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Nonlinear elliptic systems with general growth

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Abstract

We prove *local Lipschitz-continuity* and, as a consequence, C^k*and* C^∞ *regularity* of *weak* solutions u for a class of *nonlinear elliptic differential systems* of the form $\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^\alpha(Du) = 0$, $\alpha = 1, 2, \ldots, m$. The *growth conditions* on the dependence of functions $a_i^\alpha(\cdot)$ on the gradient Du are so mild to allow us to embrace growths between the *linear* and the *exponential* cases, and they are more general than those known in the literature.

MSC: 35J50; 35D10

Keywords: Elliptic differential systems; Calculus of variations; Regularity

1. Introduction

Let $n \geqslant 2$, $m \geqslant 1$, let Ω be an open set of $\mathbb{R}^n$ and let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a *weak solution* of a *nonlinear elliptic system* of PDE's of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^\alpha(Du) = 0, \quad \alpha = 1, 2, \ldots, m, \tag{1.1}$$

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where $Du : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{m\times n}$ denotes the *gradient* of the map u, by components $x = (x_i)_{i=1,2,\ldots,n}$, $u = (u^\alpha)^{\alpha=1,2,\ldots,m}$ and $Du = (\partial u^\alpha/\partial x_i) = (u^\alpha_{x_i})^{\alpha=1,2,\ldots,m}_{i=1,2,\ldots,n}$. By using the notation $\xi = (\xi^\alpha_i)^{\alpha=1,2,\ldots,m}_{i=1,2,\ldots,n}$, $A(\xi) = (a^\alpha_i(\xi))^{\alpha=1,2,\ldots,m}_{i=1,2,\ldots,n}$ is a given *vector field* $A : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ of class C^1, satisfying the ellipticity condition

$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{m}\frac{\partial a^\alpha_i(\xi)}{\partial \xi^\beta_j}\lambda^\alpha_i\lambda^\beta_j > 0, \quad \forall \lambda, \xi \in \mathbb{R}^{m\times n}: \lambda \neq 0, \tag{1.2}$$

as well as the *variational condition* that the vector field $A(\xi)$ is the *gradient* of a function $f(\xi)$, i.e., that there exists a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ such that

$$a^\alpha_i = \frac{\partial f}{\partial \xi^\alpha_i} = f_{\xi^\alpha_i}, \quad \forall \alpha = 1, 2, \ldots, m; \quad i = 1, 2, \ldots, n. \tag{1.3}$$

Under the variational condition (1.3), the ellipticity condition (1.2) implies the (strict) *convexity* of the function *f*. Finally, we assume that $f(\xi) = g(|\xi|)$ is a function g of the *modulus* $|\xi|$ (with $g'(0) = 0$, to respect the condition that the function *f* is of class $C^1\left(\mathbb{R}^{m\times n}\right)$).

The *regularity problem* for the elliptic system (1.1) consists in asking if the solution $u = u(x) = \left(u^\alpha(x)\right)^{\alpha=1,2,\ldots,m}$, a priori only a measurable function in the Sobolev class $W^{1,1}$, in fact is of class C^∞ (or $C^{0,\alpha}$, C^1, $C^{1,\alpha}$, or C^k for some k), under the assumption that the data are smooth.

With the aim to explain the situation, let us assume, *for the moment*, that the solution $u \in W^{1,1}$ in fact is also in $W^{1,\infty}_{\mathrm{loc}}$, i.e., that *the gradient Du is locally bounded* in Ω. Then, under the ellipticity condition (1.2) and the variational condition (1.3) with $f(\xi) = g(|\xi|)$, it is possible to show that

$$u \in W^{1,\infty}_{\mathrm{loc}}, \quad A \in C^1 \quad \Longrightarrow \quad u \in C^{1,\alpha}_{\mathrm{loc}}$$

(see for instance [23]; for simplicity of notations, we write $A \in C^1$ instead of, more precisely, $A \in C^{1,\gamma}$ for some $\gamma \in (0, 1)$). Moreover, it is possible to see (cf. [3,16–18]) that u admits second derivatives in weak form and that, for every $k \in \{1, 2, \ldots, n\}$, the partial derivative $u_{x_k} = \left(u^\beta_{x_k}\right)^{\beta=1,2,\ldots,m}$ satisfies the elliptic differential linear system

$$\sum_{i,j,\beta}\frac{\partial a^\alpha_i(Du(x))}{\partial \xi^\beta_j}\left(u_{x_k}\right)^\beta_{x_j} = 0, \quad \alpha = 1, 2, \ldots, m$$

(see (4.4) and note that $\partial a^\alpha_i/\partial \xi^\beta_j = f_{\xi^\alpha_i \xi^\beta_j}$). The coefficients $\partial a^\alpha_i/\partial \xi^\beta_j(Du(x))$ are locally Hölder-continuous, since $u \in C^{1,\alpha}_{\mathrm{loc}}$; thus we can apply the regularity results in the literature for *linear* elliptic systems with smooth coefficients (see for instance Section 3 of Chapter 3 of [9]) to infer

$$u \in C^{1,\alpha}_{\mathrm{loc}}, \quad A \in C^k \qquad \Longrightarrow \qquad u \in C^{k,\alpha}_{\mathrm{loc}}, \quad \forall k = 2, 3, \dots \ .$$

In particular, $u \in C^{\infty}_{\mathrm{loc}}$ if $A \in C^{\infty}$.

Therefore the problem which remains to be solved is *under which conditions on* $A(\xi)$ *is it possible to show that the gradient* Du *is in fact locally bounded, i.e.,* $u \in W^{1,\infty}_{\mathrm{loc}}$. Why the local boundedness of the gradient Du is a so relevant condition for regularity?

Because the differential system (1.1) heavily depends on Du in a nonlinear way, in particular through $a_i^{\alpha}(Du)$ and, if $Du(x)$ is bounded, then $a_i^{\alpha}(Du(x))$ is bounded too and far away from zero. Thus the behavior of $A(\xi) = \left(a_i^{\alpha}(\xi)\right)$ for $|\xi| \to +\infty$ becomes irrelevant.

On the contrary, the local boundedness of the gradient is a property related to the behavior of $A(\xi)$ as $|\xi| \to +\infty$. This problem has been extensively studied in the literature and a detailed story is presented in the next section. Precisely, in the next section we point out in detail the assumptions made in the earlier mathematical literature on the subject, as well as the results presented in this paper.

We emphasize that the mathematical literature on the subject is large: some references are given in the next section and a good survey, as well as some new interesting regularity results, are given in the recent book by Bildhauer [2]. Our assumptions, in the context of basic elliptic systems of type (1.1) with $A(\xi) = D_{\xi} f(\xi)$ and $f(\xi) = g(|\xi|)$, are more general than those in the literature, and they allow us to consider at the same time variational problems with functions $f(\xi)$ having *linear growth* as $|\xi| \to +\infty$, as well as functions $f(\xi)$ with *either polynomial or exponential growth* at infinity.

2. Description of the problem and statement of the main results

Let Ω be an open set of $\mathbb{R}^n$ for some $n \geqslant 2$ and let $u : \Omega \to \mathbb{R}^m$ $(m \geqslant 1)$ be a vector valued *local minimizer* of an integral of the calculus of variations of the type

$$\mathcal{F} = \int_{\Omega} f(Du)\,dx, \tag{2.1}$$

related to some *convex* integrand $f : \mathbb{R}^{m \times n} \to \mathbb{R}$. Here $Du : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{m \times n}$ denotes the gradient of the map u. By a *local minimizer* of the integral (2.1) we mean a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ with the property that $\mathcal{F}(u) \leqslant \mathcal{F}(u + \phi)$ for every $\phi \in C^1_0(\Omega; \mathbb{R}^m)$; in the context of this paper this definition is consistent.

It is well known that in general we cannot expect that u, either a priori minimizer of integral (2.1) or weak solution of the differential system (1.1) in a *Sobolev class* of functions $W^{1,p}(\Omega; \mathbb{R}^m)$, is in fact a smooth function, say of class $C^{\infty}_{\mathrm{loc}}$, or even of class C^1_{loc} or $C^{0,\alpha}_{\mathrm{loc}}$ for some $\alpha \in (0,1)$. In the vector-valued case $m > 2$ examples of nonsmooth minimizers and of nonsmooth weak solutions have been given by De Giorgi [6], Giusti-Miranda [12] and by Necas [21]. A recent counterexample in three-dimensional case in the context of smooth strongly convex functionals has been also given by Sverak-Yan [22].

Even in the scalar case $m = 1$ it is possible to give examples of local minimizers $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R})$ for some $p > 1$ (this phenomenon is related to the p, q-growth condition described below, with q larger than p), which do not even belong to $L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R})$; see [10,15–17].

As already mentioned, regularity of solutions is often related to the growth of $f(\xi)$ as $|\xi| \to +\infty$. More precisely, the so-called *natural growth conditions* state that there exists a *growth exponent* $p > 1$ and positive constants m, M such that

$$m|\xi|^p \leqslant f(\xi) \leqslant M(1 + |\xi|^p), \quad \forall \xi \in \mathbb{R}^{m\times n}, \tag{2.2}$$

as well as the ellipticity conditions on the matrix $D^2 f$ of the second derivatives of *f*, of the type

$$m\left(1 + |\xi|^{p-2}\right)|\lambda|^2 \leqslant \left(D^2 f(\xi)\lambda, \lambda\right) \leqslant M(1 + |\xi|^{p-2})|\lambda|^2, \quad \forall \xi, \lambda \in \mathbb{R}^{m\times n}. \tag{2.3}$$

It was pointed out by Marcellini [16,17] that the above *natural* growth conditions, sufficient for regularity, can be weakened into *anisotropic* growth conditions, or into p, q-growth conditions, i.e., with an exponent $q \geqslant p$ in the right-hand side of (2.2), (2.3), or into more *general* growth conditions. In particular, ellipticity p, q-growth conditions of the type

$$m\big(1 + |\xi|^{p-2}\big)|\lambda|^2 \leqslant \left(D^2 f(\xi)\lambda, \lambda\right) \leqslant M\big(1 + |\xi|^{q-2}\big)|\lambda|^2, \quad \forall \xi, \lambda \in \mathbb{R}^{m\times n}, \tag{2.4}$$

with exponents $q \geqslant p > 1$ such that $\frac{q}{p} < \frac{n}{n-2}$ if $n > 2$.

In the general *vectorial setting* only few contributions are available for *general growth*: we like to refer to the papers by Giusti [11], Giusti-Miranda [13], Acerbi-Fusco [1] and by Esposito et al. [7]. A recent book by Bildhauer [2] gives a complete overview and a detailed list of references. If some additional structure conditions are assumed then several results can be found in the mathematical literature on the subject. For instance, as a generalization of the "*p-growth*" case considered by Uhlenbeck [23], Marcellini proposed in [19] an approach to the regularity of minimizers of the integral

$$F(u) = \int_\Omega g(|Du|)\, dx, \tag{2.5}$$

i.e. with the integrand in (2.1) of the form $f(\xi) = g(|\xi|)$, where $g : [0, +\infty) \to [0, +\infty)$ is an increasing convex function, without growth assumption on $g(t)$ as $t \to$

$+\infty$. For example, the regularity result can be applied to the *exponential growth*, such as any finite composition of the type

$$f(\xi) = (\exp(\dots(\exp(\exp|\xi|^2)^{p_1})^{p_2})\dots)^{p_k} \tag{2.6}$$

with $p_i \geqslant 1$, $\forall i = 1, 2, \dots, k$. However, some other restrictions ware imposed in [19], such as, for instance, the fact that $t \in (0, +\infty) \to \frac{g'(t)}{t}$ is assumed to be an increasing function. To exemplify, the model case $g(t) = t^p$ gives the restriction $p \geqslant 2$. Afterwards, in [14] Leonetti–Mascolo–Siepe consider the case of subquadratic p, q-growth conditions, i.e. in (2.4) they assume $1 < p < q < 2$. Their result includes energy densities f of the type $f(\xi) = |\xi|^p \log^\alpha(1 + |\xi|)$ with $p < 2$. In [8] Fuchs–Mingione concentrate on the case of nearly linear growth, for which (2.4) fails to be true. Typical examples are the logarithmic case $f(\xi) = |\xi| \log(1 + |\xi|)$ and its iterated version

$$\begin{cases} f_k(\xi) &= |\xi| L_k(|\xi|), \\ L_{s+1}(t) &= \log(1 + L_s(t)), \ L_1(t) = \log(1 + t) \end{cases} \tag{2.7}$$

for $k \in$ arbitrary. Bildhauer [2] considers linear behaviors for functional (2.5); he gives conditions that can keep γ-elliptic linear growth with $\gamma < 1 + \frac{2}{n}$. Examples of γ-elliptic linear integrands are given by

$$g_\gamma(t) = \int_0^t \int_0^s (1 + z^2)^{-\frac{\gamma}{2}} \, dz \, ds, \quad \forall t \geqslant 0. \tag{2.8}$$

For $\gamma = 1$, $g_\gamma(t)$ behaves like $t \log(1 + t)$ and in the limit case $\gamma = 3$, $g_\gamma(t)$ becomes $(1 + t^2)^{1/2}$. Hence the functions $g_\gamma(t)$ provide a one parameter family connecting *logarithmic examples* with the *minimal surface integrand*. As further reference see also [4].

In this paper we attempt to find conditions which include different kinds of growths. At this purpose we give a general condition on function g embracing growths moving between linear and exponential functions. The condition is the following:

Let $t_0, H > 0$ *and* $\beta \in \left(\frac{1}{n}, \frac{2}{n}\right)$. *For every* $\alpha \in \left(1, \frac{n}{n-1}\right]$ *there exists* $K = K(\alpha)$ *such that*

$$Ht^{-2\beta} \left[\left(\frac{g'(t)}{t} \right)^{\frac{n-2}{n}} + \frac{g'(t)}{t} \right] \leqslant g''(t) \leqslant K \left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t} \right)^\alpha \right], \quad \forall t \geqslant t_0. \tag{2.9}$$

The exponent α in the right-hand side is a parameter to play, i.e., to use to test more functions g. The condition in the left-hand side of (2.9) permits to achieve functions, for instance, with second derivative going to zero as a power $t^{-\gamma}$, (i.e. γ -elliptic), where γ is not too large and is related to the dimension n, i.e. $\gamma < 1 + \frac{2}{n}$. As well as functions in (2.8), other examples in the linear case include

$$g(t) = 1 + t - \sqrt{t}, \quad \forall t \geqslant 1, \ \ n < 4,$$

or more in general, for $r \in (0,1)$,

$$g_r(t) = h(t) - t^r, \quad \forall t \geqslant 1, \quad n < \frac{2}{1-r},$$

and also

$$g_r(t) = h(t) + (1-t^r)^{\frac{1}{r}}, \quad \forall t \geqslant 1, \quad n < \frac{2}{r},$$

where $h(t)$ is a convex function such that, for suitable constants C_1, C_2,

$$C_1(1+t) \leqslant h(t) \leqslant C_2(1+t).$$

We observe that the functions $g_k(t) = (1+t^k)^{\frac{1}{k}}$, related to *minimal surfaces*, are convex if $k \geqslant 1$, and $g_k^{''}(t) = (k-1)\, t^{k-2}(1+t^k)^{\frac{1}{k}-2} = \mathcal{O}\left(\frac{1}{t^{k+1}}\right)$ when $t \to +\infty$, so that they *do not satisfy* left-hand side of condition (2.9).

As far as p, q-growth is concerned, we like to remark that condition (2.9) is satisfied without assuming any restriction on p and q. For example, fixed $1 < p < q$, consider the function (cf. [5])

$$g(t) = \begin{cases} t^p & \text{if } t \leqslant \tau_0, \\ t^{\frac{p+q}{2} + \frac{q-p}{2} \sin \log \log \log t} & \text{if } t > \tau_0, \end{cases} \tag{2.10}$$

where τ_0 is such that $\sin \log \log \log \tau_0 = -1$. First of all we observe that function g oscillates between the function t^p, to which it is tangent in τ_n such that $\sin \log \log \log \tau_n = -1$, and the function t^q, to which it is tangent in σ_n such that $\sin \log \log \log \sigma_n = 1$. By a direct computation it is possible to see that one can choose τ_0 and t_0 large enough such that g is convex and satisfies (2.9). We observe that the left-hand side of (2.9) implies $g^{''}(t) > 0$ for $t \geqslant t_0$. For this reason the function in (2.10), with $p = 1$, does not satisfy condition (2.9); in fact if $p = 1$ we have $g^{''}(\sigma_n) = 0$.

Also high growths like that in (2.6) are included in condition (2.9). In other words, our results unify and generalize those obtained in the literature for integral (2.5), including in particular the linear case treated in [2], the non-standard p, q-growth, the exponential growth considered in [19] and also the new example of oscillating function in (2.10). Part of the techniques of this paper have been introduced by Marcellini [19]. The starting point is the second variational weak equation for which we need the supplementary assumption that $g^{''}(t)$ and $\frac{g'(t)}{t}$ are bounded by constants N and M for all $t > 0$. In this case we give a priori estimates for $\sup |D(u)|$ by using only the properties of function g, so that the constants in the a priori bounds do not depend on M and N. Successively we remove this assumption by approximating the original problem with regular variational ones. This is possible because the constants N and M do not enter in the a priori bounds for the L^∞-norm of the gradient.

In this paper we prove in particular the following two results, the first one valid under general growth conditions, the second one specific for the linear case.

Theorem A (*General growth*). *Let* $g : [0, +\infty) \to [0, +\infty)$ *be a convex function of class* $W^{2,\infty}_{\mathrm{loc}}$ *with* $g(0) = g'(0) = 0$, *satisfying the general growth condition* (2.9) *with* $\beta \in \left(\frac{1}{n}, \frac{2}{n}\right)$ *as before. Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega; R^m)$ *be a local minimizer of integral* (2.5). *Then* $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega; R^m)$. *Moreover, the following estimate holds: for every* $\varepsilon > 0$ *and* $R > \rho > 0$ *there exists a constant* C (*depending on* $\varepsilon, n, \rho, R, H, K$ *and* $\sup_{0 \leqslant t \leqslant t_0} g''(t)$) *such that*

$$\|Du\|^{2-\beta n}_{L^\infty(B\rho; \mathbb{R}^{m\times n})} \leqslant C \left\{ \int_{B_R} (1 + g(|Du|))\, dx \right\}^{\frac{1}{1-\beta}+\varepsilon}. \tag{2.11}$$

Theorem B (*Linear growth*). *Let* $g : [0, +\infty) \to [0, +\infty)$ *be a convex function of class* $W^{2,\infty}_{\mathrm{loc}}$ *with* $g(0) = g'(0) = 0$. *Let us assume that* g *has the linear behavior at infinity*

$$\lim_{t\to+\infty} \frac{g(t)}{t} = l \in (0, +\infty) \tag{2.12}$$

and that its second derivative satisfies the inequalities

$$H \frac{1}{t^\gamma} \leqslant g''(t) \leqslant K \frac{1}{t}, \quad \forall t \geqslant t_0, \tag{2.13}$$

for some positive constants H, K, t_0 *and for some* $\gamma \in \left[1, 1 + \frac{2}{n}\right)$. *Then every local minimizer* u *of integral* (2.5) *is of class* $W^{1,\infty}_{\mathrm{loc}}(\Omega; R^m)$ *and, for every* $R > \rho > 0$, *the following estimate is satisfied*:

$$\|Du\|^{2-\beta n}_{L^\infty(B\rho; \mathbb{R}^{m\times n})} \leqslant C \int_{B_R} (1 + g(|Du|))\, dx, \tag{2.14}$$

where $\beta = \frac{\gamma}{2} - \frac{n-2}{2n}$ *and the constant* C *depends on* n, ρ, R, l, H, K *and* $\sup_{0 \leqslant t \leqslant t_0} g''(t)$.

Note that $2 - \beta n \in (0, 1]$ since $\gamma \in \left[1, 1 + \frac{2}{n}\right)$. Note also that estimate (2.14) in Theorem B is sharper than estimate (2.11) of Theorem A, when we reduce the general assumption (2.9) of Theorem A to linear growth, since in the second case the proof is more direct, as explained at the end of this paper. Therefore, Theorem B cannot be considered a particular case of Theorem A.

The plan of the paper is the following. In Section 2 we discuss some consequences of assumption (2.9) and we prove for g some estimates that will be used in Section 3, where we get a priori bounds for the gradient of local minimizers of functional (2.5). In Section 4 we define the approximating regular variational problems and we obtain a priori bounds for the gradient of their minimizers. Finally in Section 5 we go to the limit and we obtain the regularity Theorems A and B.

3. Ellipticity estimates and their consequences

With the aim to study integrals of the Calculus of Variations of the type (2.5), we consider $f(\xi) = g(|\xi|)$, for $\xi \in \mathbb{R}^{m\times n}$, $(\xi = (\xi_i^\alpha), i = 1, 2, \ldots, n,\ \alpha = 1, 2, \ldots, m)$, where

$$g : [0, +\infty) \to [0, +\infty) \quad \begin{array}{l}\text{is a convex function of class}\\ W^{2,\infty}[0, T], \forall T > 0, \quad g(0) = g'(0) = 0.\end{array} \tag{3.1}$$

By the representation $f(\xi) = g(|\xi|)$, we have

$$f_{\xi_i^\alpha} = g'(|\xi|)\frac{\xi_i^\alpha}{|\xi|}, \quad f_{\xi_i^\alpha \xi_j^\beta} = \left(\frac{g''(|\xi|)}{|\xi|^2} - \frac{g'(|\xi|)}{|\xi|^3}\right) \cdot \xi_i^\alpha \xi_j^\beta + \frac{g'(|\xi|)}{|\xi|}\delta_{\xi_i^\alpha \xi_j^\beta}. \tag{3.2}$$

Since

$$\sum_{i,j,\alpha,\beta} \xi_i^\alpha \xi_j^\beta \lambda_i^\alpha \lambda_j^\beta = \left(\sum_{i,\alpha} \xi_i^\alpha \lambda_i^\alpha\right)^2 \leqslant (|\xi||\lambda|)^2, \quad \forall \lambda, \xi \in \mathbb{R}^{m\times n}$$

(and the equality holds when λ is proportional to ξ), we easily obtain the following ellipticity estimates:

$$\min\left\{g''(|\xi|), \frac{g'(|\xi|)}{|\xi|}\right\} \leqslant \frac{\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}\lambda_i^\alpha \lambda_j^\beta}{|\lambda|^2} \leqslant \max\left\{g''(|\xi|), \frac{g'(|\xi|)}{|\xi|}\right\}, \quad \forall \lambda, \xi \in \mathbb{R}^{m\times n}. \tag{3.3}$$

Let us define

$$\mathcal{H}(t) = \max\left\{g''(t), \frac{g'(t)}{t}, \quad \forall t > 0\right\}. \tag{3.4}$$

We observe that, since $g'(t) = \int_0^t g''(s)\,ds \leqslant M_T t,\ \forall t \leqslant T$, the function $\frac{g'(t)}{t}$ (and consequently $\mathcal{H}(t)$) is bounded on $(0, T], \forall T > 0$. We observe that in (3.1) we do not assume $g'(0) > 0$ but, more generally, we allow $g'(t)$ and $g(t)$ to be equal to zero in

$(0, \bar{t}]$, with $\bar{t} > 0$. The sequel of this section is devoted to derive some useful estimates on the function g, starting by the general assumption (2.9). With this aim we begin with the following lemma (where by 2^* we denote the Sobolev's exponent, i.e $2^* = \frac{2n}{n-2}$ if $n \geqslant 3$, while 2^* is any fixed number greater than 2 if $n = 2$).

Lemma 3.1. *Let g be as in (3.1). Let β, H be positive constants such that $\frac{1}{n} < \beta < \frac{2}{n}$. Let us assume that for every $\alpha \in \left(1, \frac{n}{n-1}\right]$ there exists a constant K (depending on α) such that*

$$H t^{-2\beta}\left[\left(\frac{g'(t)}{t}\right)^{\frac{2}{2^*}} + \frac{g'(t)}{t}\right] \leqslant g''(t) \leqslant K\left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t}\right)^{\alpha}\right], \quad \forall t \geqslant t_0. \tag{3.5}$$

Then for every δ with $\frac{2\alpha}{2-\alpha} \leqslant \delta \leqslant 2^*$ and for every $\gamma \geqslant 0$ there exists a constant C such that

$$1 + \int_0^t s^{\gamma}\sqrt{g''(s)}\,ds \geqslant C\left[1 + \left(\frac{t^{\gamma+1-\beta}}{\gamma+1}\right)^{\delta}\mathcal{H}(t)\right]^{\frac{1}{\delta}}, \quad \forall t \geqslant 0. \tag{3.6}$$

Proof. In order to simplify this proof, up to a rescaling, we will assume, without loss of generality, that $t_0 = 1$ and $g(t_0) > 0$. We observe that

$$\left[1 + \left(\frac{t^{\gamma+1-\beta}}{\gamma+1}\right)^{\delta}\mathcal{H}(t)\right]^{\frac{1}{\delta}} \leqslant \left[1 + \left(\frac{t^{\gamma+1-\beta}}{\gamma+1}\right)\mathcal{H}(t)^{\frac{1}{\delta}}\right], \quad \forall t > 0, \forall \gamma \geqslant 0. \tag{3.7}$$

Now, let us call

$$F_1(t,\gamma) = 1 + \int_0^t s^{\gamma}\sqrt{g''(s)}\,ds, \tag{3.8}$$

$$G_1(t,\gamma) = 1 + \left(\frac{t^{\gamma+1-\beta}}{\gamma+1}\right)\mathcal{H}(t)^{\frac{1}{\delta}}, \tag{3.9}$$

and let us define the quotient

$$Q_1(t,\gamma) = \frac{F_1(t,\gamma)}{G_1(t,\gamma)}. \tag{3.10}$$

It is easy to see that $Q_1(t,\gamma)$ is lower bounded in the strip $(t,\gamma) \in [0, t_0] \times [0, +\infty]$ (we remember that $t_0 = 1$) by the constant $C_1 = \left(1 + \max_{0<t\leqslant 1}\ [\frac{g'(t)}{t} + g''(t)]^{\frac{1}{\delta}}\right)^{-1}$.

From this (3.6) follows for $0 \leqslant t \leqslant t_0$. Now let $t \geqslant t_0$. By definition (3.4) of function $\mathcal{H}(t)$ we get

$$\mathcal{H}(t) \leqslant \frac{g'(t)}{t} + g''(t)$$

and by the right-hand side of (3.5) we can write

$$\mathcal{H}(t) \leqslant (K+1)\left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t}\right)^{\alpha}\right]. \tag{3.11}$$

From this, instead proving (3.6) we can prove the following:

$$1 + \int_0^t s^{\gamma}\sqrt{g''(s)}\,ds \geqslant C\left[1 + \frac{t^{\gamma+1-\beta}}{\gamma+1}\left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t}\right)^{\alpha}\right]^{\frac{1}{\delta}}\right], \quad \forall t \geqslant t_0, \tag{3.12}$$

where we still denote by C the new constant. At this end it is sufficient to show the inequality between the derivatives side to side with respect to t of (3.12)

$$\begin{aligned}\sqrt{g''(t)} \;\geqslant\; C_1\, t^{-\beta}\Bigg[&\left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}} + \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}} + \left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}-1} g''(t)\\ &+ \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}-1} g''(t)\Bigg],\end{aligned} \tag{3.13}$$

or, since $\alpha > 1$,

$$\begin{cases} \sqrt{g''(t)} \geqslant 2C_1\, t^{-\beta}\left[\left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}} + \left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}-1} g''(t)\right] & \text{if } \frac{g'(t)}{t} \leqslant 1\\ \sqrt{g''(t)} \geqslant 2C_1\, t^{-\beta}\left[\left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}} + \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}-1} g''(t)\right] & \text{if } \frac{g'(t)}{t} \geqslant 1. \end{cases}$$

If $\frac{g'(t)}{t} \leqslant 1$, by the left-hand side of (3.5) we get, since $\frac{1}{\delta} \geqslant \frac{1}{2^*}$,

$$\sqrt{g''(t)} \geqslant \sqrt{H}\, t^{-\beta}\left(\frac{g'(t)}{t}\right)^{\frac{1}{2^*}} \geqslant \sqrt{H}\, t^{-\beta}\left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}} \tag{3.14}$$

and also, by the right-hand side of (3.5)

$$g''(t)\left(\frac{g'(t)}{t}\right)^{-1} \leqslant 2K.$$

As a result we have

$$\sqrt{g''(t)} \geqslant \frac{\sqrt{H}}{2K} t^{-\beta} \left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}} 2K \geqslant \frac{\sqrt{H}}{2K} t^{-\beta} \left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}-1} g''(t). \tag{3.15}$$

Adding (3.14) to (3.15) we get, if $\frac{g'(t)}{t} \leqslant 1$,

$$\sqrt{g''(t)} \geqslant \frac{\sqrt{H}}{4K} t^{-\beta} \left[\left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}} + \left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}-1} g''(t) \right]. \tag{3.16}$$

If $\frac{g'(t)}{t} \geqslant 1$, with similar arguments we have

$$\sqrt{g''(t)} \geqslant \sqrt{H}\, t^{-\beta} \left(\frac{g'(t)}{t}\right)^{\frac{1}{2}}, \tag{3.17}$$

and since $\delta \geqslant \frac{2\alpha}{2-\alpha} > 2\alpha$, i.e. $\frac{\alpha}{\delta} < \frac{1}{2}$, we get

$$\sqrt{g''(t)} \geqslant \sqrt{H}\, t^{-\beta} \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}}. \tag{3.18}$$

Moreover, by the right-hand side of (3.5) we get

$$\sqrt{g''(t)} \leqslant \sqrt{2K} \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{2}},$$

equivalently

$$g''(t) \leqslant \sqrt{2K} \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{2}} \sqrt{g''(t)}$$

and, since $t \geqslant t_0 = 1$, we can also write

$$\sqrt{g''(t)} \geqslant \frac{1}{\sqrt{2K}} \left(\frac{g'(t)}{t}\right)^{-\frac{\alpha}{2}} g''(t) \geqslant \left(\frac{1}{\sqrt{2K}}\right) t^{-\beta} \left(\frac{g'(t)}{t}\right)^{-\frac{\alpha}{2}} g''(t).$$

Since $\delta \geqslant \frac{2\alpha}{2-\alpha}$, i.e. $\frac{\alpha}{\delta} \leqslant \frac{2-\alpha}{2}$, we have $\frac{\alpha}{\delta} - 1 \leqslant -\frac{\alpha}{2}$; hence

$$\sqrt{g''(t)} \geqslant \left(\frac{1}{\sqrt{2K}}\right) t^{-\beta} \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}-1} g''(t). \tag{3.19}$$

Therefore, in the case $\frac{g'(t)}{t}\geqslant 1$, from (3.18) and (3.19) we obtain

$$\sqrt{g''(t)}\geqslant \min\left\{\frac{\sqrt{H}}{2},\frac{1}{2\sqrt{2K}}\right\} t^{-\beta}\left[\left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}}+\left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}-1} g''(t)\right]. \tag{3.20}$$

Therefore (3.13) holds for $t\geqslant t_0$ too, as a consequence of (3.16) and (3.20). □

Lemma 3.2. *Let g be as in (3.1). Suppose that g satisfies the right-hand side of condition (3.5). Then there exists a constant C, depending on K, $g'(t_0)$, t_0, α, such that*

$$g'(t)t\leqslant C(1+g(t))^{\frac{1}{2-\alpha}},\quad \forall t\geqslant 0. \tag{3.21}$$

Proof. Let $t\geqslant t_0 = 1$. A multiplication for t and an integration side to side in the right-hand side of (3.5) give

$$\int_{t_0}^{t} s g''(s)\,ds\leqslant K\int_{t_0}^{t} g'(s)\,ds + K\int_{t_0}^{t} s\left(\frac{g'(s)}{s}\right)^{\alpha} ds.$$

An integration by parts of the left-hand side in the previous inequality gives

$$g'(t)t\leqslant g'(t_0)t_0+(K+1)\int_{t_0}^{t} g'(s)\,ds + K\int_{t_0}^{t} s^{2-2\alpha}g'(s)\left(g'(s)s\right)^{\alpha-1} ds.$$

Since $g(t_0)\geqslant 0$ and $t\geqslant t_0$ we have

$$g'(t)t\leqslant g'(t_0)t_0+(K+1)g(t)+K{t_0}^{2-2\alpha}\left(g'(t)t\right)^{\alpha-1} g(t).$$

By dividing both sides for $\left(g'(t)t\right)^{\alpha-1}$ we obtain

$$\left(g'(t)t\right)^{2-\alpha}\leqslant \left(g'(t_0)t_0\right)^{2-\alpha}+\left(\frac{K+1}{(g'(t_0)t_0)^{\alpha-1}}+K{t_0}^{2-2\alpha}\right) g(t).$$

Let ${C_1}^{2-\alpha}=\max\{\left(g'(t_0)t_0\right)^{2-\alpha}, \frac{K+1}{(g'(t_0)t_0)^{\alpha-1}}+K{t_0}^{2-2\alpha}\}$. Then we have for all $t\geqslant t_0$

$$g'(t)t\leqslant C_1(1+g(t))^{\frac{1}{2-\alpha}}.$$

Finally (3.21) follows with $C\geqslant C_1$ because $g'(t)t\leqslant g'(t_0)t_0$, for all $t\leqslant t_0$. □

Lemma 3.3. *Let g be as in (3.1) and let $\mathcal{H}$ be the function defined in (3.4). Suppose that g satisfies the right-hand side of condition (3.5). Then there exists a constant C such that for any η, $1 < \eta \leqslant \frac{3n}{3n-4}$,*

$$1 + \mathcal{H}(t)t^2 \leqslant C(1 + g(t))^\eta, \quad \forall t \geqslant 0, \tag{3.22}$$

where $\eta = \eta(\alpha) = \frac{\alpha}{2-\alpha}$ and the constant C depends on K, $\sup_{0 \leqslant t \leqslant t_0} g''(t)$, α.

Proof. Since $\mathcal{H}(t) = \max\left\{\frac{g'(t)}{t}, g''(t)\right\}$ we have that $\mathcal{H}(t)t^2 \leqslant g'(t)t + g''(t)t^2 \quad \forall t \geqslant 0$. Let $t \geqslant t_0 \geqslant 1$. By the right-hand side of (3.5) and by 3.2 we obtain

$$g''(t)t^2 \leqslant KC(1 + g(t))^{\frac{1}{2-\alpha}} + KC^\alpha(1 + g(t))^{\frac{\alpha}{2-\alpha}} t^{-2\alpha+2}.$$

Let $C_1 = \max\{KC, KC^\alpha \ t_0{}^{-2\alpha+2}\}$. Then we have that for all $t \geqslant t_0$

$$g''(t)t^2 \leqslant 2C_1(1 + g(t))^{\frac{\alpha}{2-\alpha}}. \tag{3.23}$$

On the other hand, if $t \leqslant t_0$, we have

$$g''(t)t^2 \leqslant \sup_{0 \leqslant t \leqslant t_0} g''(t)t^2 \leqslant t_0{}^2 \sup_{0 \leqslant t \leqslant t_0} g''(t) \leqslant C_{t_0}, \tag{3.24}$$

By putting together (3.23), (3.24) and Lemma 3.2, from the definition of $\mathcal{H}(t)$, we obtain the result. □

4. A priori estimates

In this section we consider the integral of the Calculus of Variations

$$F(u) = \int_\Omega f(Du)\, dx, \tag{4.1}$$

with $f(Du) = g(|Du|)$, where g satisfies (3.1). We make the following assumption:

Assumption 4.1. There exist two positive constants N and M such that

$$N|\lambda|^2 \leqslant \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(\xi)\lambda_i^\alpha \lambda_j^\beta \leqslant M|\lambda|^2, \qquad \forall \lambda, \xi \in \mathbb{R}^{m\times n}, \quad f(\xi) = g(|\xi|). \tag{4.2}$$

This is equivalent to say that both $\frac{g'(t)}{t}$ and $g''(t)$ are bounded by constants $N, M, \forall t > 0$. This assumption allows us to consider u as a function of class $W^{1,\infty}_{\text{loc}}\left(\Omega, \mathbb{R}^m\right) \cap$

$W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Similarly in [19], assumption (4.2) will be successively removed. The reason that will make this removal possible relies on the fact that the constants N and M do not enter in the a priori bound obtained for the L^∞-norm of the gradient. We will denote by B_ρ and B_R balls of radii, respectively, ρ and R ($\rho < R$) contained in Ω and with the same center.

Lemma 4.1. *Let g be as in (3.1), satisfying (4.2) and (3.6). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ be a minimizer of integral (4.1). Then there exists a constant C, which does not depend on N and M, such that (the function $\mathcal{H}$ is defined in (3.4))*

$$\|Du\|^{2-\beta n}_{L^\infty(B_\rho;\mathbb{R}^{m\times n})} \leqslant \frac{C}{(R-\rho)^n} \int_{B_R} \left(1 + |Du|^2 \mathcal{H}(|Du|)\right) dx.$$

Proof. Let u be a local minimizer of (4.1). By the left-hand side of (4.2), $u \in W^{1,2}(\Omega; \mathbb{R}^m)$ and by the right-hand side of (4.2) it satisfies the weak Euler first variation:

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(Du)\varphi^\alpha_{x_i}\, dx = 0, \quad \forall \varphi = (\varphi^\alpha) \in W^{1,2}_0(\Omega, \mathbb{R}^m). \tag{4.3}$$

Using some known techniques (see for example [3,9,16–18]) we can prove that u admits second-order weak partial derivatives, precisely that $u \in W^{2,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and it satisfies the second variation

$$\int_\Omega \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} \varphi^\alpha_{x_i}\, dx = 0, \quad \forall k = 1, 2, \ldots, n,$$

$$\forall \varphi = (\varphi^\alpha) \in W^{1,2}_0(\Omega, \mathbb{R}^m). \tag{4.4}$$

For $k \in \{1, 2, \ldots, n\}$ we consider $\varphi \in W^{1,2}_0(\Omega, \mathbb{R}^m)$ (we do not denote explicitly the dependence on k) defined by

$$\varphi := \eta^2 u_{x_k} \Phi(|Du|),$$

where $\eta \in C^1_0(\Omega)$ and $\Phi : [0, +\infty) \to [0, +\infty)$ is an increasing bounded Lipschitz continuous function. We plug φ in (4.4) and, since

$$\varphi^\alpha_{x_i} = 2\eta\eta_{x_i} u^\alpha_{x_k} \Phi(|Du|) + \eta^2 u^\alpha_{x_i x_k} \Phi(|Du|) + \eta^2 u^\alpha_{x_k} \Phi'(|Du|)(|Du|)_{x_i}$$

we obtain

$$\int_\Omega 2\eta\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} u^\beta_{x_j x_k} u^\alpha_{x_k} \eta_{x_i}\, dx$$

$$+\int_\Omega \eta^2\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} u^\beta_{x_j x_k} u^\alpha_{x_i x_k}\, dx$$

$$+\int_\Omega \eta^2\Phi' \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i}\, dx = 0. \tag{4.5}$$

Defining

$$A_k = \int_\Omega 2\eta\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} u^\beta_{x_j x_k} u^\alpha_{x_k} \eta_{x_i}\, dx, \tag{4.6}$$

$$B_k = \int_\Omega \eta^2\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} u^\beta_{x_j x_k} u^\alpha_{x_i x_k}\, dx, \tag{4.7}$$

$$C_k = \int_\Omega \eta^2\Phi' \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i}\, dx, \tag{4.8}$$

Eq. (4.5) takes the concise form of

$$A_k + B_k + C_k = 0. \tag{4.9}$$

We start estimating the first addendum A_k in (4.9) with the inequality $2ab \leqslant \frac{1}{2}a^2 + 2b^2$

$$\begin{aligned} |A_k| &\leqslant \int_\Omega 2\Phi \left[\eta^2 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} u^\beta_{x_j x_k} u^\alpha_{x_i x_k} \right]^{\frac{1}{2}} \\ &\quad \times \left[\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} \eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k} \right]^{\frac{1}{2}} dx \\ &\leqslant \int_\Omega \Phi \left[\frac{\eta^2}{2} \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} u^\beta_{x_j x_k} u^\alpha_{x_i x_k} \right. \\ &\quad \left. + 2 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} \eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k} \right] dx. \end{aligned} \tag{4.10}$$

From (4.9) and (4.10) we obtain

$$\frac{1}{2} B_k + C_k \leqslant 2 \int_\Omega \Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) \eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k}\, dx. \tag{4.11}$$

We use the expression of the second derivatives of f in (3.2) to estimate C_k in the left-hand side. Since

$$(|Du|)_{x_i} = \frac{1}{|Du|} \sum_{\alpha,k} u^\alpha_{x_i x_k} u^\alpha_{x_k} \tag{4.12}$$

we obtain

$$\begin{aligned}
&\sum_k \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i} \\
&= \left(\frac{g''}{|Du|^2} - \frac{g'}{|Du|^3} \right) \sum_{k,i,j,\alpha,\beta} u^\alpha_{x_i} u^\beta_{x_j} u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i} \\
&\quad + \frac{g'}{|Du|} \sum_{k,i,\alpha} u^\alpha_{x_i x_k} u^\alpha_{x_k} (|Du|)_{x_i} \\
&= \left(\frac{g''}{|Du|} - \frac{g'}{|Du|^2} \right) \sum_{k,i,\alpha} u^\alpha_{x_i} (|Du|)_{x_i} u^\alpha_{x_k} (|u|)_{x_k} + g' \sum_i (|Du|)^2_{x_i} \\
&= \left(\frac{g''}{|Du|} - \frac{g'}{|Du|^2} \right) \sum_\alpha \left[\sum_i u^\alpha_{x_i} \, (|Du|)_{x_i} \right]^2 + g' |D(|Du|)|^2.
\end{aligned} \tag{4.13}$$

Now we recall definition (4.8) for C_k. The previous equality shows that

$$\begin{aligned}
\sum_k C_k = \int_\Omega \eta^2 \Phi'(|Du|) \Bigg\{ &\left(\frac{g''(|Du|)}{|Du|} - \frac{g'(|Du|)}{|Du|^2} \right) \\
&\cdot \sum_\alpha \left[\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right]^2 + g'(|Du|) |D(|Du|)|^2 \Bigg\} \, dx
\end{aligned} \tag{4.14}$$

Now we consider the first term $\frac{1}{2} B_k$ in inequality (4.11). From (3.2) we get

$$\begin{aligned}
\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_i x_k} = &\left(\frac{g''(|Du|)}{|Du|^2} - \frac{g'(|Du|)}{|Du|^3} \right) \left(\sum_{i,\alpha} u^\alpha_{x_i x_k} u^\alpha_{x_i} \right)^2 \\
&+ \frac{g'(|Du|)}{|Du|} \sum_{i,\alpha} (u^\alpha_{x_i x_k})^2.
\end{aligned}$$

By (4.12), summing with respect to k,

$$\sum_k \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_i x_k} = \left(g''(|Du|) - \frac{g'(|Du|)}{|Du|} \right) |D(|Du|)|^2 + \frac{g'(|Du|)}{|Du|} |D^2 u|^2.$$

By definition (4.7) we can write

$$\sum_k B_k = \int_\Omega \eta^2 \Phi(|Du|) \left(\left(g''(|Du|) - \frac{g'(|Du|)}{|Du|} \right) |D(|Du|)|^2 + \frac{g'(|Du|)}{|Du|} |D^2 u|^2 \right) dx. \tag{4.15}$$

By (4.12) and applying the Cauchy–Schwarz inequality we have

$$|D(|Du|)|^2 = \sum_i (|Du|)^2_{x_i} = \frac{1}{|Du|^2} \sum_i \left(\sum_{\alpha,k} u^\alpha_{x_i x_k} u^\alpha_{x_k} \right)^2 \leqslant \sum_{i,\alpha,k} (u^\alpha_{x_i x_k})^2 = |D^2 u|^2,$$

from which we deduce that

$$\sum_k B_k \geqslant \int_\Omega \eta^2 \Phi(|Du|) g''(|Du|) |D(|Du|)|^2 \, dx. \tag{4.16}$$

Now, we consider $\sum_k C_k$ in formula (4.14). We can write that

$$\sum_k C_k = \int_\Omega \eta^2 \Phi'(|Du|) \left(\frac{g''(|Du|)}{|Du|} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 + g'(|Du|) |D(|Du|)|^2 - \frac{g'(|Du|)}{|Du|^2} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 \right) dx.$$

Since, by Cauchy–Schwarz inequality, we get

$$\sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 \leqslant \sum_{i,\alpha} (u^\alpha_{x_i})^2 \sum_i (|Du|)^2_{x_i} \leqslant |Du|^2 |D(|Du|)|^2,$$

then we can conclude that

$$\sum_k C_k \geqslant \int_\Omega \eta^2 \Phi'(|Du|) \frac{g''(|Du|)}{|Du|} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 dx \geqslant 0. \tag{4.17}$$

By using the inequalities obtained for $\sum_k B_k$ and $\sum_k C_k$ in (4.16) and (4.17) we obtain from formula (4.11) where we sum on k

$$\begin{aligned}
&\frac{1}{2} \int_\Omega \eta^2 \Phi(|Du|) g''(|Du|) |D(|Du|)|^2 \, dx \\
&\quad \leqslant \frac{1}{2} \sum_k B_k \leqslant \frac{1}{2} \sum_k B_k + \sum_k C_k \\
&\quad \leqslant 2 \int_\Omega \Phi(|Du|) \sum_{i,j,\alpha,\beta,k} f_{\xi^\alpha_i \xi^\beta_j}(Du) \eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k} \, dx.
\end{aligned} \tag{4.18}$$

By the right-hand side in (3.3), finally we obtain

$$\int_\Omega \eta^2 \Phi(|Du|) g''(|Du|) |D(|Du|)|^2 \, dx \leqslant 4 \int_\Omega \Phi(|Du|) \mathcal{H}(|Du|) |D\eta|^2 |Du|^2 \, dx \tag{4.19}$$

for every $\Phi : [0, +\infty) \to [0, +\infty)$, increasing, local Lipschitz continuous function with Φ and Φ' bounded on $[0, +\infty)$. If we consider a more general Φ not bounded, with derivative Φ' not bounded too, then we can approximate it by a sequence of functions Φ_r, each of them being equal to Φ in the interval $[0, r]$, and then extended to $[r, +\infty)$ with the constant value $\Phi(r)$. We insert $\Phi(r)$ in (4.19) and we go to the limit as $r \to +\infty$ by the monotone convergence theorem. So we obtain that (4.19) is true for every Φ positive, increasing, local Lipschitz continuous function in $[0, +\infty)$. Let us define

$$G(t) = 1 + \int_0^t \sqrt{\Phi(s) g''(s)} \, ds, \qquad \forall t \geqslant 0. \tag{4.20}$$

By Hölder inequality, since function Φ is increasing and $g'(0) = 0$, we get

$$\begin{aligned}
[G(t)]^2 &= \left(1 + \int_0^t \sqrt{\Phi(s) g''(s)} \right)^2 \leqslant 2 + 2\Phi(t) t \int_0^t g''(s) \, ds \\
&= 2 + 2\Phi(t) t g'(t) \leqslant 2 + 2\Phi(t) \mathcal{H}(t) t^2.
\end{aligned}$$

Then we can write the following estimate for the gradient of function $\eta G(|Du|)$:

$$\begin{aligned}
|D(\eta G(|Du|))|^2 &= |(D\eta) G(|Du|) + \eta G'(|Du|) D(|Du|)|^2 \\
&\leqslant 2|D\eta|^2 |G(|Du|)|^2 + 2\eta^2 |G'(|Du|)|^2 \cdot |D(|Du|)|^2
\end{aligned}$$

$$\leqslant\ 4|D\eta|^2\left(1+\Phi(|Du|)\mathcal{H}(|Du|)|Du|^2\right)$$
$$+\,2\eta^2\Phi(|Du|)g''(|Du|)|D(|Du|)|^2.$$

By integrating over Ω the previous inequality we obtain

$$\int_\Omega |D(\eta G(|Du|))|^2\,dx$$
$$\leqslant 4\int_\Omega |D\eta|^2\left(1+\Phi(|Du|)\mathcal{H}(|Du|)|Du|^2\right)\,dx$$
$$+\,2\int_\Omega \eta^2\Phi(|Du|)g''(|Du|)|D(|Du|)\,dx.$$

Now we use inequality (4.19) and we get

$$\int_\Omega |D(\eta G(|Du|))|^2\,dx\leqslant 4\int_\Omega |D\eta|^2\left(1+3\Phi(|Du|)\mathcal{H}(|Du|)|Du|^2\right)\,dx. \tag{4.21}$$

As a consequence of (4.2), Du is locally bounded; hence we can apply Sobolev's inequality: there exists a constant C_1 such that

$$\left\{\int_\Omega \left[\eta G(|Du|)\right]^{2^*}\,dx\right\}^{\frac{2}{2^*}}\leqslant C_1\int_\Omega |D(\eta G(|Du|))|^2\,dx. \tag{4.22}$$

Let us define $\Phi(t)=t^{2\gamma}$, with $\gamma\geqslant 0$ (so that Φ is increasing). Since g satisfies (3.6) we can choose $\delta=2^*$ and combining (4.22) and (4.21) we have that there exist constants C_3, C_4 and same β, $0\leqslant\beta<\frac{2}{n}$ such that

$$\int_\Omega |D\eta|^2\left(1+3\Phi(|Du|)\mathcal{H}(|Du|)|Du|^2\right)\,dx\geqslant\frac{1}{4}\int_\Omega |D(\eta G(|Du|))|^2\,dx$$
$$\geqslant C_3\left\{\int_\Omega \left[\eta G(|Du|)\right]^{2^*}\,dx\right\}^{\frac{2}{2^*}}$$
$$\geqslant\frac{C_4}{(\gamma+1)^2}\left\{\int_\Omega \eta^{2^*}\left(1+|Du|^{(\gamma+1-\beta)2^*}\mathcal{H}(|Du|)\right)\,dx\right\}^{\frac{2}{2^*}}. \tag{4.23}$$

Substituting in the left-hand side of inequality (4.23) the expression of function Φ we get that there exist a constant C_5 and same numbers β, $0\leqslant\beta<\frac{2}{n}$ such that for every $\gamma\geqslant 0$

$$\left\{\int_\Omega \eta^{2^*}\left(1+|Du|^{(\gamma+1-\beta)2^*}\mathcal{H}(|Du|)\right)\,dx\right\}^{\frac{2}{2^*}}$$
$$\leqslant C_5(\gamma+1)^2\int_\Omega |D\eta|^2\left(1+|Du|^{2\gamma}\mathcal{H}(|Du|)|Du|^2\right)\,dx.$$

Let the test function η be equal to 1 in B_ρ, with support contained in B_R and such that $|D\eta|\leqslant\frac{2}{(R-\rho)}$. Let us denote by $\delta=2(\gamma+1)$ (note that, since $\gamma\geqslant 0$, then $\delta\geqslant 2$). We have

$$\left\{\int_{B_\rho}\left(1+|Du|^{(\delta-2\beta)\frac{2^*}{2}}\mathcal{H}(|Du|)\right)dx\right\}^{\frac{2}{2^*}} \leqslant C_5\left(\frac{\delta}{R-\rho}\right)^2\int_{B_R}\left(1+|Du|^{\delta}\mathcal{H}(|Du|)\right)dx. \tag{4.24}$$

Fixed $\overline{R}$ and $\overline{\rho}$, with $\overline{R}>\overline{\rho}$ we define the decreasing sequence of radii $\{\rho_i\}_{i\geqslant 0}$

$$\rho_i=\overline{\rho}+\frac{\overline{R}-\overline{\rho}}{2^i},\quad \forall i\geqslant 0.$$

We observe that $\rho_0=\overline{R}>\rho_i>\rho_{i+1}>\overline{\rho}$. We define also the increasing sequence of exponents $\{\delta_i\}_{i\geqslant 0}$, $\delta_0=2$, $\delta_{i+1}=(\delta_i-2\beta)\frac{2^*}{2}$, $i\geqslant 0$, and we rewrite the (4.24) with $R=\rho_i$, $\rho=\rho_{i+1}$ and $\delta=\delta_i$. Then we obtain for every $i\geqslant 0$,

$$\left\{\int_{B_{\rho_{i+1}}}\left(1+|Du|^{\delta_{i+1}}\mathcal{H}(|Du|)\right)dx\right\}^{\frac{2}{2^*}} \leqslant C_5\left(\frac{\delta_i 2^{i+1}}{\overline{R}-\overline{\rho}}\right)^2\int_{B_{\rho_i}}\left(1+|Du|^{\delta_i}\mathcal{H}(|Du|)\right)dx. \tag{4.25}$$

By iterating (4.25) we get

$$\left\{\int_{B_{\rho_{i+1}}}\left(1+|Du|^{(2-\beta n)(\frac{2^*}{2})^{i+1}+\beta n}\mathcal{H}(|Du|)\right)dx\right\}^{(\frac{2}{2^*})^{i+1}} \leqslant C_6\int_{B_{\overline{R}}}\left(1+|Du|^{2}\mathcal{H}(|Du|)\right)dx, \tag{4.26}$$

where the exponent in the first integral is given by computing

$$\delta_{i+1}=2\left(\frac{2^*}{2}\right)^{i+1}-2\beta\sum_{k=1}^{i+1}\left(\frac{2^*}{2}\right)^k=(2-\beta n)\left(\frac{2^*}{2}\right)^{i+1}+\beta n$$

and

$$C_6 \leqslant \prod_{k=0}^{+\infty}\left[\frac{C_5 8}{(\overline{R}-\overline{\rho})^2}\,(2^*)^{2k}\right]^{(\frac{2}{2^*})^k} = \left(\left(\frac{C_5 8}{(\overline{R}-\overline{\rho})^2}\right)^{\sum_{k=0}^{+\infty}(\frac{2}{2^*})^k}\right)(2^*)^{\sum_{k=0}^{+\infty}k(\frac{2}{2^*})^k}$$

$$= \left(\frac{C_5 8}{(\overline{R}-\overline{\rho})^2}\right)^{\frac{n}{2}}\cdot(2^*)^{\frac{n(n-2)}{2}} = \frac{C_7}{(\overline{R}-\overline{\rho})^n},$$

for every $n\geqslant 3$; otherwise, if $n=2$, then for every $\varepsilon>0$ we can choose 2^* so that $C_6=\frac{C_7}{(\overline{R}-\overline{\rho})^{2+\varepsilon}}$ for some constant C_7. Now, we observe that the function $1+t^{\alpha}\mathcal{H}(t)\geqslant 1+t^{\alpha-1}g'(t)$, since $\mathcal{H}(t)\geqslant\dfrac{g'(t)}{t}$ for every $t>0$. Now, if $t\geqslant 1$, since $g'(t)$ is increasing we have $1+t^{\alpha-1}g'(t)\geqslant t^{\alpha-1}g'(1)$ and, if $t\leqslant 1$ we have $1+t^{\alpha-1}g'(t)\geqslant 1\geqslant t^{\alpha-1}$. Hence, we can write

$$\left\{\int_{B_{\overline{\rho}}}|Du|^{(2-\beta n+\frac{\beta n-1}{(\frac{2^*}{2})^{i+1}})(\frac{2^*}{2})^{i+1}}\,dx\right\}^{(\frac{2}{2^*})^{i+1}} \leqslant \frac{C_7}{(\overline{R}-\overline{\rho})^n}\int_{B_{\overline{R}}}\Big(1+|Du|^2\mathcal{H}(|Du|)\Big)\,dx. \tag{4.27}$$

Finally we go to the limit as $i\to+\infty$ and we obtain

$$\sup\left\{|Du(x)|^{2-\beta n} : x\in B_{\overline{\rho}}\right\} = \lim_{i\to+\infty}\left\{\int_{B_{\overline{\rho}}}|Du|^{(2-\beta n+\frac{\beta n-1}{(\frac{2^*}{2})^{i+1}})(\frac{2^*}{2})^{i+1}}\,dx\right\}^{(\frac{2}{2^*})^{i+1}} \leqslant \frac{C_7}{(\overline{R}-\overline{\rho})^n}\int_{B_{\overline{R}}}\Big(1+|Du|^2\mathcal{H}(|Du|)\Big)\,dx. \qquad \square$$

Lemma 4.2. *Let g be as in (3.1). Let us assume that g satisfies (4.2) and (3.5). Let $u\in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ be a minimizer of integral (4.1). Then, for every $\varepsilon>0$ and for every ρ, R $(0<\rho<R)$, there exists a constant $C=C(n,\varepsilon,\rho,R)$ such that*

$$\int_{B_\rho}\Big(1+|Du|^2\mathcal{H}(Du)\Big)\,dx\leqslant C\left\{\int_{B_R}(1+g|Du|)\,dx\right\}^{\frac{1}{1-\beta}+\varepsilon},$$

the constant C depends also on $g(t_0)$, $g'(t_0)$, K, H, $\sup_{0\leqslant t\leqslant t_0} g''(t)$, $\inf_{0\leqslant t\leqslant t_0} g''(t)$, but it does not depend on the constants N and M in (4.2).

Proof. In Lemma 3.1 we considered parameters α and γ such that $\alpha \in \left(1, \frac{n}{n-1}\right]$ and $\gamma \geqslant 0$. Here we restrict ourselves to the case $1 < \alpha \leqslant \frac{2n}{2n-1} = 1 + \frac{1}{2n-1}$ and $\gamma = 0$. Then (3.6) holds for any $\delta \in \left[\frac{2\alpha}{2-\alpha}, 2^*\right]$. We define $v = \frac{2^*}{\delta}$, so that $v \in \left[1, 2^* \frac{2-\alpha}{2\alpha}\right]$. The condition $\beta < \frac{2}{n}$ is equivalent to $1 < (1-\beta)\frac{2^*}{2}$; therefore it is possible to limit v (and δ) to satisfy the conditions $1 < v < (1-\beta)\frac{2^*}{2}$ too. Finally, since $\beta > \frac{1}{n}$ we have $\alpha \leqslant \frac{2n}{2n-1} < \frac{2}{2-\beta}$ and this implies $1 - \beta < \frac{2-\alpha}{\alpha}$. Thus

$$v \in \left[1, (1-\beta)\frac{2^*}{2}\right] \implies v \in \left[1, 2^* \frac{2-\alpha}{2\alpha}\right] \Leftrightarrow \delta \in \left[\frac{2\alpha}{2-\alpha}, 2^*\right]$$

so that the parameter δ satisfies the condition of Lemma 3.1. Therefore there exists a constant C_1 (we still denote by C_1, C_2, etc.. the constants in this proof) such that

$$(G(t))^{2^*} = \left[\left(1 + \int_0^t \sqrt{g''(s)}\, ds\right)^{\frac{2^*}{v}}\right]^v \geqslant C_2 \left[1 + t^{(1-\beta)\frac{2^*}{v}} \mathcal{H}(t)\right]^v.$$

Under the notations of the previous Lemma 4.1, let us consider again estimates (4.23) with ϕ identically equal to 1 (or, equivalently, with $\gamma = 0$); we have

$$\left\{\int_\Omega (\eta G(|Du|))^{2^*}\, dx\right\}^{\frac{2}{2^*}} \leqslant 4C_2 \int_\Omega |D\eta|^2 \left(1 + 3\mathcal{H}(|Du|)|Du|^2\right) dx$$

and thus

$$\left\{\int_\Omega \eta^{2^*} \left[1 + |Du|^{(1-\beta)\frac{2^*}{v}} \mathcal{H}(|Du|)\right]^v dx\right\}^{\frac{2}{2^*}} \leqslant C_3 \int_\Omega |D\eta|^2 \left(1 + \mathcal{H}(|Du|)|Du|^2\right) dx. \tag{4.28}$$

Since $v < (1-\beta)\frac{2^*}{2}$, we have $(1-\beta)\frac{2^*}{v} > 2$. Under the notation $V = V(x) = 1 + |Du|^2 \mathcal{H}(|Du|)$ (4.28) becomes

$$\left\{\int_\Omega \eta^{2^*} V^v\, dx\right\}^{\frac{2}{2^*}} \leqslant C_3 \int_\Omega |D\eta|^2 V\, dx. \tag{4.29}$$

As in the previous Lemma 4.1 we consider a test function η equal to 1 on B_ρ with support contained in B_R and such that $|D\eta| \leqslant \frac{2}{R-\rho}$, we obtain

$$\left\{\int_{B_\rho} V^v\, dx\right\}^{\frac{2}{2^*}} \leqslant \frac{4C_4}{(R-\rho)^2} \int_{B_R} V\, dx. \tag{4.30}$$

Let $\gamma > \frac{2^*}{2}$. By the Holder inequality we have

$$\left\{ \int_{B_\rho} V^v \, dx \right\}^{\frac{2}{2^*}} \leqslant \frac{4C_4}{(R-\rho)^2} \int_{B_R} V^{\frac{v}{\gamma}} V^{1-\frac{v}{\gamma}} \, dx$$

$$\leqslant \frac{4C_4}{(R-\rho)^2} \left\{ \int_{B_R} V^v \, dx \right\}^{\frac{1}{\gamma}} \left\{ \int_{B_R} V^{\frac{\gamma-v}{\gamma-1}} \, dx \right\}^{\frac{\gamma-1}{\gamma}}. \tag{4.31}$$

Let R_0 and ρ_0 be fixed. For any $i \in \mathbb{N}$ we consider (4.31) with $R = \rho_i$ and $\rho = \rho_{i-1}$, where $\rho_i = R_0 - \frac{R_0-\rho_0}{2^i}$. By iterating (4.31), since $R - \rho = \frac{R_0-\rho_0}{2^i}$, similar to the computation in [19, p. 19], we can write

$$\int_{B_{\rho_0}} V^v \, dx \leqslant \left\{ \int_{B_{\rho_i}} V^v \, dx \right\}^{(\frac{2^*}{2\gamma})^i} C_5 \left\{ \frac{1}{(R_0-\rho_0)^2} \right\}^{\left(\frac{2^*\gamma}{2\gamma-2}\right)^i}$$

$$\times \left\{ \int_{B_{\rho_0}} V^{\frac{\gamma-v}{\gamma-1}} \, dx \right\}^{\frac{2^*(\gamma-1)}{2\gamma-2^*}}. \tag{4.32}$$

Since $\frac{\gamma-v}{\gamma-1} < 1$ we can apply Lemma (3.3) with $\eta = \frac{\gamma-1}{\gamma-v}$ and we obtain

$$\int_{B_\rho} V^v \, dx \leqslant \left\{ \int_{B_{\rho_i}} V^v \, dx \right\}^{\left(\frac{2^*}{2\gamma}\right)^i} C_5 \left\{ \frac{1}{(R_0-\rho_0)^2} \right\}^{\frac{2^*\gamma}{2\gamma-2^*}}$$

$$\times \left\{ \int_{B_{\rho_0}} [1 + g(|Du|)] \, dx \right\}^{\frac{2^*(\gamma-1)}{2\gamma-2^*}}.$$

In the limit as $i \to +\infty$ we get

$$\int_{B_{\rho_0}} V^v \, dx \leqslant C_6 \left\{ \frac{1}{(R_0-\rho_0)^2} \right\}^{\frac{2^*\gamma}{2\gamma-2^*}} \left\{ \int_{B_{R_0}} [1 + g(|Du|)] \, dx \right\}^{\frac{2^*(\gamma-1)}{2\gamma-2^*}}.$$

Finally

$$\int_{B_{\rho_0}} V \, dx \leqslant \text{meas} \left\{ B_{\rho_0} \right\}^{1-\frac{1}{v}} \left\{ \int_{B_{\rho_0}} V^v \, dx \right\}^{\frac{1}{v}}$$

$$\leqslant C_7 \left\{ \frac{1}{(R_0-\rho_0)^2} \right\}^{\frac{2^*\gamma}{(2\gamma-2^*)v}} \left\{ \int_{B_{R_0}} [1 + g(|Du|)] \, dx \right\}^{\frac{2^*(\gamma-1)}{(2\gamma-2^*)v}}. \tag{4.33}$$

As $v \to (1-\beta)\frac{2^*}{2}$ and $\gamma \to +\infty$ the two exponents in (4.33) converge to $\frac{1}{1-\beta}$ and we have the result. □

By combining together Lemmas 3.1, 4.1 and 4.2 we proved the following theorem.

Theorem 4.1. *Let g be as in (3.1). Suppose that g satisfies (4.2) and (3.5). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ be a minimizer of integral (4.1). Then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ and for every $\varepsilon > 0$ and for every ρ, R $(0<\rho<R)$, there exists a constant $C = C(n,\varepsilon,\rho,R)$ such that*

$$\|Du\|^{2-\beta n}_{L^\infty(B\rho;\mathbb{R}^{m\times n})} \leqslant C\left\{\int_{B_R}(1+g(|Du|))\,dx\right\}^{\frac{1}{1-\beta}+\varepsilon},$$

the constant C depends also on H, K, $\sup_{0\leqslant t\leqslant t_0} g''(t)$, but does not depend on the constants N and M in (4.2).

5. The approximating regular problems

Let us consider a function g with the properties described in (3.1). Now we consider the function $\frac{g'(t)}{t}$. It is possible to have one and only one of the following three cases:

(i) There exists a sequence $\{t_n\}$, $\lim_{n\to+\infty} t_n = +\infty$ such that $\frac{g'(t_n)}{t_n} = 1$.
(ii) There exists T such that for all $t \geqslant T$ it follows that $\frac{g'(t)}{t} > 1$.
(iii) There exists T such that for all $t \geqslant T$ it follows that $\frac{g'(t)}{t} < 1$.

Let $\bar t = \inf\{t > 0 : \frac{g'(t)}{t} > 0\}$; up to a rescaling we can assume that $0 \leqslant \bar t < 1 \leqslant t_0$. We consider a sequence ε_n, $\lim_{n\to+\infty}\varepsilon_n = 0$, in the following way. In case (i) we put $\varepsilon_n = \frac{1}{t_n}$, in case (ii) or (iii) we consider any sequence $\varepsilon_n \to 0$, with $\frac{1}{\varepsilon_n} \geqslant T$. It is obvious that we can choose n sufficiently large such that $\bar t + \varepsilon_n < 1$ and $\frac{1}{\varepsilon_n} \geqslant \max\{T, \bar t + \varepsilon_n\}$. Now we define the function

$$g'_{\varepsilon_n}(t) = \begin{cases} \dfrac{g'(\bar t+\varepsilon_n)}{\bar t+\varepsilon_n}t, & 0\leqslant t\leqslant \bar t+\varepsilon_n,\\ g'(t), & \bar t+\varepsilon_n < t \leqslant \dfrac{1}{\varepsilon_n},\\ \min\left\{\varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right)t,\, g'(t)+\varepsilon_n t - 1\right\}, & t > \dfrac{1}{\varepsilon_n}. \end{cases} \tag{5.1}$$

Then obviously we can define

$$g_{\varepsilon_n}(t) = \int_0^t g'_{\varepsilon_n}(s)\,ds. \tag{5.2}$$

The function $g_{\varepsilon_n}(t)$ results to be a convex function of class $C^1([0,+\infty))$, satisfying (3.1) and (4.2) with suitable constants $N(\varepsilon_n)$ and $M(\varepsilon_n)$.

Lemma 5.1. *Let g be as in (3.1) satisfying the left-hand side of (3.5). Let $g_{\varepsilon_n}(t)$ be defined in (5.2). Then there exists a constant $H_1>0$ such that we have*

$$H_1 t^{-2\beta}\left[\left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\frac{2}{2^*}}+\frac{g'_{\varepsilon_n}(t)}{t}\right]\leqslant g''_{\varepsilon_n}(t),\quad \forall t\geqslant t_0. \tag{5.3}$$

Proof. Let $\bar{t}+\varepsilon_n<1\leqslant t_0$ and $t\geqslant t_0$.
(1) If $\bar{t}+\varepsilon_n<t\leqslant\frac{1}{\varepsilon_n}$, then (5.3) holds because $g'_{\varepsilon_n}(t)=g'(t)$ and $g''_{\varepsilon_n}(t)=g''(t)$.
(2) Let $t>\frac{1}{\varepsilon_n}$.
(2a) If $g'_{\varepsilon_n}(t)=\varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right)t$, then $g''_{\varepsilon_n}(t)=\varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right)$ and we have

$$\begin{aligned} t^{-2\beta}\left(\varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right)\right)^{\frac{2}{2^*}}+t^{-2\beta}\varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right) &\leqslant {\varepsilon_n}^{2\beta+\frac{2}{2^*}}\left[\frac{g'\left(\frac{1}{\varepsilon_n}\right)}{g'(t_0)}\right]^{\frac{2}{2^*}}(g'(t_0))^{\frac{2}{2^*}} \\ &\quad +\varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right). \end{aligned}$$

Since $2\beta+\frac{2}{2^*}>1$ and $\varepsilon_n\leqslant 1$ we have ${\varepsilon_n}^{2\beta+\frac{2}{2^*}}\leqslant\varepsilon_n$; moreover, from the monotonicity of function $g'(t)$ we get $\frac{g'(\frac{1}{\varepsilon_n})}{g'(t_0)}\geqslant 1$. As a consequence we can write

$$\begin{aligned} t^{-2\beta}\left[\left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\frac{2}{2^*}}+\frac{g'_{\varepsilon_n}(t)}{t}\right] &\leqslant \left((g'(t_0))^{\frac{2}{2^*}-1}+1\right)\varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right) \\ &= \left((g'(t_0))^{\frac{2}{2^*}-1}+1\right)g''_{\varepsilon_n}(t), \end{aligned}$$

i.e. (5.3) holds with $H_1\leqslant\left((g'(t_0))^{\frac{2}{2^*}-1}+1\right)^{-1}$.
(2b) Let $g'_{\varepsilon_n}(t)=g'(t)+\varepsilon_n t-1$. Then

$$\frac{g'(t)}{t}\leqslant\frac{g'(t)}{t}+\varepsilon_n-\frac{1}{t}=\frac{g'_{\varepsilon_n}(t)}{t}\leqslant\varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right).$$

If we are in case (i) or (iii), then we have as a consequence $\frac{g'(t)}{t} \leqslant 1$. Hence by the left-hand side of (3.5) we can write

$$t^{-2\beta}\left[\left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\frac{2}{2^*}} + \frac{g'_{\varepsilon_n}(t)}{t}\right] \leqslant 2t^{-2\beta}\left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\frac{2}{2^*}} = 2t^{-2\beta}\left(\frac{g'(t)}{t} + \varepsilon_n - \frac{1}{t}\right)^{\frac{2}{2^*}}$$

$$\leqslant 2t^{-2\beta}\left(\frac{g'(t)}{t}\right)^{\frac{2}{2^*}} + 2t^{-2\beta}\varepsilon_n^{\frac{2}{2^*}}$$

$$\leqslant \frac{2}{H}g''(t) + 2\varepsilon_n^{\frac{2}{2^*}+2\beta}. \tag{5.4}$$

Since the exponent $\frac{2}{2^*} + 2\beta > 1$ and $\varepsilon_n < 1$, then

$$\frac{2}{H}g''(t) + 2\varepsilon_n^{\frac{2}{2^*}+2\beta} < \frac{2}{H}g''(t) + 2\varepsilon_n < \left(2 + \frac{2}{H}\right)(g''(t) + \varepsilon_n)$$

$$= \left(2 + \frac{2}{H}\right) g''_{\varepsilon_n}(t). \tag{5.5}$$

By (5.4) and (5.5) we have the estimate in (5.3) with $H_1 \leqslant \left(2 + \frac{2}{H}\right)^{-1}$.

If we are in case (ii) then $\frac{g'(t)}{t} > 1$ and again by the left-hand side of (3.5) we obtain

$$t^{-2\beta}\left[\left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\frac{2}{2^*}} + \frac{g'_{\varepsilon_n}(t)}{t}\right] \leqslant 2t^{-2\beta}\frac{g'_{\varepsilon_n}(t)}{t} = 2t^{-2\beta}\left(\frac{g'_{\varepsilon_n}(t)}{t} + \varepsilon_n - \frac{1}{t}\right)$$

$$\leqslant 2\left(t^{-2\beta}\frac{g'(t)}{t} + \varepsilon_n\right) \leqslant \left(2 + \frac{2}{H}\right)(g''(t) + \varepsilon_n).$$

This last inequality completes the proof. □

Lemma 5.2. *Let g be as in (3.1) satisfying the right-hand side of (3.5). Let $g_{\varepsilon_n}(t)$ be defined in (5.2). Then there exists a constant $K_1 > 0$ such that for any $\alpha > 1$ we have*

$$g''_{\varepsilon_n}(t) \leqslant K_1\left[\frac{g'_{\varepsilon_n}(t)}{t} + \left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\alpha}\right], \quad \forall t \geqslant t_0. \tag{5.6}$$

Proof. Let $\bar{t} + \varepsilon_n < 1 \leqslant t_0$ and $t \geqslant t_0$.

(1) If $\bar{t} + \varepsilon_n < t \leqslant \frac{1}{\varepsilon_n}$, then (5.6) holds because $g'_{\varepsilon_n}(t) = g'(t)$ and $g''_{\varepsilon_n}(t) = g''(t)$.

(2) Let $t > \frac{1}{\varepsilon_n}$.

(2a) If $g'_{\varepsilon_n}(t) = \varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right) t$, then $g''_{\varepsilon_n}(t) = \varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right)$ and (5.6) is obviously satisfied.
(2b) Let $g'_{\varepsilon_n}(t) = g'(t) + \varepsilon_n t - 1$. Then $g''_{\varepsilon_n}(t) = g''(t) + \varepsilon_n$ and

$$\frac{g'(t)}{t} \leqslant \frac{g'(t)}{t} + \varepsilon_n - \frac{1}{t} \leqslant \varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right).$$

If we are in case (i) or (iii), then we have as a consequence $\frac{g'(t)}{t} \leqslant 1$. By the right-hand side of (3.5) we can write

$$g''_{\varepsilon_n}(t) = g''(t) + \varepsilon_n \leqslant 2K\,\frac{g'(t)}{t} + \varepsilon_n \leqslant 2K\left(\frac{g'(t)}{t} + \varepsilon_n\right). \tag{5.7}$$

Since $g'(t)$ is an increasing function, we have

$$\begin{aligned} g''_{\varepsilon_n}(t) \;&\leqslant\; 2K\left(\frac{g'(t)}{t} + \varepsilon_n + \left(\frac{1+g'(t_0)}{g'(t_0)}\right)\frac{g'(t)}{t} - \frac{1}{t}\right) \\ &\leqslant\; 4K\left(\frac{1+g'(t_0)}{g'(t_0)}\right)\left(\frac{g'(t)}{t} + \varepsilon_n - \frac{1}{t}\right) = 4K\left(\frac{1+g'(t_0)}{g'(t_0)}\right)\frac{g'_{\varepsilon_n}(t)}{t} \end{aligned} \tag{5.8}$$

i.e. (5.6) with $K_1 \geqslant 4K\left(\frac{1+g'(t_0)}{g'(t_0)}\right)$.

If case (ii) is realized, then $\frac{g'(t)}{t} > 1$ and we can write

$$\left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\alpha} = \left(\frac{g'(t)}{t} + \varepsilon_n - \frac{1}{t}\right)^{\alpha} \geqslant \frac{1}{2^{\alpha}}\left(\frac{g'(t)}{t} + \varepsilon_n\right)^{\alpha} = \frac{1}{2^{\alpha}}\left(\frac{g'(t)}{t}\right)^{\alpha}\left(1 + \frac{\varepsilon_n}{\frac{g'(t)}{t}}\right)^{\alpha}.$$

Since

$$\left(1 + \frac{\varepsilon_n}{\frac{g'(t)}{t}}\right)^{\alpha} \geqslant 1 + \alpha\frac{\varepsilon_n}{\frac{g'(t)}{t}}$$

we can write

$$\left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\alpha} \geqslant \frac{1}{2^{\alpha}}\left[\left(\frac{g'(t)}{t}\right)^{\alpha} + \varepsilon_n\right] \geqslant \frac{1}{2^{\alpha+1}K}\,\left(g''(t) + \varepsilon_n\right) = \frac{1}{2^{\alpha}K}\,g''_{\varepsilon_n}(t), \tag{5.9}$$

i.e. (5.6) with $K_1 \geqslant K2^{\alpha+1}$. Combining (5.8) and (5.9) we obtain the result. □

Lemma 5.3. *Let g be as in (3.1). Let $g_{\varepsilon_n}(t)$ be defined in (5.2). Then there exists a constant C such that*

$$g_{\varepsilon_n}(t) \leqslant C\,(1+g(t))+\varepsilon_n t^2, \quad \forall t \geqslant 0. \tag{5.10}$$

Proof. Let $\bar{t}+\varepsilon_n<1$ and $0\leqslant t\leqslant \bar{t}+\varepsilon_n$. We have

$$g_{\varepsilon_n}(t)=\frac{g'(\bar{t}+\varepsilon_n)}{\bar{t}+\varepsilon_n}\frac{t^2}{2}\leqslant\frac{1}{2}\frac{g'(1)}{\bar{t}}. \tag{5.11}$$

If $\bar{t}+\varepsilon_n<t\leqslant\frac{1}{\varepsilon_n}$, we have

$$g_{\varepsilon_n}(t)=g(t)-g(\bar{t}+\varepsilon_n)+\frac{g'(\bar{t}+\varepsilon_n)(\bar{t}+\varepsilon_n)}{2}\leqslant g(t)+\frac{1}{2}g'(1). \tag{5.12}$$

If $t>\frac{1}{\varepsilon_n}$, we have $g'_{\varepsilon_n}(t)\leqslant g'(t)+\varepsilon_n t$ from which

$$\begin{aligned} g_{\varepsilon_n}(t) &\leqslant g(t_0)+\int_{t_0}^{t}\left(g'(s)+\varepsilon_n s\right)\,ds \\ &\leqslant g(t_0)+\int_{0}^{t}\left(g'(s)+\varepsilon_n s\right)\,ds=g(t_0)+g(t)+\frac{\varepsilon_n t^2}{2}. \end{aligned} \tag{5.13}$$

By (5.11), (5.12) and (5.13) we obtain the result with the constant

$$C\geqslant\max\left\{\frac{1}{2}\,g'(1),\,g(t_0),\,\frac{1}{2}\,\frac{g'(1)}{\bar{t}}\right\}. \qquad \square$$

6. Passage to the limit

Let us consider for every ε_n (ε_n is the sequence defined in the previous Section 5) the sequence of integral functionals

$$F_{\varepsilon_n}(v)=\int_\Omega g_{\varepsilon_n}\left(|Dv|\right)\,dx, \tag{6.1}$$

where $g_{\varepsilon_n}(t)$ is defined through its derivative $g'_{\varepsilon_n}(t)$ by (5.1) and (5.2). Let $u\in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ be a local minimizer of integral (4.1), i.e. $g(|Du|)\in L^1_{\mathrm{loc}}(\Omega)$ and $F(u)\leqslant F(u+\phi)$ for every $\phi\in C^1_0(\Omega;\mathbb{R}^m)$. Let B_R be a ball of radius R such that $B_{2R}\subset\subset\Omega$ and let $0<\sigma<\min\{1,R\}$. We indicate by u_σ a sequence of smooth functions defined from u by means of standard mollifiers. Then $u_\sigma\in W^{1,2}(B_R;\mathbb{R}^m)$. Let

$u_{\varepsilon_n,\sigma}$ be a minimizer of the integral $F_{\varepsilon_n}(v)$ in (6.1) that satisfies the Dirichlet condition $u_{\varepsilon_n,\sigma} = u_\sigma$ on the boundary ∂B_R, i.e., since F_{ε_n} has a quadratic growth,

$$\int_{B_R} g_{\varepsilon_n}\left(|Du_{\varepsilon_n,\sigma}|\right) dx \leqslant \int_{B_R} g_{\varepsilon_n}(|Dv|)\, dx, \quad \forall v \in W_0^{1,2}(B_R;\mathbb{R}^m) + u_\sigma. \tag{6.2}$$

By results of previous Section 5, for every ε_n, g_{ε_n} satisfies conditions (3.1), (4.2) (with suitable constants $N(\varepsilon_n)$ and $M(\varepsilon_n)$) and (3.5) with constants H and K not depending on ε_n. Therefore we can apply to g_{ε_n} the a priori estimate obtained in Theorem 4.1 obtaining that for every ε_n and for every ball B_ρ of radius $\rho < R$ there exists a constant C_1 (independent on N, M, ε_n, σ) such that, for some constants β, $\frac{1}{n} < \beta < \frac{2}{n}$, we have

$$\|Du_{\varepsilon_n,\sigma}\|^{2-\beta n}_{L^\infty(B\rho;\mathbb{R}^{m\times n})} \leqslant C_1 \left\{ \int_{B_R} \left(1 + g_{\varepsilon_n}\left(|Du_{\varepsilon_n,\sigma}|\right)\right) dx \right\}^{\frac{1}{1-\beta}+\varepsilon}. \tag{6.3}$$

By the minimality of $u_{\varepsilon_n,\sigma}$ we can write that

$$\int_{B_R} g_{\varepsilon_n}\left(|Du_{\varepsilon_n,\sigma}|\right) dx \leqslant \int_{B_R} g_{\varepsilon_n}(|Du_\sigma|)\, dx, \tag{6.4}$$

and by (5.10) and the properties of mollifiers we obtain

$$\begin{aligned}\int_{B_R} g_{\varepsilon_n}(|Du_\sigma|)\, dx &\leqslant C_2 \left\{ \int_{B_R} (1 + g(|Du_\sigma|))\, dx + \varepsilon_n \int_{B_R} |Du_\sigma|^2\, dx \right\} \\ &\leqslant C_2 \left\{ \int_{B_{R+\sigma}} (1 + g(|Du|))\, dx + \varepsilon_n \int_{B_R} |Du_\sigma|^2\, dx \right\} \\ &\leqslant C_3(\sigma).\end{aligned} \tag{6.5}$$

From this chain of inequalities and (6.3) we obtain as a consequence

$$\begin{aligned}\|Du_{\varepsilon_n,\sigma}\|^{2-\beta n}_{L^\infty(B_\rho;\mathbb{R}^{m\times n})} &\leqslant C_4 \left\{ \int_{B_{R+\sigma}} (1 + g(|Du|))\, dx + \varepsilon_n \int_{B_R} |Du_\sigma|^2\, dx \right\}^{\frac{1}{1-\beta}+\varepsilon} \\ &\leqslant C_5(\sigma).\end{aligned} \tag{6.6}$$

Then for every fixed σ, $|Du_{\varepsilon_n,\sigma}|$ is equibounded with respect to ε_n. Hence, up to a subsequence, $u_{\varepsilon_n,\sigma}$ converges in the weak* topology of $W^{1,\infty}(B_\rho;\mathbb{R}^{m\times n})$ to a function w_σ for some w_σ. Going to the limit for $\varepsilon_n \to 0$ in (6.6) we obtain

$$\|Dw_\sigma\|^{2-\beta n}_{L^\infty(B_\rho,\mathbb{R}^{m\times n})} \leqslant C_4 \left\{ \int_{B_{R+\sigma}} (1 + g(|Du|))\, dx \right\}^{\frac{1}{1-\beta}+\varepsilon}. \tag{6.7}$$

Hence, we also have that $|Dw_\sigma|$ is equibounded in $L^\infty(B_\rho, \mathbb{R}^{m\times n})$ and it is still possible to take a subsequence which converges in the weak* topology of $L^\infty(B_\rho, \mathbb{R}^{m\times n})$ to a function Dw for some w.

We will prove that $w = u$. Let us consider ε_n sufficiently small in dependence on σ; more precisely, fixed σ, we consider $\varepsilon_n \leqslant \overline{\varepsilon_n}(\sigma)$, with $\overline{\varepsilon_n}(\sigma)$ such that $\frac{1}{\overline{\varepsilon_n}(\sigma)} > [C_5(\sigma)]^{\frac{1}{2-\beta n}}$ where $C_5(\sigma)$ is the constant obtained in estimate (6.6). Then we have by (6.6) that $|Du_{\varepsilon_n,\sigma}| < \frac{1}{\varepsilon_n}$. By the definition of $g_{\varepsilon_n}(t)$ we can calculate

$$g_{\varepsilon_n}(t) = \begin{cases} \frac{g'(\bar t+\varepsilon_n)}{\bar t+\varepsilon_n}\frac{t^2}{2} & \text{if } 0 \leqslant t \leqslant \bar t + \varepsilon_n, \\ g(t) - g(\bar t + \varepsilon_n) + \frac{g'(\bar t+\varepsilon_n)(\bar t+\varepsilon_n)}{2} & \text{if } \bar t + \varepsilon_n < t \leqslant \frac{1}{\varepsilon_n}, \end{cases} \tag{6.8}$$

and hence we can write that

$$g(t) \leqslant g(\bar t + \varepsilon_n) + g_{\varepsilon_n}(t)\,, \quad \bar t + \varepsilon_n \leqslant t \leqslant \frac{1}{\varepsilon_n}. \tag{6.9}$$

By lower semicontinuity and (6.9) we obtain

$$\begin{aligned} \int_{B_\rho} g\left(|Dw_\sigma|\right)\,dx &\leqslant \liminf_{\varepsilon_n\to 0} \int_{B_R} g\left(|Du_{\varepsilon_n,\sigma}|\right)\,dx \\ &\leqslant \liminf_{\varepsilon_n\to 0} \int_{B_R} g_{\varepsilon_n}\left(|Du_{\varepsilon_n,\sigma}|\right)\,dx. \end{aligned}$$

From (6.4) and (5.10) we can deduce that $g_{\varepsilon_n}\left(|Du_{\varepsilon_n,\sigma}|\right)$ is bounded with respect to ε_n and then we can apply in (6.4) the dominant convergence theorem obtaining

$$\liminf_{\varepsilon_n\to 0} \int_{B_R} g_{\varepsilon_n}\left(|Du_{\varepsilon_n,\sigma}|\right)\,dx \leqslant \int_{B_R} g\left(|Du_\sigma|\right)\,dx \leqslant \int_{B_{R+\sigma}} g\left(|Du|\right)\,dx.$$

By resuming we have, for every $\rho < R$,

$$\int_{B_\rho} g\left(|Dw_\sigma|\right)\,dx \leqslant \int_{B_{R+\sigma}} g\left(|Du|\right)\,dx. \tag{6.10}$$

Again by lower semicontinuity and by (6.10) we have

$$\int_{B_R} g\left(|Dw|\right)\,dx \leqslant \liminf_{\sigma\to 0} \int_{B_R} g\left(|Dw_\sigma|\right)\,dx \leqslant \int_{B_R} g\left(|Du|\right)\,dx.$$

Now, our assumptions on g do not guarantee uniqueness of the minimizer for the Dirichlet problem. However $g(|\xi|)$ is locally strictly convex for $|\xi| > 1$, then we can conclude as in [19] that $w = u$. Going to the limit for $\sigma \to 0$ in (6.7) we get

$$\|Dw\|_{L^\infty(B_\rho;\mathbb{R}^{m\times n})}^{2-\beta n} \leqslant C_4 \left\{ \int_{B_R} (1 + g(|Du|))\, dx \right\}^{\frac{1}{1-\beta}+\varepsilon}. \tag{6.11}$$

Hence estimate (6.11) holds also for Du. Therefore we completed the proof of Theorem A.

Theorem B follows by Theorem A with some simplifications; below we give an outline of its proof.

Outline of the proof of Theorem B. We first observe that assumption (2.12) implies that $\lim_{t\to+\infty} g'(t) = l \in (0, +\infty)$ and hence there exist t_0 such that $\frac{g'(t)}{t} < 1$ for every $t \geqslant t_0$. Thus, condition (2.13) can be rewritten as

$$H \left(\frac{g'(t)}{t} \right)^{\frac{2}{2^*}} t^{-2\beta} \leqslant g''(t) \leqslant K\, \frac{g'(t)}{t}, \quad \forall t \geqslant t_0,$$

where $\beta = \frac{\gamma}{2} - \frac{1}{2^*}$. Since the case $\gamma = 1$, corresponding to the assumption $H\,\frac{1}{t} \leqslant g''(t) \leqslant K\,\frac{1}{t}$, is easier to be treated, we limit ourselves to consider here $\gamma > 1$; in this case we have $\beta = \frac{\gamma}{2} - \frac{1}{2^*} > \frac{1}{n}$ and we are in the conditions of Theorem A. Moreover, the function $g'(t)$ has the Δ_2-*property*. This make immediate Lemma 4.2 and that is why in the right-hand side of final estimate (2.14) there does not appear the exponent β (see also Remarks 1.2 and 5.1 in [19]).

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A referee pointed out to us that the first version of this paper presented for publication contained a mistake in the proof of Lemma 3.1. We corrected the proof of Lemma 3.1 and its application in the proof of Lemma 4.2. The main Theorems A and B (stated at the end of Section 2) remain unchanged with respect to the first version. We thank the referee for having read the paper in detail. The research was partially supported by the Italian *Ministero dell'Istruzione*, *dell'Università e della Ricerca* (MIUR).

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