ottiene la tesi (7.4).

Notiamo che, se dal punto di vista del Calcolo delle Variazioni si può esprimere l'ipotesi di coercitività in (7.3) in modo equivalente richiedendo soltanto che $f(\xi) \geqslant c_0 |\xi|^p$ (sommando ad f una costante, non si cambiano le funzioni di minimo), ciò non è ammesso nella proposizione 7.2; per avere un controesempio, basta infatti considerare una funzione $f(\xi)$ uguale a $|\xi|$ per ξ vicino a zero.