Stationary isothermic surfaces

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Outline

References

- R. Magnanini-S. Sakaguchi, *Matzoh ball soup: heat conductors with a stationary isothermic surface,* Ann. Math. 156 (2002).
- R. Magnanini-J. Prajapat-S. Sakaguchi, *Stationary isothermic surfaces and uniformly dense domains,* to appear in Transactions of AMS.
- R. Magnanini-S. Sakaguchi, *Interaction between degenerate diffusion and shape of domain,* in preparation (2005).

Assumptions

Heat equation

We consider the **heat equation:**

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0, \infty).$$

Assumptions on Ω.

- $\Omega \subset \mathbb{R}^N$ is a domain
- $\partial\Omega$ is connected (to avoid certain ambiguities)
- $\partial\Omega$ is **of class** C^2 (this assumption can be weakened in various ways, depending on circumstances).

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Stationary isothermic surfaces

Isothermic surface (IS)

A surface $\Gamma \subset \Omega$ of codimension 1 such that $u(x,t) =$ constant for some $t > 0$ and every $x \in \Gamma$.

In general, isothermic surfaces evolve with time. We want to study the occurrence of invariant surfaces.

Stationary isothermic surface (SIS)

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where $c(t)$ is some function of t.

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Examples

Eigenmodes

Let $U = U_\lambda(x)$ be such that $\Delta U + \lambda U = 0$ in Ω for some $\lambda > 0$. Then $u(x) = U(x)\ e^{-\lambda t}$ solves the heat equation and **all its IS are SIS**.

Symmetric domains

Let Ω be a **ball**, an **infinite solid spherical cylinder** or a **halfspace**. Then the solution of the initial-boundary value problem

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0, \infty),$$
$$u = 1 \quad \text{on} \quad \Omega \times \{0\}, \qquad u = 0 \quad \text{in} \quad \partial\Omega \times (0, \infty), \tag{1}$$

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Statements of the problems

In this talk e will consider the occurrence of SIS for the solutions of two relevant problems:

Initial-boundary value problem

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0, \infty),$$
$$u = \varphi \quad \text{on} \quad \Omega \times \{0\}, \qquad u = 0 \quad \text{in} \quad \partial\Omega \times (0, \infty), \tag{2}$$

where φ is a given $L^2(\Omega)$ function. Clearly, in problem (1) we have $\varphi \equiv 1$.

Initial value problem

$$\begin{aligned} u_t &= \Delta u && \text{in} && \mathbb{R}^N \times (0, \infty), \\ u &= \mathcal{X}_\Omega && \text{on} && \mathbb{R}^N \times \{0\}, \end{aligned} \tag{3}$$

where $\mathcal{X}_\Omega$ is the characteristic function of a given domain $\Omega \subset \mathbb{R}^N$.

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Problem's history

Klamkin's question (1964)

Consider problem (1). If all IS are SIS and Ω is bounded, is Ω a ball? What happens when Ω is unbounded?

L. Zalcman (1987) included this question in a list of problems about the ball and named it **Matzoh Ball Soup**.

G. Alessandrini (1990)

Let Ω be bounded and let every point of $\partial\Omega$ be regular for the Dirichlet problem. Let u be the solution of (1).

If all IS of u are SIS then Ω must be a ball.

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Problem's history cont'd

Serrin's result

Alessandrini's proof is based on Serrin's celebrated symmetry result:

$$\begin{array}{lll} \Delta v = f(v) & \text{in} & \Omega, \\ v = 0 & \text{on} & \partial\Omega, \\ \frac{\partial v}{\partial \nu} = \text{constant} & \text{on} & \partial\Omega, \end{array} \quad \Rightarrow \quad \Omega \text{ is a ball.}$$

Alessandrini (1991)

Let u be the solution of problem (2) (initial value $= \varphi$).

If all IS of u are SIS, then either $u(x,t) = U_\lambda(x)e^{-\lambda t}$ for some $\lambda > 0$ or Ω is a ball.

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S. Sakaguchi, (1999)

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where Ω is a Lipschitz domain, $\varphi \in L^2(\Omega)$ and $\int_\Omega \varphi dx = 0$.

If all IS of u are SIS, then either φ is a Neumann eigenfunction or, modulo a rotation of coordinates, u depends on either

- (x_1, t),
- $(\sqrt{x_1^2 + \cdots + x_k^2}, t)$ with $2 \le k \le N - 1$,
- or $(|x|, t)$.

Problem's history cont'd

Isoparametric functions

Such a theorem is based on the classification theorem for **isoparametric functions** in $\mathbb{R}^N$ (which satisfy both equations $\Delta f = \alpha(f)$ and $|\nabla f|^2 = \beta(f)$ and turn out to have planar, cylindrical or spherical symmetry) due to **B. Segre** and **T. Levi-Civita**.

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Only one stationary isothermic surface

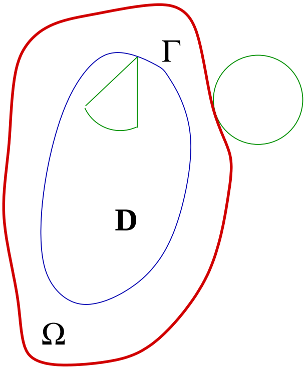
R.M.-S. Sakaguchi, 2002

Let Ω (or $\mathbb{R}^N \setminus \Omega$) be a bounded domain with boundary $\partial\Omega$ that satisfies the **exterior sphere condition**.

Let $\Gamma = \partial D$ with $\overline{D} \subset \Omega$ and D satisfying the **interior cone condition.**

Let u be the solution of (1).

If Γ is a SIS for u, then $\partial\Omega$ must be a sphere.



Sketch of the proof

Balance law

If $v_t = \Delta v$ in $G \times (0, \infty)$ and $x_0 \in G$ then $v(x_0, t) = 0$ for every $t > 0$ if and only if

$$\int_{B(x_0,r)} v(y,t)\ dy = 0,$$

for every $0 \le r < \operatorname{dist}(x_0, \partial G)$ and $t > 0$.

Remark

I stress the fact that this property does not depend on any boundary condition. It is obtained by proving that the spherical means of v satisfy a a parabolic PDE in $(0, +\infty) \times (0, +\infty)$ and certain overdetermined boundary conditions.

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Consequences of balance law

1^{st} consequence

We apply the balance law with $x_0 = 0$ and $v(x,t) = u(p+x,t) - u(q+x,t)$, where $p, q \in \Gamma$ and obtain that if Γ is a SIS then

$$\int_{B(x,r)} u(y,t)dy = C(r,t),$$

for every $0 \le r < \mathrm{dist}(x_0, \partial\Omega)$, $t > 0$, and $x \in \Gamma$.

2^{nd} consequence

$\partial\Omega$ must be **analytic**, since u is analytic and $\nabla u \neq 0$ on Γ for some t. (We argue by contradiction: if $\nabla u(x_0,t) \neq 0$ for some $x_0 \in \Gamma$ and every t, by the balance law, it must be $\int_{B(x_0,r)}(y - x_0)u(y,t)dy = 0$ for every r and t; we get a contradiction by using the exterior sphere condition for Ω and the interior cone condition for D.)

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Short-time behavior

Short-time behavior

If u is a solution of (1) and $d(x) = \mathrm{dist}(x, \partial\Omega)$, then

$$1 - u(x,t) = \frac{2}{\sqrt{\pi}} \int_{d(x)/\sqrt{4t}}^{\infty} e^{-\sigma^2} d\sigma \times \{1 + o(1)\} \quad \text{as} \quad t \to 0^+. \qquad (4)$$

1^{st} consequence

Γ must be **parallel** to $\partial\Omega$; in fact

$$-4t \log[1 - u(x,t)] \to d(x)^2$$

as $t \to 0^+$.

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Formula (4) gives a quantitative estimate of the **boundary layer** occurring for $t \to 0^+$, that we exploit to get information about the **heat content of balls tangent to** $\partial\Omega$.

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Asymptotic formula for heat content

Barriers

In fact, we construct **upper and lower barriers** of the form $F_\pm(d(x)/\sqrt{4t})$ for u as $t \to 0^+$. This helps us to prove the following crucial formula:

Asymptotic formula for heat content

$$t^{-(N+1)/4} \int_{B(x,R)} u(z,t)\, dz \to C_N \left\{ \prod_{j=1}^{N-1} [1/R - \kappa_j(y)] \right\}^{-1/2} \qquad (5)$$

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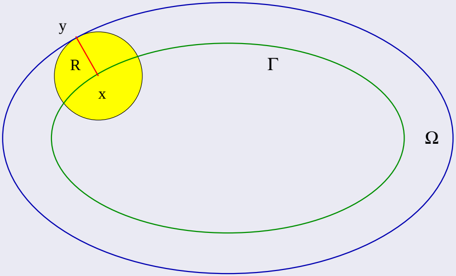
Here $R = \mathrm{dist}(x, \partial\Omega) = \mathrm{dist}(\Gamma, \partial\Omega)$ and $\kappa_j(y)$ is the j-th principal curvature of $\partial\Omega$ evaluated at the unique point y such that $|x - y| = \mathrm{dist}(x, \partial\Omega)$.



Conclusion

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The asymptotic formula (5) for heat content and the 1^{st} consequence of the balance law (i.e. constant heat content on Γ) imply that

$$\prod_{j=1}^{N-1} [1/R - \kappa_j] = \text{ constant on } \partial\Omega. \qquad (6)$$

Conclusion

(6) is a Monge-Ampère equation; by the **moving planes technique,** V.I. Aleksandrov (in the Soap Bubble Theorem's version for Wirtinger surfaces) showed a result that implies that, if $\partial\Omega$ is bounded and satisfies (6), then it must be a sphere. Therefore: **if Γ is a SIS for u, then $\partial\Omega$ must be a sphere.**

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Remarks

The asymptotic formula (5) can be extended (with a constant different from C_N) to solutions of the **porous medium equation** or of the **evolution p-Laplace equation.**

If $\partial\Omega$ is **unbounded** and admits $N-1$ distinct SIS, $\Gamma_1, \dots, \Gamma_{N-1}$, then we find the system $\prod_{j=1}^{N-1} [1/R_i - \kappa_j] = c_i, \ i = 1, \dots, N-1, \quad \text{on } \partial\Omega,$ which implies that all principal curvatures $\kappa_1, \dots, \kappa_{N-1}$ are constant on $\partial\Omega$, and hence $\partial\Omega$ is either a spherical cylinder or a hyperplane, thus giving a definitive answer to Klamkin's original question.

The previous item may set the difference between Alessandrini's and our assumptions and results: while only isoparametric surfaces are possible if **all** IS are SIS, it may be possible to find **"isolated SIS"** under the latter assumptions.

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Using global assumptions

When $\partial\Omega$ is unbounded, one way to get information about the symmetry of $\partial\Omega$ is to couple our formula (6) with some a priori information on Ω.

Adapting a technique of Caffarelli, Nirenberg and Spruck

Let

$$\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > \varphi(x'), x' \in \mathbb{R}^{N-1}\},$$

where $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ is Lipschitz continuous and $|\nabla\varphi(x')| = o(\sqrt{|x'|})$ as $|x'| \to \infty$. If Ω contains a SIS, then $\partial\Omega$ is a hyperplane.

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Let Ω be as in the previous item with $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ Lipschitz continuous and such that $\varphi(x' + h) - \varphi(x') \to 0$ as $|x'| \to \infty$ uniformly in h. If Ω contains a SIS, then $\partial\Omega$ is a hyperplane.

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A different setting

We have not been able to obtain further symmetry results for the Matzoh Ball Soup problem with only one SIS. In what follows I will present results and remarks that can help us to have a better insight into the problem.

We now consider the Cauchy problem (3):

$$u_t = \Delta u \quad \text{in} \quad \mathbb{R}^N \times (0, \infty), \qquad u = \mathcal{X}_\Omega \quad \text{on} \quad \mathbb{R}^N \times \{0\}.$$

and suppose as usual that $\Gamma \subset \mathbb{R}^N$ is a SIS for u.

This setting is more favourable than the initial-boundary value problem, since we can write the solution u explicitly:

$$u(x,t) = (4\pi t)^{-N/2} \int_{\mathbb{R}^N} \mathcal{X}_\Omega(y)\, e^{-|x-y|^2/4t} dy.$$

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$$u(x,t) = (4\pi t)^{-N/2} \int_{\mathbb{R}^N} \mathcal{X}_\Omega(y)\; e^{-|x-y|^2/4t} dy.$$

An improvement of a result by Chavel and Karp

We can rewrite $u(x,t)$ as a $\mathbf{p = 1/4t}$ Lebesgue norm:

$$u(x,t) = (4\pi t)^{-N/2}|\Omega| \; ||e^{-|x-\cdot|^2}||^{4t}_{L^{1/4t}(\Omega, dy/|\Omega|)}.$$

If Γ is a SIS, then

$$||e^{-|x-\cdot|^2}||_{L^{1/4t}(\Omega, dy/|\Omega|)} = C(t) \quad \text{for} \quad x \in \Gamma,$$

and hence, when $t \to \infty$ we have:

$$\exp\left\{\int_\Omega \log\left[e^{-|x-y|^2}\right]\right\}\frac{dy}{|\Omega|} = C_\infty, \quad x \in \Gamma,$$

and, when $t \to 0^+$ we obtain: $\quad e^{-\inf_{y\in\Omega}|x-y|^2} = C_0, \quad x \in \Gamma.$

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In conclusion, if Ω has a **finite moment of inertia** $\int_\Omega |y|^2 dy/|\Omega|$, then

$$\left|x - \frac{1}{|\Omega|}\int_\Omega y\ dy\right|^2 = \left(\frac{1}{|\Omega|}\int_\Omega y\ dy\right)^2 - \frac{1}{|\Omega|}\int_\Omega |y|^2 dy - C_\infty, \quad x \in \Gamma,$$

that is Γ **is a sphere,** and $\mathrm{dist}(x, \Omega) = \sqrt{-\log C_0}$, $x \in \Gamma$, that is $\partial\Omega$ **is a sphere.**

Remark

This result seems to suggest that the necessary global information to infer the symmetry of Ω should be looked for at $t = \infty$.

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Uniformly dense domains

Necessary and sufficient condition

The same representation formula implies that Γ is a SIS if and only if for fixed $r > 0$

$$|\Omega \cap B(x,r)| = c^*(r) \quad \text{for every} \quad x \in \Gamma.$$

Uniformly dense domains

We name the domains with such a property **uniformly dense in** Γ and define the **density** of Ω at x by

$$\rho(x,r) = \frac{|\Omega \cap B(x,r)|}{|B(x,r)|}.$$

Short times means small r

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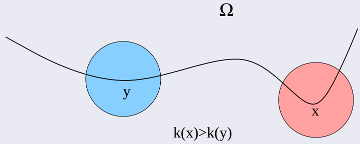
2-D

If $N = 2$ and Γ is uniformly dense, then a straightforward geometrical argument tells us that the curvature of $\partial\Omega$ must be constant.

Proof for $\Gamma = \partial\Omega$



Asymptotic formula for $r \to R^+$ (here $R = \mathrm{dist}(x, \partial\Omega) = |x - y|$)

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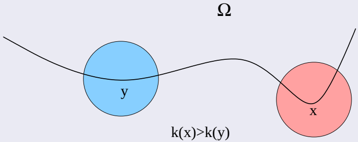

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The case $\Gamma = \partial\Omega$

Something more, if $\Gamma = \partial\Omega$

If Ω is uniformly dense in $\partial\Omega$, then for $r \to 0^+$

$$\rho(x,r) = \frac{1}{2} + c_1\ K_1\ r + \left[c_3' K_1^3 + c_3'' K_1^2 K_2 + c_3''' K_3\right]\ r^3 + O(r^5)$$

at x, where $K_j = \sum_{i_1 < \cdots < i_j} \kappa_{i_1} \cdots \kappa_{i_j}$ is the j-th symmetric invariant of $\partial\Omega$ (e.g. $K_1 = (N-1)H$ and $K_{N-1} = K$).

1^{st} consequence

If $\partial\Omega$ is a SIS, then it **has constant mean curvature** and $c_3'' K_1^2 K_2 + c_3''' K_3$ **is constant on** $\partial\Omega$.

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If $N = 3$, $K_3 = 0$ and hence either $H = 0$ or both H and K are constant. Therefore, $\partial\Omega$ is either a sphere, a spherical cylinder, or a **minimal surface.**

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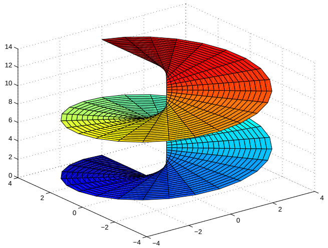
About minimal surfaces

An example

A **right helicoid** is a minimal surface that is the boundary of a uniformly dense domain Ω and hence a SIS.

In fact, $|\Omega \cap B(x,r)| = |(\mathbb{R}^N \setminus \Omega) \cap B(x,r)|$ for every $x \in \partial\Omega$; in other words $\rho = 1/2$. ($\partial\Omega$ is symmetric with respect to any straight line contained in it.)



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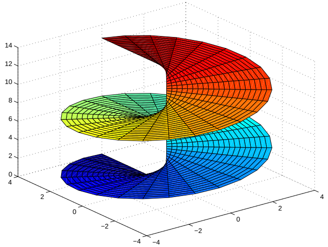

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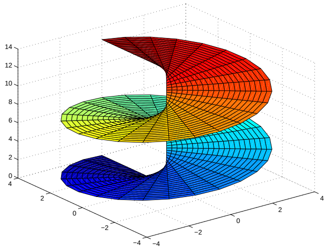

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If Ω is uniformly dense in $\Gamma = \partial\Omega$ and Γ is a complete minimal surface with finite total curvature $\int_\Gamma K \, dS$, then Γ must be a plane. (The total curvature of a helicoid is infinite, because of its periodicity.)

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The helicoid is an example of “isolated” SIS, in the sense that a sufficiently small neighborhood of it does not contain any other SIS (differently from the case of the isoparametric surfaces).

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On isolated SIS

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Are isolated SIS possible for the initial-boundary value problem (the Matzoh Ball Soup setting)?

A possible stategy for Q1

The function u solves (1) iff $W(x,s) = s^{-2} - s^{-2} \int_0^\infty u(x,t)\, e^{-s^2 t} dt$ solves the problem:

$$\Delta W - s^2 W = 0 \;\text{ in }\; \Omega, \qquad W = 1 \;\text{ on }\; \partial\Omega.$$

By the **ansatz** (here, $d(x) = \mathrm{dist}(x, \partial\Omega)$)
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If Γ is a SIS for u, then $W(x,s) = C(s)$ on Γ for every $s > 0$ and hence all the coefficients A_n must be constant on $\partial\Omega$.

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Compute A_1 in terms of geometric quantities of the surface $\partial\Omega$.

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The use of the **ansatz** dispenses us from using the **balance law** (except for the regularity issue).

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What about the case of a general $L^2(\Omega)$ initial data φ (problem (2), solved by Alessandrini in the case of infinitely many SIS)?

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Write the sol. of (2) by the **spectral formula:** $u = \sum_{n \in \mathbb{N}} \hat{\varphi}_n \, u_n(x) \, e^{-\lambda_n t}$.

Suppose that $\hat{\varphi}_n \neq 0$ for every $n \in \mathbb{N}$. If Γ is a SIS for u, then every u_n must be constant on Γ and hence Γ is also a SIS for the solution of (1): $\sum_{n \in \mathbb{N}} \hat{1}_n \, u_n(x) \, e^{-\lambda_n t}$.

This implies that Ω must be a ball (if bounded).

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Suppose that $\hat{\varphi}_n \neq 0$ for infinitely many $n \in \mathbb{N}$ and that Γ is a SIS for u. Does this imply that Ω is a ball?

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On uniformly dense domains

Question 5

Compute the $O(r^5)$ term of the expansion for $\rho(x, r)$ to get further information about the SIS $\Gamma = \partial\Omega$.