Critical points of solutions of elliptic and parabolic PDE's

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Outline

References

- G. Alessandrini-R. Magnanini, *The index of isolated critical points and solutions of elliptic equations in the plane*, Ann. Scuola Norm. Sup. Pisa Cl. Sc. 19 (4) (1992), 567-589.
- G. Alessandrini-R. Magnanini, *Elliptic equations in divergence form, geometric critical points of solutions, and Stekloff eigenfunctions*, SIAM J. Math. Anal. 25 (1994), 1259-1268.
- R. Magnanini-S. Sakaguchi, *On heat conductors with a stationary hot spot*, Annali Matematica Pura ed Applicata 183, no. 1 (2004), 1-23.
- R. Magnanini-S. Sakaguchi, *Polygonal heat conductors with a stationary hot spot*, submitted (2006).

Problem (G. Alessandrini, 1988?)

First eigenfunction and sectional torsion

Let u be the **first Dirichlet eigenfunction for the Laplace operator**:

$$\Delta u + \lambda_1 \, u = 0, \;\; u > 0 \;\; \text{in} \;\; \Omega,$$
$$u = 0 \;\; \text{on} \;\; \partial\Omega,$$

or the solution for the **torsion problem:**

$$\Delta u = -1, \;\; \text{in} \;\; \Omega, \quad u = 0 \;\; \text{on} \;\; \partial\Omega.$$

Ω convex

If Ω is **convex**, every level set $\{x \in \Omega : u(x) > t\}$ is strictly convex and hence u has only one critical point — a maximum point.

Question

If Ω is not convex, how the topology and geometry of Ω determine the number and (maybe) the position of the critical points of u?

More in general

Given an elliptic equation

$$\mathcal{L}u = -f \;\; \text{in} \;\; \Omega,$$

how do the topology and geometry of Ω, the boundary values of u, and/or the coefficients of $\mathcal{L}$ influence the formation of the critical points of u?

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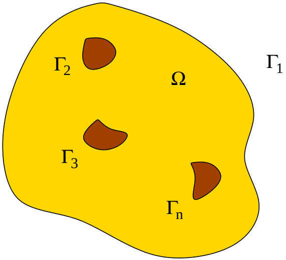
Capacity potential



Capacity potential

$$\Delta u = 0 \quad \text{in} \quad \Omega$$
$$u = a_j \quad \text{on} \quad \Gamma_j, \; j = 1, \dots, n,$$

where the a_j's are constants, not all equal ($n - 1 =$ number of holes).

Index $m(z_k)$ of ∇u at z_k

If z_k is an isolated critical point of u and $\omega = \arg \nabla u$,

$$m(z_k) := \frac{1}{2\pi} \Delta_{+\gamma}(\omega);$$

$+\gamma = \partial B(z_k, \varepsilon)$ counterclockwise.

Perfect counting

The critical points z_k of u are isolated and

$$\sum_{z_k \in \Omega} m(z_k) + \frac{1}{2} \sum_{z_k \in \partial\Omega} m(z_k) = n - 2.$$

N.B.: If Ω is doubly connected ($n = 2$), $\nabla u \neq 0$.

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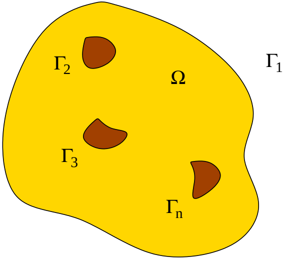

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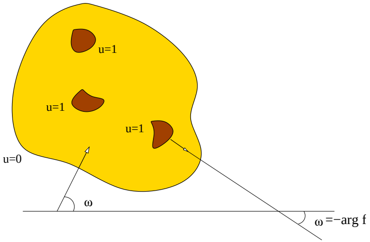
Sketch of the proof



Argument principle

If $\Delta u = 0$, then

$$f = u_x - i u_y = |\nabla u|\, e^{-i\omega}, \quad \omega = \arg \nabla u,$$

is **holomorphic** and hence, if $\nabla u \neq 0$ on $\partial\Omega$,

$$\sharp\,\{\text{zeroes of } f\} - \sharp\,\{\text{poles of } f\} = \frac{1}{2\pi}\,\Delta_{\partial\Omega} \arg f.$$

If $\{z_k \in \partial\Omega\} = \emptyset$

1. f is holomorphic in Ω $\Rightarrow$ no poles in Ω;
2. $m(z_k) = \frac{1}{2\pi}\Delta_{\partial B(z_k,\varepsilon)}\omega$ $\Rightarrow$ $\sharp\,\{\text{zeroes of } f\} = -\sum_{z_k \in \Omega} m(z_k)$;
3. $\frac{1}{2\pi}\,\Delta_{\partial\Omega} \arg f = \frac{1}{2\pi}\,\Delta_{\partial\Omega} \arg(\text{exterior normal}) = 2 - n$ $\Rightarrow$ QED.

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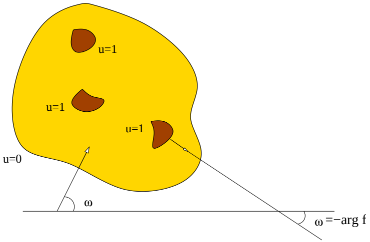

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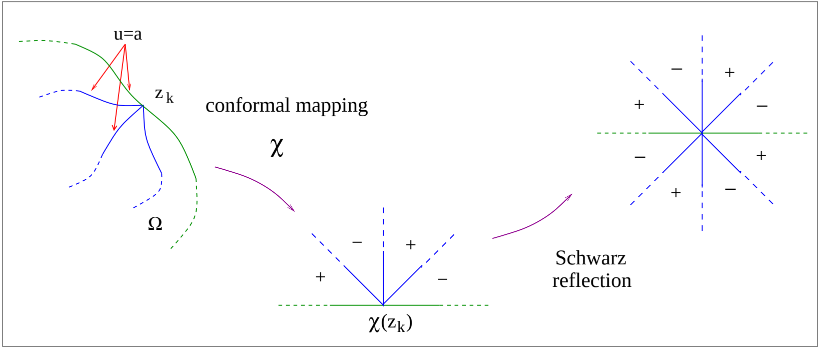
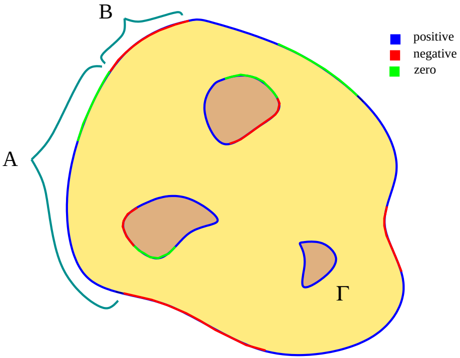
How do we treat the case $z_k \in \partial\Omega$?



Calcolo di $m(z_k)$

$$2m(z_k) = \text{number of components of the set } \{z : u(z) > a\} - 1$$

A more general result (Alessandrini-M., 1992)



α=ν	D=1−3=−2
M=3+1+1=5	M−D=7

1. $\vec{\alpha} : \partial\Omega \to \mathbb{S}^1$ unitary vector field, $\vec{\alpha} \in C^1(\partial\Omega)$;
2. $D := \frac{1}{2\pi}\Delta_{\partial\Omega} \arg(\vec{\alpha}) =$ index of $\vec{\alpha}$ on $\partial\Omega$;
3. $\Delta u = 0$ in Ω;
4. $\nabla u \neq 0$ on $\partial\Omega$.
5. M is the **minimum number** of connected components of the set

$$\mathcal{J}^+ = \{z \in \partial\Omega : \nabla u(z) \cdot \vec{\alpha}(z) \geq 0\}$$

which are a **proper** subset of $\partial\Omega$.

Then

$$\sum_{z_k \in \Omega} m(z_k) \leq M - D.$$

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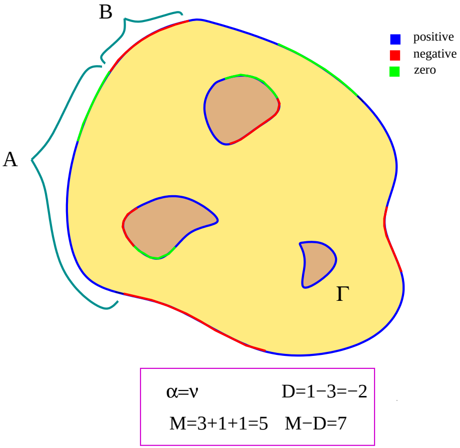

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Sketch of the proof

Key remark

$$\frac{\nabla u}{|\nabla u|} \cdot \alpha = \cos(\omega - \theta)$$

where $\theta = \arg \vec{\alpha}$; hence

$$|\omega - \theta| \leq \frac{\pi}{2} \quad \text{on} \quad \mathcal{J}^+,$$

$$|\omega + \theta| \leq \frac{\pi}{2} \quad \text{on} \quad \mathcal{J}^-.$$

First consequence

On every component Γ of $\partial\Omega$ with $\Gamma \subseteq \mathcal{J}^+$ we have:

$$\left| \frac{1}{2\pi} \Delta_\Gamma \, \omega - \frac{1}{2\pi} \Delta_\Gamma \, \theta \right| \leq \frac{1}{2};$$

hence

$$\frac{1}{2\pi} \Delta_\Gamma \, \omega = \frac{1}{2\pi} \Delta_\Gamma \, \theta.$$

Second consequence

$$\left| \frac{1}{2\pi} \Delta_{A \cup B} \, \omega - \frac{1}{2\pi} \Delta_{A \cup B} \, \theta \right| \leq 1 \Rightarrow -\frac{1}{2\pi} \Delta_{A \cup B} \, \omega \leq -\frac{1}{2\pi} \Delta_{A \cup B} \, \theta + 1.$$

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Sketch of the proof

Conclusion

Summing up these contributions gives

$$\sum_{z_k \in \Omega} m(z_k) = -\frac{1}{2\pi}\Delta_{\partial\Omega}\ \omega \le -\frac{1}{2\pi}\Delta_{\partial\Omega}\ \theta + M = M - D.$$

Extension

These results can be extended to elliptic equations of the form

$$\mathcal{L}\mathbf{u} = \operatorname{div}\{\mathbf{A}(\mathbf{x})\ \nabla\mathbf{u}\} + \mathbf{b}(\mathbf{x}) \cdot \nabla\mathbf{u} = \mathbf{0},$$

$$\text{where } A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{12}(x) & a_{22}(x) \end{bmatrix}, \quad a_{ij} \in \mathrm{Lip}(\Omega),$$

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The two ingredients of the proof

Uniformization principle

For every solution of $\mathcal{L}u = 0$ in Ω, there is a **quasi-conformal mapping** ζ, $\zeta = \xi + i\eta$, such that the function U such that $u = U \circ \zeta$ satisfies

$$\Delta U + P\ U_\xi + Q\ U_\eta \quad \text{in} \quad \zeta(\Omega),$$

with $P, Q \in L^\infty(\zeta(\Omega))$.

Similarity principle

There exist a **holomorphic** function $G(\zeta)$ and a function $s(\zeta)$, Hölder continuous on $\mathbb{C}$, such that

$$2\partial_\zeta U = U_\xi - iU_\eta = e^{s(\zeta)} G(\zeta) \quad \text{in} \quad \zeta(\Omega).$$

($s(\zeta)$ can be chosen real-valued on $\partial\Omega$.)

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Extensions

Further extension

We can even extend to the case

$$\mathrm{div}\{A(x)\ \nabla u\} = 0.$$

with $A \in L^\infty(\Omega)$.

Lack of regularity

In this case, since u is in general only Hölder continuous, we must change the definition of critical point and its multiplicity.

For every non-constant solution $u \in W^{1,2}(\Omega)$, we can write

$$u(z) = h(\chi(z)),\ \ z \in \Omega.$$

where $\chi : \Omega \to B(0,1)$ is a **quasi-conformal** mapping and h is **harmonic.**

"Geometric" critical point

1. $z_0 \in \Omega$ is a critical point of u if $\nabla h(\chi(z_0))$;
2. multiplicity of $z_0 = \frac{1}{2\pi}\Delta_{\partial B(\chi(z_0),\varepsilon)} \arg(\nabla h)$.

Application: inverse problem

$$\mathrm{div}(\sigma \nabla u) = 0 \ \text{ in } \ \Omega \subset \mathbb{R}^2,$$
$$0 < \sigma_0 \le \sigma = \ \text{unknown}$$

The equation is a 1^{st} order PDE for σ. The previous theorems help to establish whether $\nabla u \neq 0$ by examining, for instance, the sign of $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$. This can be done even if σ is **discontinuous**.

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First eigenfunction

Assumptions

$\Delta u + \lambda_1 u = 0$ in Ω, $\quad u = 0$ on $\partial\Omega$

$$\partial\Omega = \bigcup_{j=1}^{n} \Gamma_j, \quad \Gamma_j \in C^{1,\alpha}.$$

Alessandrini-M., 1992

If the critical points of u are isolated, then

$\sharp\{\text{saddle pts}\} - \sharp\{\text{maximum pts}\} = n - 2.$

Extension 1

The theorem holds also for

$$-\Delta u = f(u), \quad u > 0 \quad \text{in} \quad \Omega,$$

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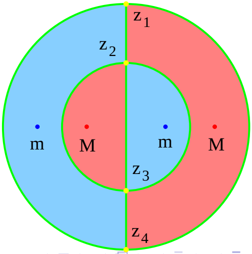
Extension 2: removing assumption $u > 0$

If $f(0) = 0$ and $f(t)/t > 0$ for $t \neq 0$,

$$\sharp\{\text{saddle pts}\} - \sharp\{\text{extremum pts}\} + \sum_{z_k \in \Omega} m(z_k) + \frac{1}{2} \sum_{z_k \in \partial\Omega} m(z_k) = n - 2,$$

where the z_k's are the **nodal** points of u.
(Below $0 - 4 + 2 + \frac{1}{2} \cdot 2 = 1 - 2$.)



Critical points: extension to $\mathbb{R}^N$, $N \geq 3$

Difficulties

1. Lack of complex variables.
2. Critical points are in general **not isolated**.
3. The “number” of critical points does not only depend on the topology (and the values of u on $\partial\Omega$): curvature (and/or something else) should also be taken into account.

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Example 1

The function

$$u(x, y, z) = J_0(\sqrt{x^2 + y^2})\ ch(z),$$

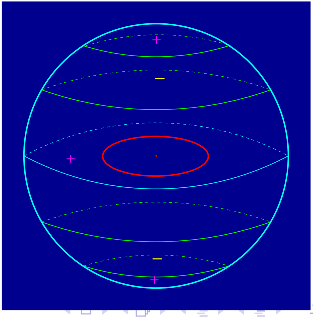
$$J_0(r) \ \text{ Bessel function:}$$

$$J_0'' + \frac{1}{r}\ J_0' + J_0 = 0,$$

is harmonic in $\mathbb{R}^3$ and

$$\{\nabla u = 0\} = \{z = 0\} \cap \bigcup_{n=0}^{\infty} \partial B(0, \kappa_{1,n}),$$

where $0 = \kappa_{1,0} < \kappa_{1,1} < \kappa_{1,2} < \dots$ are the zeroes of the Bessel function $J_1 = -J_0$.



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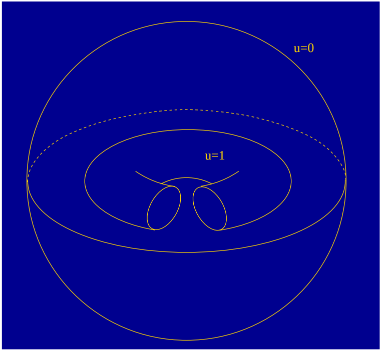

Example 2

1. The function u is harmonic in the region Ω between the sphere and the toroidal surface.
2. The toroidal surface is placed in such a way that is symmetric with respect to 2 coordinate planes, say the xy and xz plane.
3. With this choice, $u_y(x,0,0) = u_z(x,0,0) = 0$ for all $(x,0,0) \in \Omega$.
4. The value $u(0,0,0)$ is bounded by a constant $c < 1$ independent on the position of the two ends of the torus.
5. The values of u between the two ends of the torus are close to 1 if the two ends are close to one another.
6. Hence, $x \mapsto u(x,0,0)$ must have a relative maximum and a relative minimum, i.e. u has 2 **critical points in** Ω.

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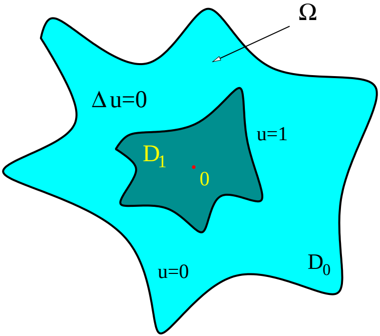

Star-shaped condensers

D_0 and D_1 star-shaped w.r.t. 0, then

1. ∇u never vanishes in Ω and,
2. for every $t \in (0, 1)$,

$$\{x \in \Omega : u(x) = t\} = \partial D_t,$$

where D_t is star-shaped w.r.t. 0.

Extensions (Longinetti, 1985; Francini, 1998)

Instead of Laplace equation one can consider general nonlinear equations

$$F(\nabla^2 u, \nabla u, u, x) = 0$$

with suitable assumptions on F.

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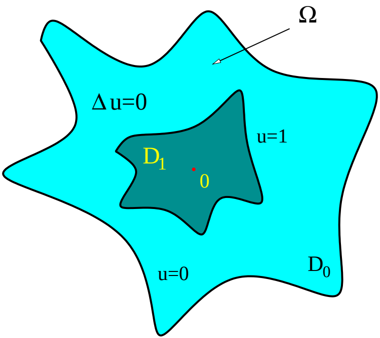

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Hot spots

Problem for heat equation

Consider the problem:

$$\partial_t u = \Delta u \quad \text{in} \quad \Omega \times (0, \infty),$$
$$u = \varphi \quad \text{on} \quad \Omega \times \{0\},$$
$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty).$$

The set of hot spots

$$M(t) := \{x \in \Omega : u(x, t) = \max_{y \in \Omega} u(y, t)\}$$

Spectral formula

$$u(x, t) = \sum_{n=1}^{\infty} \widehat{\varphi}_n u_n(x) e^{-\lambda_n t}$$

Notations

1. u_n and λ_n Dirichlet eigenfunctions and eigenvalues of $-\Delta$ in Ω;
2. $\overline{\text{span}\{u_n\}} = L^2(\Omega)$;
3. $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$;
4. $\widehat{\varphi}_n = (\varphi, u_n)$.

Behavior for large t's

If $\widehat{\varphi}_1 \neq 0$, then as $t \to \infty$

$$e^{\lambda_1 t}\, u(x, t) \to \widehat{\varphi}_1\, u_1(x)$$

uniformly in $\overline{\Omega}$ and with all derivatives up to the second order on compact subsets of Ω.

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Behaviour for large and small times

Brascamp-Lieb (1975)

If Ω is **convex**, $\log u_1$ is strictly concave ; hence u_1 has a **unique** maximum point x_∞ in Ω.

Behaviour of hot spots for large times

Therefore, there exixts a $T > 0$ such that the set $M(t)$ contains exactly one point $x(t)$ for $t > T$ and $x(t) \to x_\infty$ as $t \to \infty$.

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The function $x \mapsto \log u(x, t)$ is concave for each $t > 0$; thus $M(t)$ is made of only one point $\mathbf{x(t)}$ and $x(t) \to x_\infty$ as $t \to \infty$.

Varadahn (1967)

If $\varphi \equiv 1$, then as $t \to 0^+$:

$$-4t \log\{1 - u(x,t)\} \to \text{dist}(x, \partial\Omega)^2.$$

Behaviour of hot spots for small times

$$\text{dist}(x(t), M_d) \to 0 \quad \text{as} \quad t \to 0^+,$$

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$$M_d = \{x \in \Omega : d(x) = \max_{\overline{\Omega}} d\},$$
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If Ω is **strictly convex**, then $M_d = \{x_0\}$ for some $x_0 \in \Omega$.

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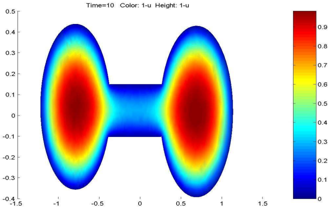
Pictures

Numerical proof of Varadahn's result



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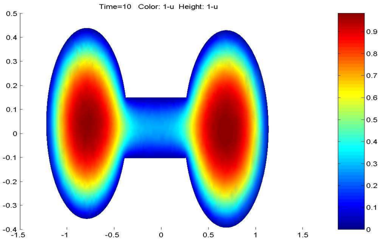

A possible bifurcation

The next picture shows that $M(t)$ may initially contain **two points** and, later, collapse to **one single** point.

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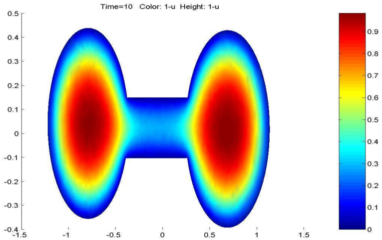
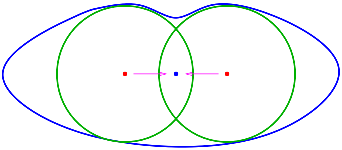




A possible bifurcation

The next picture shows that $M(t)$ may initially contain **two points** and, later, collapse to **one single** point.

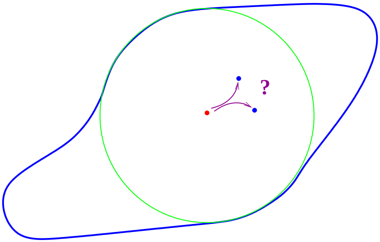
In fact

1. Initially, the level curves of $u(x, t)$ look like those of $d(x)$.
2. If our domain is a slight perturbation of a convex one, we expect that $u_1(x)$ has only one maximum point.

Problems

Problem 1

If M_d is made of one single point is it so for $M(t)$?



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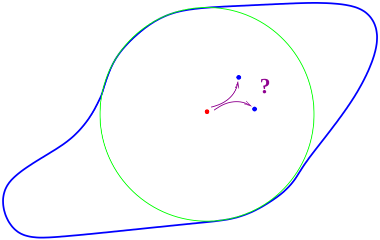
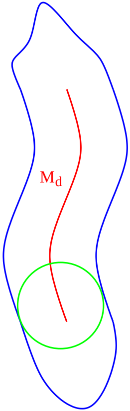
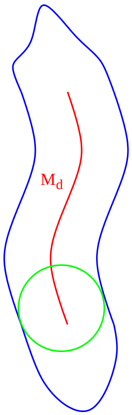




Problem 2

If $M(t) = \{x(t)\}$ for every $t > 0$, is M_d **contractible**?

Problems



Problem 3: global version

Let M_d be contractible; find extra assumptions on d such that $M(t)$ is made of only one point $x(t)$ for every $t > 0$.

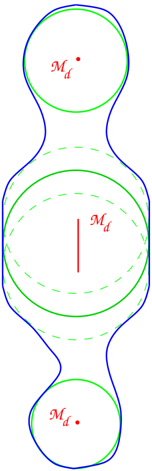
Problem 3: local version

Let M_d be contractible; find extra assumptions on d such that there exists $T > 0$ for which $M(t)$ is made of only one point $x(t)$ for every $t \in (0, T]$.

Hint

A non-smooth version of Dini's implicit functions theorem could be useful.

More problems



The sets of local maximum points

Let $\mathcal{M}(t)$ and $\mathcal{M}_d$ be the sets of local maximum points of $u(x,t)$ and $d(x)$, respectively.

Problem 4: global version

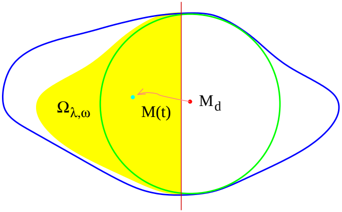
If $\mathcal{M}_d$ has n contractible components is it true that the number of points of $\mathcal{M}(t)$ does not exceeds n?

In particular, the number of local maximum points of u_1 does not exceeds that of contractible components of $\mathcal{M}_d$.

Problem 4: local version

If $\mathcal{M}_d$ has n contractible connected components find extra assumptions on d such that there exists $T > 0$ for which $\mathcal{M}(t)$ is made of n points for every $t \in (0, T]$.

The location of hot spots



Problem 5

Another interesting problem is that of locating the hot spot (when it is unique).

Notations

$$\omega \in \mathbb{S}^{N-1}, \ x^{\lambda,\omega} = x - 2(x \cdot \omega - \lambda)\omega,$$
$$\pi_\lambda = \{x \in \mathbb{R}^N : x \cdot \omega = \lambda\},$$
$$v^\lambda(x,t) = u(x,t) - u(x^\lambda, t).$$

Conditions on v^λ

$$v^\lambda_t = \Delta v^\lambda \ \text{ in } \ \Omega_{\lambda,\omega} \times (0,\infty),$$
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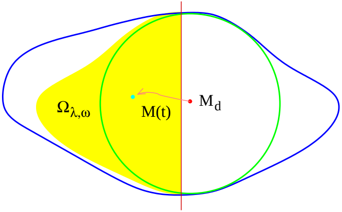

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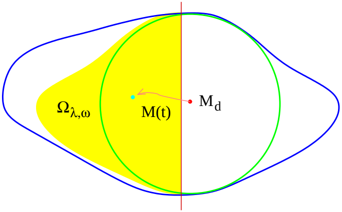

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The function $\lambda(\omega)$

Conclusion

Therefore, as long as $\Omega_{\lambda,\omega}$ stays in Ω, $\pi_{\lambda,\omega}$ cannot contain critical points of u and hence,

$$C(t) := \{\nabla u(\cdot, t) = 0\} \subseteq \bigcap_{\substack{\Omega_{\lambda,\omega} \subset \Omega \\ \omega \in \mathbb{S}^{N-1}}} H^{\lambda,\omega},$$

where $H^{\lambda,\omega} = \{x \in \mathbb{R}^N : x \cdot \omega < \lambda\}$.

In particular, if Ω is convex

1. If Ω is symmetric w.r.t. a hyperplane, then $C(t)$ is contained in that hyperplane.
2. If Ω has N **independent** hyperplanes of symmetry through a point 0,

$$x(t) = 0 \quad \text{for any} \quad t > 0.$$

Ω bounded and convex

Let $M_d = \{0\}$ and, for any $\omega \in \mathbb{S}^{N-1}$, let us define:

$$\lambda(\omega) = \inf\{\lambda : \Omega_{\mu,\omega} \subset \Omega, \mu > \lambda\}.$$

We observed that

$$x(t) \in \bigcap_{\omega \in \mathbb{S}^{N-1}} \{x \in \mathbb{R}^N : x \cdot \omega < \lambda(\omega)\}.$$

Problem 6

Compute or estimate $\lambda(\omega)$.

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Stationary hot spot: sufficient conditions

Klamkin (1994)

If Ω is centrally symmetric w.r.t. a point $0 \in \Omega$, then for every $t > 0$ $u(-x, t) = u(x, t)$ and hence

$$-\nabla u(-x, t) = \nabla u(x, t),$$

that implies:

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Ω convex and centrally symmetric

$C(t) = M(t) = \{0\}$ for every $t > 0$.

Chamberland-Siegel (1997)

Let Ω be convex and G-invariant, where is an **essential** subgroup of $O(N)$. Then $C(t) = M(t) = \{0\}$ for every $t > 0$.

Definition

$G \subset O(N)$ **essential** if, for every $x \in \Omega$, $x \neq 0$, there is a $g \in G$ such that $gx \neq x$.

Proof

$$g(\Omega) = \Omega \;\Rightarrow\; u(gx, t) = u(x, t)$$

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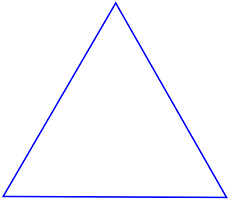
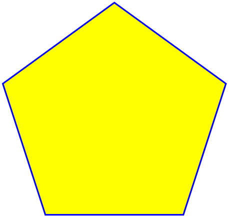
Examples



Centrally symmetric, $G = \{1, -1\}$



Non centrally symmetric
$G = \{e^{2\pi ik/3}\}_{k=0,1,2}$

Non centrally symmetric
$G = \{e^{2\pi ik/5}\}_{k=0,1,2,3,4}$

Stationary hot spot: a general necessary condition

Klamkin (1994)

Let Ω be convex and let $x(t) = 0$ for every $t > 0$. Does Ω have any kind of symmetry?

R.M. - Sakaguchi (2004)

1. Ω bounded (not necessarily convex), $0 \in \Omega$;
2. $\partial\Omega$ Lipschitz continuous;
3. R=d(0);
4. $x(t) = 0$ for every $t > 0$.

Then

$$\int_{\partial\Omega \cap \partial B(0,R)} x \, dS_x = 0.$$

If $\mathcal{H}^{N-1}(\partial\Omega \cap \partial B(0,R)) = 0$

1. $\partial\Omega$ piecewise of class C^2;
2. $\partial\Omega \cap \partial B(0,R) = \bigcup_{k=1}^{K} M_k^{m_k}$, where $M_k^{m_k}$ are pairwise disjoint m_k-submanifolds with $0 \le m_k \le N-2$ and $\partial M_k^{m_k} = \emptyset$;
3. $m = \max\{m_k : \mathcal{H}^{m_k}(M_k^{m_k}) > 0\}$;
4. $\mathcal{K}_m(x) = \left\{ \prod_{j=m+1}^{N-1} \left[\frac{1}{R} - \kappa_j(x)\right] \right\}^{-\frac{1}{2}}$.

Then

$$\sum_{m_k = m} \int_{M_k^{\mu}} x \, \mathcal{K}_m(x) \, d\mathcal{H}^m = 0.$$

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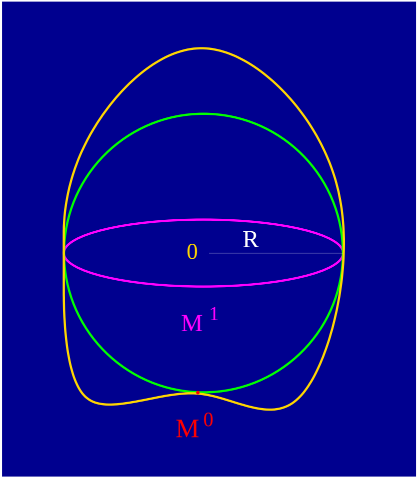
Corollary



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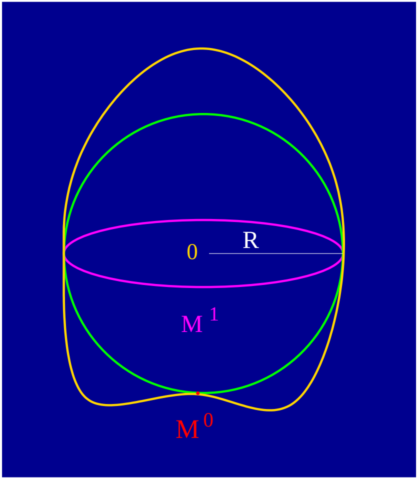

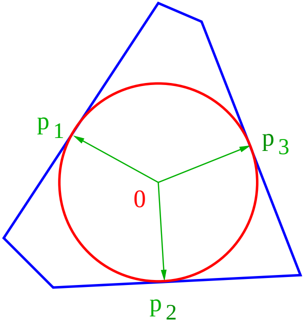

Ω convex polyhedron

$\partial\Omega \cap \partial B(0, R) = \{p_i\}_{i=1,\dots,m}$ and hence

$$\sum_{i=1}^{m} p_i = 0.$$

Proof: first part

Mean value property (for stationary points)

If $v_t = \Delta v$ in $D \times (0, \infty)$ and $v(0, t) = c$ for every $t > 0$, then

$$\frac{1}{|B(0,r)|} \int_{B(0,r)} v(x,t)\, dx = c = v(0,t),$$

for every $r < \operatorname{dist}(0, \partial D)$ and $t > 0$.

Corollary

Since every $\partial_{x_i} v$ also satisfies the heat equation, if $\nabla v(p,t) = 0$ for every $t > 0$, we have:

$$\int_{B(0,r)} x\, v(x,t)\, dx = 0,$$

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Boundary layer

Let us choose

1. $v(x,t) = u(x,t)$ (i.e. $u = 1$ on $\Omega \times \{0\}$; $u = 0$ on $\partial\Omega \times (0, \infty)$);
2. $r = R = d(0)$, so that $B(p, R)$ touches $\partial\Omega$;

and let us use the "boundary layer" produced by $u(x,t)$ as $t \to 0^+$.

Barriers for small t; $\varepsilon > 0$ is a parameter

The Varadhan's formula

$$\lim_{t \to 0^+} (-4t) \log\{1 - u(x,t)\} = d(x)^2,$$

suggests the construction of two barriers:

$$F_-^\varepsilon\left(\frac{d(x)}{\sqrt{t}}\right) \le u(x,t) \le F_+^\varepsilon\left(\frac{d(x)}{\sqrt{t}}\right)$$

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Proof: second part

Fubini's theorem implies

If F is BV and $d\nu(dx) = \varphi(x)\ dx$ with $\varphi \in C_0^0(\mathbb{R}^N)$, then

$$\int_{B(p,R)} F\left(\frac{d(x)}{\sqrt{t}}\right)\ \nu(dx) =$$

$$\int_0^{2R/\sqrt{t}} F'(\sigma)\ \nu(\{x \in B(p,R) : d(x) > \sigma\sqrt{t}\})d\sigma$$

Crucial lemma

$$\lim_{s\to 0^+} s^{-\frac{N+1-\mu}{2}} \nu(\{x \in B(p,R) : d(x) < s\}) =$$

$$C(N,\mu) \sum_{m_k=m}^{K_m} \int_{M_k^{m_k}} \varphi(x)\mathcal{K}_m(x) d\mathcal{H}^m := A^{m,N}_{\Omega,R}$$

Therefore

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Choose $\varphi(x) = (x_i - p_i)^\pm$, $F = F^\varepsilon_\pm$, and use the fact that, as $\varepsilon \to 0^+$,

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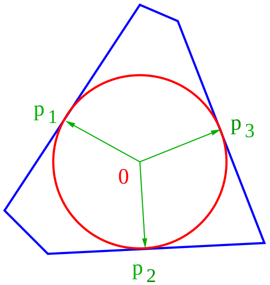
Symmetry: triangles and quadrangles



Polygons

In this case, our theorem implies that

$$\sum_{i=1}^{n} p_i = 0.$$

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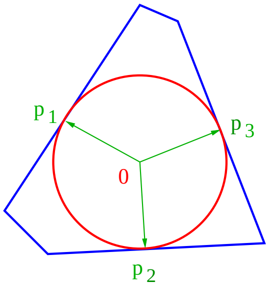

Symmetry

Let $x(t) = 0$ for every $t > 0$. Then

1. if Ω is a triangle $\Rightarrow$ Ω is equilateral;
2. if Ω is a quadrangle $\Rightarrow$ Ω is a parallelogram.

In particular, a non convex quadrangle does not admit a stationary hot spot.

The most diffucult case to treat concerns the picture here below.

Polygons

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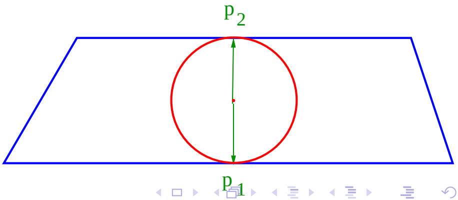
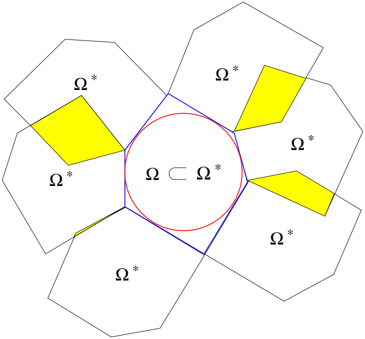


Polygons: another condition



Ω polygon

By applying the **Schwarz reflection principle** w.r.t. **each side**, we can extend u to a solution u^* of $u^*_t = \Delta u^*$, in a larger domain (the **white domain** Ω^*)

Since still $u^*(0,t) = 0$ for every $t > 0$, we again find:

$$\int_{B(0,r)} x \; u^*(x,t) \; dx = 0;$$

this time for $r \le R^*$, where

R^* is the distance of 0 from the **closest vertices** of Ω.

Choose $r = R^*$; then

$$\int_{B(0,R^*)} x \; u^*(x,t) \; dx = 0 \quad \text{per ogni} \quad t > 0,$$

but it is not convenient to work with this integral.

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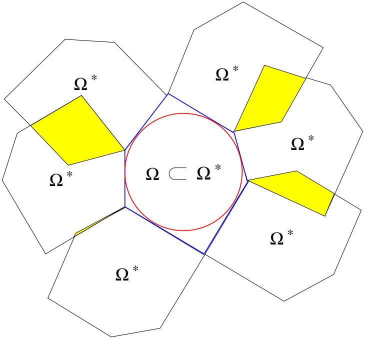

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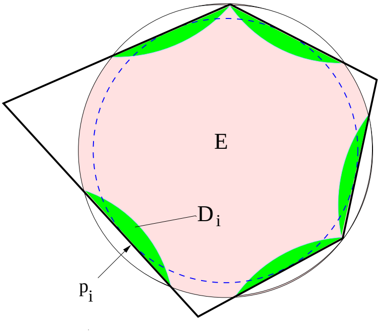


It is instead convenient to fold back into Ω the parts of $B(0, R^*)$ that are outside Ω.

It turns out that

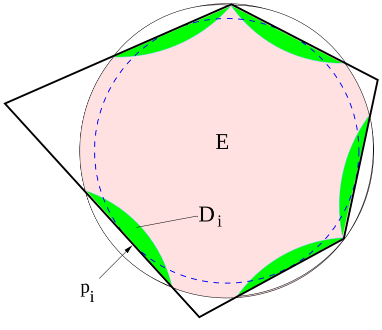
$$\int_{B(0,r)} x\, u^*(x,t)\, dx = 0$$

if and only if

$$\int_E x\, u(x,t)\, dx + \sum_i \int_{D_i} (x - x_i^*)\, u(x,t)\, dx = 0,$$

where E is the **pink** set, D_i the **green** ones and x_i^* is the reflection of x w.r.t. the side of Ω contained in D_i.

In this way we can again work with u in place of u^*.



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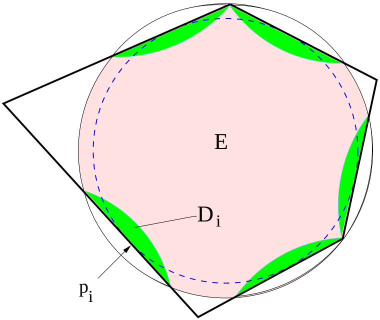

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Symmetry: pentagons ed exagons

If all sides of Ω touch $\partial B(0, R^*)$

By an asymptotic analysis similar to (but more complicated than) that already seen, by sending t to 0, we show that

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Symmetry

If Ω is as specified, then

1. if Ω is a pentagon, Ω is regular;
2. if Ω is an exagon, Ω is invariant w.r.t. rotations by the angles $\frac{\pi}{3}$, $\frac{2\pi}{3}$ e π.