On the complex eikonal equation

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Outline

1. Formal resolution
   - Decoupling
   - Variational formulations
   - Non divergence equations
2. Analysis of solutions
   - Examples
   - Critical points of the real part
3. Existence results
   - A boundary value problem
   - Find u first and then v
   - Approximating process and convergence

References

- R. Magnanini-G. Talenti, *On complex-valued solutions to a 2D eikonal equation. Part one: qualitative properties*, in Nonlinear Partial Differential Equations, G.-Q. Chen and E. DiBenedetto eds., Contemporary Mathematics, AMS 1999, Providence, RI. (USA).
- R. Magnanini-G. Talenti, *On complex-valued solutions to a 2D eikonal equation. Part two: existence theorems*, SIAM J. Math. Anal. 34 (2003), 805-835.
- R. Magnanini-G. Talenti, *On complex-valued solutions to a 2D eikonal equation. Part three: analysis of a Bäcklund transformation*, Applicable Analysis 85 (2006), 249-276.

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$$w_x^2 + w_y^2 + n(x,y)^2 = 0$$

or

$$w_x^2 + w_y^2 - n(x,y)^2 = 0$$

Index of refraction

$n(x,y)$ is the **index of refraction** and is supposed to be positive and bounded away from zero:

$$n(x,y) \geq n_0 > 0.$$

We will study the first version to stress the fact that we are interested in **complex-valued solutions**

$$w = u + iv.$$

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Geometrical optics: classical asymptotics

Maxwell equations

$$\frac{\partial}{\partial t}\left(\epsilon \vec{E}\right) = \mathrm{curl}\left(\vec{H}\right), \quad \frac{\partial}{\partial t}\left(\mu \vec{H}\right) = -\mathrm{curl}\left(\vec{E}\right),$$
$$\mathrm{div}\left(\epsilon \vec{E}\right) = 0, \qquad \mathrm{div}\left(\mu \vec{H}\right) = 0,$$

Eliminate $\vec{H}$, and obtain an equation for $\vec{E}$ only.

If $\ln(\varepsilon/\varepsilon_0)$ is small

Look for solutions harmonic in time, $\vec{E} = \vec{E}(x, y, z)\ e^{i\omega t}$. The z-component E_z of $\vec{E}(x, y, z)$ is a solution of the Helmholtz equation.

Helmholtz equation

$$\Delta u + k^2\ n(x, y, z)^2\ u = 0,$$

where n and k are the **refraction coefficient** and **wave number**:

$$k^2 = \varepsilon_0\ \mu_0\ \omega^2, \quad \varepsilon\ \mu = n(x, y, z)^2\ \varepsilon_0\ \mu_0.$$

Here,

$$\varepsilon_0 = \textbf{backgrnd dielectric constant},$$
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Further approximation

If k is large

Look for $u(x,y,z) = e^{ik\ \phi(x,y,z)} \sum_{n=0}^{+\infty} A_n(x,y,z)(ik)^{-n}$ and obtain:

$$\phi_x^2 + \phi_y^2 + \phi_z^2 = n(x,y,z)^2,$$ **eikonal eq.**

$$2\nabla\phi \cdot \nabla A_n + \left[\mu \mathrm{div}\left(\frac{\nabla\phi}{\mu}\right)\right] A_n + 2(\nabla \log n \cdot A_n)\nabla\phi = 0, \quad n = 0, 1, \cdots .$$ **transport eqs.**

The **eikonal** ϕ describes the propagation of light in terms of **rays.**

In $\mathbb{R}^2$

$$w_x^2 + w_y^2 = n(x,y)^2$$

Evanescent Wave Tracking (EWT)

If ϕ is allowed to take complex values, we have EWT (Felsen), a theory which **extends** geometrical optics **beyond caustics.**

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Ludwig, Kravtsov

$$\vec{E} = e^{iku} \cdot \{Ai(-k^{2/3}v)\vec{U} + ik^{-2/3}Ai'(-k^{2/3}v)\vec{V} + \text{remainder}\},$$

where Ai is an **Airy function** satisfying $Ai''(t) - tAi(t) = 0.$

We obtain:

$$|\nabla u|^2 + v|\nabla v|^2 = n^2, \qquad \nabla u \cdot \nabla v = 0,$$

$$+\text{transport eqns. for} \qquad \vec{U} \ \& \ \vec{V}$$

Transform by

$$\phi = u + \tfrac{2}{3}\, v^{3/2}, \quad \vec{A} = v^{-1/4}(\vec{U} + \sqrt{v}\ \vec{V}),$$

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ϕ, ψ complex-valued if $v < 0$.

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Decoupling: first step

A nonlinear system

Looking for complex-valued solutions $w = u + iv$ of

$$w_x^2 + w_y^2 + n(x, y)^2 = 0$$

gives two first-order equations:

$$\begin{aligned} |\nabla u|^2 - |\nabla v|^2 + n(x, y)^2 &= 0, \\ \nabla u \cdot \nabla v &= 0. \end{aligned}$$

Notice that $|\nabla v| \geq n$.

Ellipticity

By a classical analysis this system is **degenerate elliptic** because

$$-\left(u_x v_y - u_y v_x\right)^2 = |\nabla u|^2 \left(n^2 + |\nabla u|^2\right) \leq 0;$$

the system degenerates only at **critical points** of u.

Orthogonality

$\nabla u \cdot \nabla v = 0$ implies that

$$\frac{\nabla v}{|\nabla v|} = \pm \frac{(\nabla u)^\perp}{|\nabla u|}.$$

Bäcklund transformations

Hence, $|\nabla u|^2 - |\nabla v|^2 + n(x, y)^2 = 0$ gives:

$$\nabla v = \pm\sqrt{n^2 + |\nabla u|^2}\, \frac{(\nabla u)^\perp}{|\nabla u|}$$

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Use of Bäcklund transformations

If we know u then we can recover v and viceversa.

NOTE

If $n \equiv 0$, then the Bäcklund transformations read:

$$v_x = \mp u_y, \quad v_y = \pm v_x,$$

the **Cauchy-Riemann** (or anti Cauchy-Riemann) equations.

Equations in divergence form

From the Bäcklund transformations, since $\mathrm{curl}(\nabla v) = \mathrm{curl}(\nabla u) = 0$, we obtain two second order differential equations in **divergence form**:

$$\mathrm{div}\left\{\sqrt{n^2 + |\nabla u|^2}\, \frac{\nabla u}{|\nabla u|}\right\} = 0,$$

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Functionals for u and v

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are (formally) **Euler equations** of suitable functionals.

Functional for u

$$J(u)=\int_\Omega j\left(\frac{|\nabla u|}{n}\right)\,n^2\,dxdy$$

where

$$j'(\rho)=\sqrt{1+\rho^2}.$$

Functional for v

$$K(v)=\int_\Omega k\left(\frac{|\nabla v|}{n}\right)\,n^2\,dxdy$$

where

$$k(r)=\max_{\rho\ge 0}\{r\rho-j(\rho)\},$$

is the **Young conjugate** of j.

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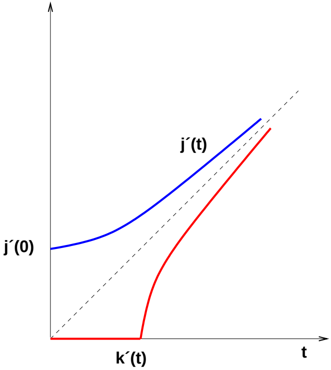
Energy densities



j´ versus $k' = (j')^{-1}$

Two quasilinear equations

Performing the divergence

From the two divergence equations, we obtain:

$$\left(|\nabla u|^4+n^2 u_y^2\right) u_{xx}-2n^2 u_x u_y u_{xy}+\left(|\nabla u|^4+n^2 u_x^2\right) u_{yy}+n|\nabla u|^2\nabla n\cdot\nabla u=0,$$

$$\left(|\nabla v|^4-n^2 v_y^2\right) v_{xx}+2n^2 v_x v_y v_{xy}+\left(|\nabla v|^4-n^2 v_x^2\right) v_{yy}-n|\nabla v|^2\nabla n\cdot\nabla v=0.$$

Structure

These two quasilinear equations have the structure:

$$a\ u_{xx}+2b\ u_{xy}+c\ u_{yy}+f=0;$$

here a, b, c, and f depend on x, u_x, u_y.
If λ and Λ are the eigenvalues of

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

we have that

Type

$$\frac{\lambda_u}{\Lambda_u}=\frac{|\nabla u|^2}{n^2+|\nabla u|^2},$$

$$\frac{\lambda_v}{\Lambda_v}=\frac{|\nabla v|^2-n^2}{|\nabla v|^2},$$

hence the 1^{st} equation is **elliptic** and degenerates when $\nabla u=0$, while the 2^{nd} is **elliptic** for $|\nabla v|>n$ and **hyperbolic** for $|\nabla v|<n$.

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Structure

These two quasilinear equations have the structure:

$$a\;u_{xx}+2b\;u_{xy}+c\;u_{yy}+f=0;$$

here a, b, c, and f depend on x, u_x, u_y.
If λ and Λ are the eigenvalues of

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

we have that

Type

$$\frac{\lambda_u}{\Lambda_u}=\frac{|\nabla u|^2}{n^2+|\nabla u|^2},$$

$$\frac{\lambda_v}{\Lambda_v}=\frac{|\nabla v|^2-n^2}{|\nabla v|^2},$$

hence the 1^{st} equation is **elliptic** and degenerates when $\nabla u=0$, while the 2^{nd} is **elliptic** for $|\nabla v|>n$ and **hyperbolic** for $|\nabla v|<n$.

Two quasilinear equations

Performing the divergence

From the two divergence equations, we obtain:

$$\left(|\nabla u|^4 + n^2 u_y^2\right) u_{xx} - 2n^2 u_x u_y u_{xy} + \left(|\nabla u|^4 + n^2 u_x^2\right) u_{yy} + n|\nabla u|^2 \nabla n \cdot \nabla u = 0,$$

$$\left(|\nabla v|^4 - n^2 v_y^2\right) v_{xx} + 2n^2 v_x v_y v_{xy} + \left(|\nabla v|^4 - n^2 v_x^2\right) v_{yy} - n|\nabla v|^2 \nabla n \cdot \nabla v = 0.$$

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Example 1: method of characteristics

Note

So far, all computations were formal; by some example, we shall see that some computations **are not legal.**

We will consider examples for $n \equiv 1$

$$w_x^2 + w_y^2 + 1 = 0,$$

$$\left(|\nabla u|^4 + u_y^2\right) u_{xx} - 2 u_x u_y u_{xy} + \left(|\nabla u|^4 + u_x^2\right) u_{yy} = 0$$

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Characteristics

Let $s \mapsto (x_0(s), y_0(s), w_0(s))$ be a parametrization of a given function w_0 on a planar curve Γ (s=arclength of Γ).

The eikonal equation tells us that w grows linearly along the trajectories of its gradient. Hence, we obtain a parametrization for a w which takes the values w_0 on Γ :

$$\begin{aligned} x &= x_0(s) + t\ p_0(s) \\ y &= y_0(s) + t\ q_0(s) \\ w &= w_0(s) + t \end{aligned} \quad \text{where} \quad \begin{aligned} & p_0(s)^2 + q_0(s)^2 = 1 \quad \text{and} \\ & \\ & w_0'(s) = p_0(s)x_0'(s) + q_0(s)y_0'(s). \end{aligned}$$

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See Chapman, Lawry, Ockendon and Tews

Complexify

One can obtain **complex-valued** solutions by allowing the parameters s and t to take complex values.

Complex distance

When $w_0 \equiv 0$, then we can write:

$$\begin{aligned} x &= x_0(s) - t\ y_0'(s), \\ y &= y_0(s) + t\ x_0'(s), \\ w &= t, \end{aligned} \qquad (1)$$

which is a parametrization of the distance w from a given planar curve Γ (complex if t and s are complex).

For fixed s, the three equations parametrize a **ray** issuing from a point on Γ.

Complex distance from a point

If $w = u + iv$ is the (complex) distance from the point $(0, i)$,

$$w(x, y) = i\ \sqrt{x^2 + (y - i)^2},$$

we find solutions of our non-divergence equations:

$$u = \sqrt{\frac{1 - x^2 - y^2}{2} + \sqrt{\left(\frac{1 - x^2 - y^2}{2}\right)^2 + y^2}},$$

$$v(x, y) = y/u(x, y).$$

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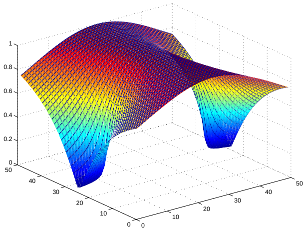
Example 1: properties of u



Properties of u

1. The line $x = 0$ is a line of **critical points** for u.
2. No strict maximum principle holds.
3. J is differentiable at u.
4. u **solves** the non-divergence equation but is **not** a solution the divergence equation in the sense of distributions.

Fréchet derivative of J at u

$$\int \sqrt{1+|\nabla u|^2}\, \frac{\nabla u}{|\nabla u|} \cdot \nabla \phi \, dxdy = 2 \int \phi(0,y)\, dy - 2 \int_{1<|x|<\infty} \phi(x,0)\, \frac{|x|}{\sqrt{x^2-1}}\, dx,$$

for every $\phi \in C_0^\infty$. Hence, u is not a solution of the divergence equation **at the critical points of** u (where it is smooth).

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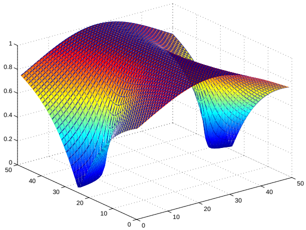

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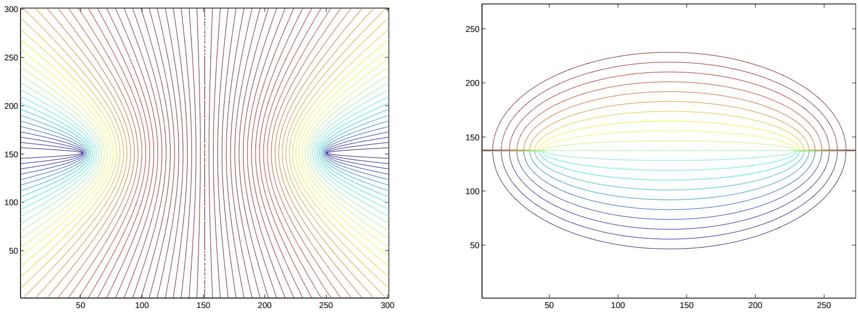
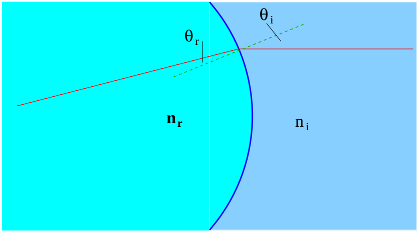
Pictures: contour plots for u and v



Example 1: non-homogeneous media



Non-homogeneous media

This idea may also be used to construct solutions when the refraction coefficient n is **piecewise constant**.

Snell's law

In order to do this, we must patch each (complex) **incident ray** to the interface, to the corresponding **refracted ray**, by using the well-known Snell's law that is nothing else then **Euler's equation** for the **minimum path problem**:

$$n_i \sin\theta_i = n_r \sin\theta_r.$$

Here θ_i and θ_r are the (complex) angles formed with the normal to the interface by the respective rays.

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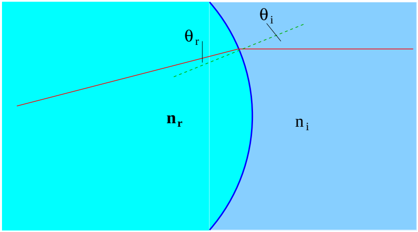

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Example 2: solutions by Legendre transformation

Legendre transformation

A change of variables from the (x, y) plane to the (p, q) plane, where p and q are the components of the gradient of u (hence the transformation depends on each single u considered):

$$p = u_x(x, y) \qquad x = U_p(p, q)$$
$$q = u_y(x, y) \qquad y = U_q(p, q)$$
$$U = p\ x + q\ y - u \qquad u = x\ p + y\ q - U$$

If the *Hessian* $u_{xx} u_{yy} - u_{xy}^2 \neq 0$, then the application $(x, y) \mapsto (p, q)$ is a local diffeomorphism.

Parametrization for u

Once U is computed, the second set of equations gives a parametrization of u.

Quasilinear equations

This transformation changes **quasilinear equations** into **linear ones**.

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Transforming into a **linear PDE**

$$\left(|\nabla u|^4 + u_y^2\right) u_{xx} - 2 u_x u_y u_{xy} + \left(|\nabla u|^4 + u_x^2\right) u_{yy} = 0$$

$$\Downarrow$$

$$\{(p^2+q^2)^2+p^2\}\ U_{pp} + 2pq\ U_{pq} + \{(p^2+q^2)^2+q^2\}\ U_{qq} = 0.$$

Lucky change of variables

By the further transformation

$$p = \sinh\lambda\cos\mu, \quad q = \sinh\lambda\sin\mu$$

we see that

$$U^*(\lambda,\mu) = U(p,q)$$

is **harmonic.**

Plotting u

Then we can easily plot u by the parametrization:

$$x = \frac{\cos\mu}{\cosh\lambda}\ U^*_\lambda - \frac{\sin\mu}{\sinh\lambda}\ U^*_\mu,$$

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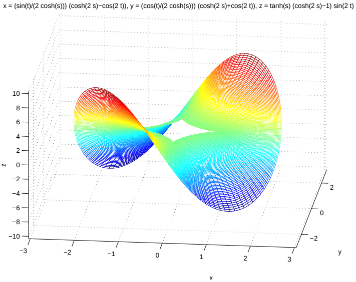
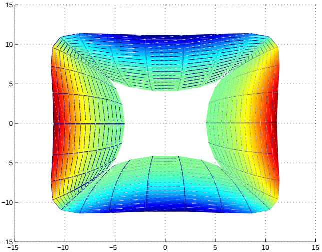
Pictures: real part, second harmonic

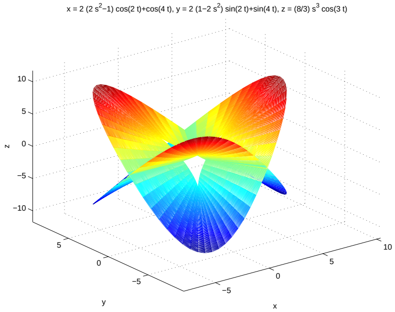
It is natural to expect that u is identically constat in the "hole".

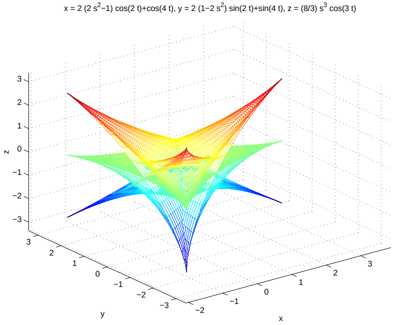




Pictures: imaginary part; third harmonic.

The pictures display v in the elliptic ($|\nabla v| > 1$) and hyperbolic ($|\nabla v| < 1$) zone.





Example 3: solutions by a rational parametrization

Remark

$$w_x^2 + w_y^2 + 1 = 0$$
$$\Leftrightarrow$$
$$i\,(w_x, w_y) \in \text{ unit circle in } \mathbb{C}^2.$$

Rational parametrization

Hence, we parametrize:

$$w_x = i\,\frac{2\omega}{1+\omega^2}, \quad w_y = i\,\frac{1-\omega^2}{1+\omega^2};$$
$$\Downarrow \; w_{xy} = w_{yx}$$

A semilinear equation

$$2\omega\;\omega_x + (1-\omega^2)\;\omega_y = 0,$$

or, in complex notation,

$$(1+i\omega)\;\omega_{\overline{z}} - (1-i\omega)\;\omega_z = 0$$

Change of variables

If $|\nabla u| \neq 0$, then the **jacobian** $|\omega_z|^2 - |\omega_{\overline{z}}|^2$ is different from zero.

We can locally invert $\omega = \omega(z)$ by a function $z = z(\omega)$.

In other words, we regard ω as the independent variable.

Notice that
$$|\nabla u| = 0 \Leftrightarrow Im(\omega) = 0.$$

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Linearization

Linear equation

We obtain:

$$\partial_{\overline{\omega}} \left\{ (1+i\omega)^2 \, z(\omega) + (1-i\omega)^2 \, \overline{z(\omega)} \right\} = 0.$$

That is, we can write

$$(1+i\omega)^2 \, z(\omega) + (1-i\omega)^2 \, \overline{z(\omega)} = 2\, f(\omega), \text{ or}$$

$$(1-\omega^2)\, x - 2\omega \, y = f(\omega),$$

with $f(\omega)$ **holomorphic**.

Let $\omega = \xi + i\eta$ and recall that $|\nabla u| = 0$ if and only if $\eta = Im(\omega) = 0$.
If $\eta = 0$, then

$$(1-\xi^2)\, x - 2\xi \, y = \mathrm{Re}[f(\xi)] \quad \text{and} \quad \mathrm{Im}[f(\xi)] = 0.$$

For fixed ξ, the first is the equation of a straight line.

Linearization

Linear equation

We obtain:

$$\partial_{\overline{\omega}} \left\{ (1+i\omega)^2 \ z(\omega) + (1-i\omega)^2 \ \overline{z(\omega)} \right\} = 0.$$

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For fixed ξ, the first is the equation of a straight line.

Geometric information

Hence,

(i) if equation $\mathrm{I}m[f(\xi)] = 0$ has no roots, ∇u does not vanish;

(ii) if equation $\mathrm{I}m[f(\xi)] = 0$ has roots, ∇u does vanish on **straight lines or segments**;

(iii) if $\overline{f(\omega)} = f(\overline{\omega})$, then $\mathrm{I}m[f(\xi)]$ vanishes identically; this means that ∇u vanishes on a **pencil of segments** and, hence, on the **region** Ω_0 **swept out** by these segments; the boundary of Ω_0 is the **envelope** of the segments.

For instance, if $f(\omega) = 1 + \omega^2$, we obtain:

$$w(z) = -i\sqrt{z\overline{z} - 1} - \log(\sqrt{z\overline{z} - 1} + i) + \log \overline{z}.$$

Conjecture

The above remarks may lead the path to a proof of the following **conjecture:**

the gradient of real part u of the eikonal w
either does not vanish or it vanishes on a continuum.

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(i) if equation $Im[f(\xi)] = 0$ has no roots, ∇u does not vanish;

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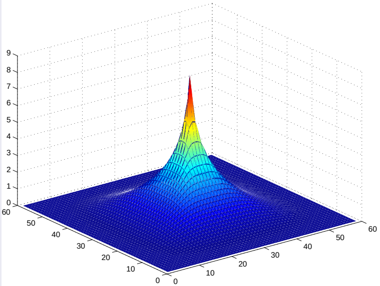
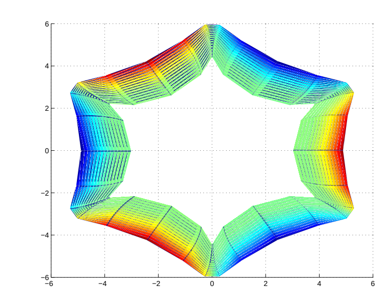
Pictures

Real part: radial



Real part: third harmonic



Non-isolated critical points: partial results

THEOREM 1

Assume n is strictly positive and that $w = u + iv$ is a **smooth** (C^2) solution of

$$w_x^2 + w_y^2 + n(x,y)^2 = 0.$$

If $\nabla u = \underline{0}$ at some point, then $\nabla u \equiv \underline{0}$ on a **ray** through that point.

Definition of a ray

A path $t \mapsto (x(t), y(t))$, between two points P and Q, that has **minimal length** (in the metric induced by n):

$$\int_0^1 n(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt, \to \min$$

where $(x(0), y(0)) = P$, $(x(1), y(1)) = Q$.

THEOREM 2

Assume n is strictly positive and that u is a smooth solution of

$$\left(|\nabla u|^4 + n^2 u_y^2\right) u_{xx} - 2n^2 u_x u_y u_{xy} +$$
$$\left(|\nabla u|^4 + n^2 u_x^2\right) u_{yy} + n|\nabla u|^2 \nabla n \cdot \nabla u = 0.$$

- If $\nabla u(z_0) = \underline{0}$, then $\det \nabla^2 u(z_0) = 0$.
- If $\nabla u(z_0) = \underline{0}$ and $\left|\nabla^2 u(z_0)\right| > 0$, then $\nabla u \equiv \underline{0}$ on a smooth curve through z_0, and this curve is a ray.

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"Cauchy-Riemann" boundary value problem

Problem

We want to find solutions of the system:

$$\begin{aligned} |\nabla u|^2 - |\nabla v|^2 + n(x,y)^2 &= 0 \\ \nabla u \cdot \nabla v &= 0 \end{aligned} \quad \text{in } \Omega,$$

$$u = \phi \quad \text{on} \quad \partial\Omega, \quad \int_{\partial\Omega} v dx = 0.$$

Here, $\Omega \subseteq \mathbb{R}^2$ is simply connected.

Note

When $n \equiv 0$, this problem amounts to find a function w, holomorphic (or anti-holomorphic) in Ω, when its real part is given on the boundary.

1^{st} step

We find a solution $u \in \phi + H_0^1(\Omega)$ of the equation

$$\left(|\nabla u|^4 + n^2 u_y^2\right) u_{xx} - 2n^2 u_x u_y u_{xy} + \left(|\nabla u|^4 + n^2 u_x^2\right) u_{yy} + n|\nabla u|^2 \nabla n \cdot \nabla u = 0.$$

2^{nd} step

We use the Bäcklund transformation to recover v :

$$\nabla v = \pm\sqrt{n^2 + |\nabla u|^2}\, \frac{(\nabla u)^{\perp}}{|\nabla u|}, \text{ in } \Omega$$

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Finding u

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We minimize the functional

$$J(u) = \int_\Omega j\left(\frac{|\nabla u|}{n}\right) n^2 \; dx \; dy$$

for $u \in \phi + H_0^1(\Omega)$. Here

$$j'(\rho) = \sqrt{1+\rho^2},$$

A possible obstruction

J is **not** differentiable: the one-sided directional derivative of J is:

$$\lim_{\epsilon \to 0^+} \frac{J(u+\epsilon\eta) - J(u)}{\epsilon} =$$

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This is due to the fact thet $j'(0) = 1 > 0$.

No problem

However,

J is strictly convex and coercive;

Then a **unique** minimizing function exists.

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We would like to solve the Bäcklund transformation **but**, as we have seen, u may be constant on open sets and hence the right-hand side of the system **may not be defined**.

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Approximating J

We approximate j

We choose j_ϵ such that j_ϵ is convex, $j'_\epsilon(0) = 0$, and j_ϵ converges to j uniformly on $[0, +\infty)$.

There exists a **unique** function $u^\epsilon \in \phi + H^1_0(\Omega)$ minimizing the functional:

$$J_\epsilon(u) = \int_\Omega j_\epsilon \left(\frac{|\nabla u|}{n} \right) \; n^2 \; dx \; dy$$

J_ϵ is now differentiable

Since J_ϵ is differentiable, then

$$\operatorname{div} \left\{ n \; j'_\epsilon \left(\frac{|\nabla u^\epsilon|}{n} \right) \; \frac{\nabla u^\epsilon}{|\nabla u^\epsilon|} \right\} = 0$$

in the sense of distributions.

Quasilinear equation for u^ϵ

u^ϵ satisfies the quasi-linear equation:

$$\frac{\alpha_\epsilon(\rho)(u^\epsilon_x)^2 + (u^\epsilon_y)^2}{1 - \alpha_\epsilon(\rho)} \; u^\epsilon_{xx} - 2 u^\epsilon_x u^\epsilon_y u^\epsilon_{xy} + \frac{(u^\epsilon_x)^2 + \alpha_\epsilon(\rho)(u^\epsilon_y)^2}{1 - \alpha_\epsilon(\rho)} \; u^\epsilon_{yy} = -|\nabla u^\epsilon|^2 \nabla u^\epsilon \cdot \nabla \ln n,$$

where $\rho = |\nabla u^\epsilon|/n$ and the **ellipticity ratio** satisfies:

$$0 < \alpha^*_\epsilon \le \alpha_\epsilon(\rho) := \frac{\rho j''_\epsilon(\rho)}{j'_\epsilon(\rho)} < 1, \;\; \alpha_\epsilon(\rho) \to \frac{\rho^2}{1 + \rho^2}.$$

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Compactness for the sequence u^ϵ

Bernstein inequality

Let

$$r = u^\epsilon_{xx}, \;\; s = u^\epsilon_{xy} \;\; t = u^\epsilon_{yy}$$

For solutions u^ϵ of the perturbed quasi-linear equation a **Bernstein inequality** holds:

$$\frac{|\nabla u^\epsilon|^2}{n^2 + |\nabla u^\epsilon|^2} \, (r^2 + 2s^2 + t^2) \leq$$
$$2(s^2 - rt) + |\nabla n|^2.$$

A priori estimate

By integrating over the domain

$$\Omega(L) = \{(x,y) \in \Omega : \text{dist } ((x,y), \partial\Omega) > L\},$$

and by using level set analysis, we obtain the estimate:

$$\int_{\Omega(L)} \frac{|\nabla u^\epsilon|^2}{n^2 + |\nabla u^\epsilon|^2} \, |\nabla^2 u^\epsilon|^2 \, dx \, dy \leq$$
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Compactness

Our estimate gives enough compactness to show:

1. u^ϵ converges uniformly on compact subsets of Ω to a function u,
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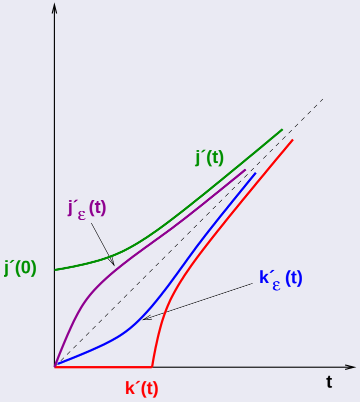
The sequence v^ϵ

Approximating j and k



We apply Bäcklund safely

By the Bäcklund transform:

$$\nabla v^\epsilon = n\, j'_\epsilon\left(\frac{|\nabla u^\epsilon|}{n}\right)\, \frac{(\nabla u^\epsilon)^\perp}{|\nabla u^\epsilon|},$$

since $d^2 v^\epsilon = 0$, we can find v^ϵ normalized by $\int_\Omega v^\epsilon dxdy = 0$.

Remark

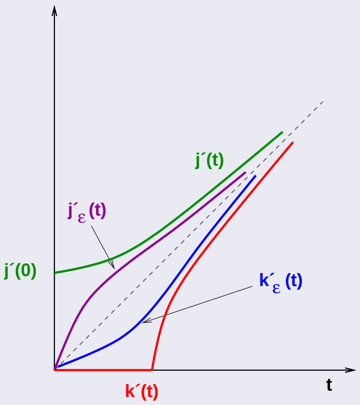
v^ϵ is a critical point of the functional

$$K_\epsilon(v) = \int_\Omega k_\epsilon\left(\frac{|\nabla v|}{n}\right)\, n^2\, dx\, dy,$$

where k_ϵ is the **Young conjugate** of j_ϵ.

The sequence v^ϵ

Approximating j and k



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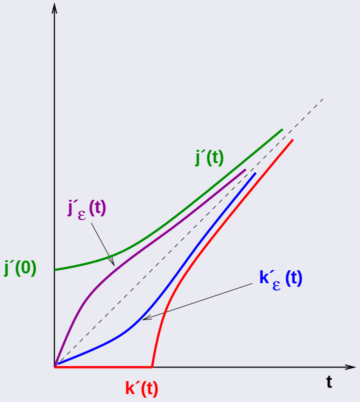

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Taking the v^ϵ to the limit

Quasilinear equation for v^ϵ

v^ϵ is a solution of the quasi-linear equation:

$$\frac{\beta_\epsilon(r)\,(v_x^\epsilon)^2+(v_y^\epsilon)^2}{\beta_\epsilon(r)-1}\,v_{xx}^\epsilon+2v_x^\epsilon v_y^\epsilon v_{xy}^\epsilon+\frac{(v_x^\epsilon)^2+\beta_\epsilon(r)\,(v_y^\epsilon)^2}{\beta_\epsilon(r)-1}\,v_{yy}^\epsilon=|\nabla v^\epsilon|^2\nabla v^\epsilon\cdot\nabla\ln n,$$

where $r=|\nabla v^\epsilon|/n$ and $\beta_\epsilon(r)=\alpha_\epsilon(g_\epsilon'(r))^{-1}$.

Compactness

The compactness properties of u^ϵ imply that

1. there exists a subsequence v^ϵ that uniformly converges on compact subsets of Ω to a v;
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Equation(s) for v

Quasilinear equation for v

v is a viscosity solution of the equation:

$$< \nabla^2 v \ \nabla v, \nabla v > + n^2 B(|\nabla v|/n) \ \Delta v = |\nabla v|^2 \nabla(\ln n) \cdot \nabla v,$$

where B is the **uniform** limit in compact subsets of $(0, +\infty)$ of the sequence: $B_\epsilon(r) = \frac{r^2}{\beta_\epsilon(r) - 1}$.

Non unique B

B depends on the way we approximate the function $j(\rho)$: different j_ϵ's may lead to different B's and hence to different equations for v.

A possible B

For a particular choice of j_ϵ :

$$B(r) = \begin{cases} 1 - r^2 & \text{if } 0 \le r \le 1, \\ r^2(r^2 - 1) & \text{if } r > 1. \end{cases}$$

K is flat for $|\nabla v| < 1$

This depends also on the fact that the limiting functional K is convex, but **not strictly convex**.

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Back to the first order system

System for (u^ϵ, v^ϵ)

As far as our first order system is concerned, we have:

$$|\nabla v^\epsilon| = n\, j_\epsilon'\left(\frac{|\nabla u^\epsilon|}{n}\right)$$
$$\nabla u^\epsilon \cdot \nabla v^\epsilon = 0.$$

Favourable occurrence

Since j_ϵ' does not converge uniformly to j', we obtain: if $\{(x,y) \in \Omega : \nabla u(x,y) = (0,0)\}$ has zero measure, then $v^\epsilon \to v$ uniformly and v satisfies the system:

$$\begin{cases} |\nabla u|^2 - |\nabla v|^2 + n(x,y)^2 & = 0 \\ \nabla u \cdot \nabla v & = 0. \end{cases}$$

Unfavourable occurrence

If $\{(x,y) \in \Omega : \nabla u(x,y) = (0,0)\}$ has positive measure, then

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Changing our point of view

$$|\nabla u^\epsilon| = n\, k_\epsilon'\left(\frac{|\nabla v^\epsilon|}{n}\right)$$
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