REGULARITY ISSUES FOR LOCAL MINIMIZERS OF THE MUMFORD & SHAH ENERGY IN 2D
QUESTIONI DI REGOLARITÀ PER MINIMI LOCALI DEL FUNZIONALE DI MUMFORD & SHAH IN DIMENSIONE 2

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Abstract. We review some issues about the regularity theory of local minimizers of the Mumford & Shah energy in the 2-dimensional case. In particular, we stress upon some recent results obtained in collaboration with C. De Lellis (Universität Zurich). On one hand, we deal with basic regularity, more precisely we survey on an elementary proof of the equivalence between the weak and strong formulation of the problem established in [16]; and on the other hand we discuss fine regularity properties by outlining an higher integrability result for the approximate gradient proved in [17], implying in turn an estimate on the Hausdorff dimension of the singular set of minimizers.

Sunto. Verranno presentati alcuni aspetti della teoria di regolarità dei minimi locali del funzionale di Mumford & Shah in dimensione 2, ottenuti recentemente in collaborazione con C. De Lellis (Università di Zurigo). In particolare, da una parte si discuterà un risultato di regolarità bassa, più precisamente l'equivalenza fra la formulazione debole e quella forte del problema dimostrata in [16]; dall'altra un risultato di regolarità alta, o meglio la maggiore integrabilità del gradiente approssimato dei minimi provata in [17], dalla quale segue una stima sulla dimensione di Hausdorff del relativo insieme singolare.

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1. Introduction

The Mumford & Shah model is a prominent example of variational problem in image segmentation (see [23]). A smoothed version of a black and white picture, whose levels of gray are represented by a function $g \in L^\infty(\Omega, [0,1])$, is obtained by minimizing the

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functional

$$(v, K) \to \mathscr{E}(v, K) + \beta \int_{\Omega \setminus K} |v - g|^2 dx, \tag{1}$$

with

$$\mathscr{E}(v, K) := \int_{\Omega \setminus K} |\nabla v|^2 \, dx + \gamma \, \mathcal{H}^1(K),$$

where $\Omega \subset \mathbb{R}^2$ is a fixed open set, K is a closed subset of Ω with finite $\mathcal{H}^1$ measure, $v \in C^1(\Omega \setminus K)$, and β and γ nonnegative constants. For the sake of simplicity we set $\beta = \gamma = 1$.

This energy has been then conveniently modified and exploited in problems in Fracture Mechanics, mainly to model quasi-static irreversible crack-growth for brittle materials according to Griffith (see [4, Section 4.6.6]).

The role of the squared L^2 distance in (1) is that of a fidelity term in order that the output of the process is close in an average sense to the original input picture g. The set K represents the set of countours of the objects in the picture, the length of which is kept controlled by the penalization of its $\mathcal{H}^1$ measure, while the Dirichlet energy of u favours sharp contours rather than zones where a thin layer of gray is used to pass smoothly from white to black or viceversa.

We stress the attention upon the fact that the set K is not assigned a priori and it is not a boundary in general. Therefore, this problem is not a free-boundary problem, and new ideas and techniques had to be developed to solve it.

Since the appearance in the late 80's to today the research on the Mumford & Shah problem, and on related fields, has been very active and different approaches have been developed. In this paper we shall focus mainly on the ideas and the setting proposed by De Giorgi limited to the 2d case of interest here. More precisely, a weak formulation of the problem, from which an existence theory for minimizers of $\mathscr{E}$ can be developed, is obtained within the space SBV of *Special functions of Bounded Variation* introduced by De Giorgi and Ambrosio: the subspace of BV functions with singular part of the distributional derivative concentrated on a 1-dimensional set (throughout the paper we will use standard notations and results concerning the spaces BV and SBV, following the book [4]). To be more precise we recall that $v \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if

Dv is a (vector-valued) Radon measure on Ω. Then, we can decompose the distributional derivative of v according to

$$Dv = \nabla v\, \mathcal{L}^2 \llcorner \Omega + (v^+ - v^-)\nu_v\; \mathcal{H}^1 \llcorner S_v + D^c v,$$

where

(i) ∇v is the density of the absolutely continuous part of Dv with respect to the Lebesgue measure on Ω (and the approximate gradient of v in the sense of Geometric Measure Theory as well),
(ii) S_v is the set of approximate discontinuities of v, an $\mathcal{H}^1$-rectifiable set (so that $\mathcal{L}^2(S_v) = 0$) endowed with approximate normal $\nu_v(x)$ for $\mathcal{H}^1$ a.e. x,
(iii) $v^\pm$ are the one-sided traces left by v on S_v.

Definition 1.1 ([14], Section 4.1 [4])**.** *$v \in BV(\Omega)$ is a* Special function of Bounded Variation*, in short $v \in SBV(\Omega)$, if $D^c v = 0$, i.e. $Dv = \nabla v\, \mathcal{L}^2 \llcorner \Omega + (v^+ - v^-)\nu_v\; \mathcal{H}^1 \llcorner S_v$.*

So, no Cantor staircase type behaviour is allowed for these functions. Simple examples are collected in the ensuing list:

(i) $W^{1,1}(\Omega) \subset SBV(\Omega)$,
(ii) $v = \sum_{i=1}^M a_i\, \chi_{E_i} \in SBV(\Omega)$ if $\chi_{E_i} \in BV(\Omega)$, i.e. E_i are sets of finite perimeter, and $a_i \in \mathbb{R}$,
(iii) the function $\sqrt{\rho} \cdot \sin(\theta/2)$ for $\theta \in (-\pi, \pi)$ and $\rho > 0$ is in $SBV(B_1)$. Thus, the direct sum of the subspaces of absolutely continuous functions and piecewise constant ones in items (i) and (ii) above is strictly included in SBV.

More interesting examples can be obtained as follows (see [4, Proposition 4.4]): if $K \subset \Omega$ is a closed set such that $\mathcal{H}^1(K) < +\infty$ and $v \in W^{1,1} \cap L^\infty(\Omega \setminus K)$, then $v \in SBV(\Omega)$ and

$$\mathcal{H}^1(S_v \setminus K) = 0. \tag{2}$$

Clearly, property (2) above is not valid for a generic member of SBV, but it does for a significant class of functions: local minimizers of the energy under consideration (see below for the definition).

Keeping in mind this example, the weak formulation of the problem is obtained naively by taking $K = S_u$. Loosely speaking in this approach the set of contours K is identified by the (Borel) set S_v of (approximate) discontinuities of the function v that is not fixed apriori. This is the reason for the terminology *free-discontinuity* problem introduced by De Giorgi. The Mumford & Shah energy of a function v in $SBV(\Omega)$ on an open subset $A \subseteq \Omega$ then reads as

$$\mathrm{MS}(v, \Omega) + \int_\Omega |v - g|^2 dx,$$

where

$$\mathrm{MS}(v, A) := \int_A |\nabla v|^2 dx + \mathcal{H}^1(S_v \cap A). \tag{3}$$

For the sake of simplicity in case $A = \Omega$ we drop the dependence on the set of integration.

Ambrosio's SBV closure and compactness theorem (see [4, Theorems 4.7 and 4.8]) ensures the existence of a minimizer in SBV. Instead, existence of minimizers for the strong formulation of the problem is obtained via a regularity property enjoyed by (the jump set of) the minimizers of the weak counterpart. To this aim we need to analyze the scaling of the energy in order to understand the local behaviour of minimizers. This operation has to be done with some care since the volume and length terms in MS scale differently under affine change of variables of the domain. Let $v \in SBV(B_\rho(x))$, set

$$v_\rho(y) := \rho^{-1/2} v(x + \rho\, y),$$

then $v_\rho \in SBV(B_1)$, with

$$\mathrm{MS}(v_\rho, B_1) = \rho^{-1}\mathrm{MS}(v, B_\rho(x))$$

and

$$\int_{B_1} |v_\rho - g_\rho|^2 dz = \rho^{-3} \int_{B_\rho(x)} |v - g|^2 dy.$$

Thus,

$$\frac{1}{\rho}\left(\mathrm{MS}(v, B_\rho(x)) + \int_{B_\rho(x)} |v - g|^2 dz\right) = \mathrm{MS}(v_\rho, B_1) + \rho^2 \int_{B_1} |v_\rho - g_\rho|^2 dy.$$

This calculation shows that at first order the leading term in the energy is that related to the MS functional, the other being a contribution of higher order that can be neglected in a preliminary analysis. Motivated by this, we introduce a notion of minimality involving only the leading part of the energy. In what follows, u will always denote a *local minimizer* of MS, that is any $u \in SBV(\Omega)$ with $\mathrm{MS}(u) < +\infty$ and such that

$$\mathrm{MS}(u) \leq \mathrm{MS}(w) \qquad \text{whenever } \{w \neq u\} \subset\subset \Omega.$$

The class of all local minimizers shall be denoted by $\mathcal{M}(\Omega)$. Actually, we shall often refer to local minimizers simply as minimizers if no confusion can arise. Regularity properties for minimizers of the whole energy can be obtained by perturbing the theory for local minimizers (cp. with Corollary 2.2 below).

As established in [15] in all dimensions (and proved alternatively in [9] and [10] in dimension two), if $u \in \mathcal{M}(\Omega)$ then the pair $(u, \overline{S_u})$ is a minimizer of $\mathscr{E}$. The main point is the identity $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$, which holds for every $u \in \mathcal{M}(\Omega)$. The groundbreaking paper [15] proves this identity via the following density lower bound estimate (actually established in any dimension with the obvious changes in the statement, see [4, Theorem 7.21]).

Theorem 1.1 (De Giorgi, Carriero & Leaci [15]). *Let $u \in \mathcal{M}(\Omega)$, then there exists a dimensional constant θ independent of u such that*

$$\frac{\mathrm{MS}(u, B_r(z))}{2r} \geq \theta \qquad \text{for all } z \in \overline{S_u}, \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{4}$$

Building upon the same ideas, in [8] it is proved a slightly more precise result (see again [4, Theorem 7.21]).

Theorem 1.2 (Carriero & Leaci [8]). *Let $u \in \mathcal{M}(\Omega)$, then for some dimensional constant θ_0 independent of u it holds*

$$\frac{\mathcal{H}^1(S_u \cap B_r(z))}{2r} \geq \theta_0 \qquad \text{for all } z \in \overline{S_u}, \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{5}$$

The argument for (4) used by De Giorgi, Carriero & Leaci in [15], and similarly in [8] for (5), is indirect: it relies on Ambrosio's SBV compactness theorem, an SBV Poincaré-Wirtinger type inequality established in [15] (see also [4, Theorem 4.14]) and the asymptotic analysis of blow-up limits of minimizers with vanishing jump energy (see [4, Theorem 7.21]). In the paper [16] a simpler proof in 2 dimensions is given, that does not require any Poincaré-Wirtinger inequality, nor any compactness argument. Indeed, the proof in [16] is based on an observation of geometric nature and on a direct variational comparison argument, it differs from those exploited in [9] and [10] to derive (5) in the two dimensional case as well (see Section 2 for more details and further comments).

Theorem 1.3 (De Lellis & Focardi [16])**.** *Let* $u \in \mathcal{M}(\Omega)$. *Then*

$$\frac{m_z(r)}{r} \geq 1 \qquad \text{for all } z \in \overline{S_u} \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{6}$$

More precisely, the set $\Omega_u := \{z \in \Omega : (6) \text{ fails}\}$ *is open and* $\Omega_u = \Omega \setminus \overline{J_u} = \Omega \setminus \overline{S_u}$.

Furthermore, Corollary 2.1 provides a similar conclusion involving only the $\mathcal{H}^1$ measure of the jump set in analogy with Theorem 5 above.

Having established the existence of (local) minimizers for $\mathscr{E}$ we discuss next more refined regularity properties of the minimizers. The interest of the researchers in this problem is motivated by the Mumford & Shah conjecture that we recall below for the readers' convenience.

Conjecture 1.4 (Mumford & Shah [23])**.** *If* $u \in \mathcal{M}(\Omega)$, *then* $\overline{J_u}$ *is the union of (at most) countably many injective* C^1 *arcs* $\gamma_i : [a_i, b_i] \to \Omega$ *with the following properties:*

- (c1) *Any compact* $K \subset \Omega$ *intersects at most finitely many arcs;*
- (c2) *Two arcs can have at most an endpoint* p *in common, and if this is the case, then* p *is in fact the endpoint of three arcs, forming equal angles of* $2\pi/3$.

So according to this conjecture only two possible singular configurations occur: either three arcs meet in an end with angles of $2\pi/3$, or an arc has a free-end. In what follows, we shall call triple junction or spider the first configuration and crack-tip the second.

It was shown by Alberti, Bouchitté & Dal Maso [1] that triple junctions are indeed local minimizers by developing a suitable theory of calibrations for free-discontinuity problems. Instead, Bonnet & David [6] have shown that crack-tip functions are *global* minimizers of the Mumford & Shah energy, a slightly different notion including a topological condition (see [5]). We do not know yet whether they are local minimizers as well or not.

Let us now review the state of the art about Conjecture 1.4.

Theorem 1.5 (Ambrosio, Fusco & Pallara [3])**.** *Let $u \in \mathcal{M}(\Omega)$, then there exists $\Sigma \subset \overline{S_u}$ relatively closed in Ω with $\mathcal{H}^1(\Sigma) = 0$, and such that $\overline{S_u} \setminus \Sigma$ is locally a $C^{1,1}$ arc.*

More precisely, there exists $\varepsilon_0 > 0$ such that

$$\Sigma = \{x \in \overline{S_u} : \liminf_{\rho \downarrow 0} (\mathscr{D}(x, \rho) + \mathscr{A}(x, \rho)) \geq \varepsilon_0\} \tag{7}$$

where

$$\mathscr{D}(x, \rho) := \rho^{-1} \int_{B_\rho(x)} |\nabla u|^2 dy, \quad \textit{(scaled Dirichlet energy)}$$

$$\mathscr{A}(x, \rho) := \rho^{-3} \min_{T \text{ line}} \int_{S_u \cap B_\rho(x)} \mathrm{dist}^2(y, T) d\mathcal{H}^1(y), \quad \textit{(scaled mean flatness).}$$

Note that the affine change of variables mapping $B_\rho(x)$ into B_1 shows that $\mathscr{D}(x, \cdot)$ and $\mathscr{A}(x, \cdot)$ are equal to the Dirichlet energy and the mean flatness of the blow-up maps u_ρ on B_1, respectively.

Theorem 1.5, or better the characterization of the *singular set* Σ in (7), can be employed to subdivide Σ according to the Mumford & Shah conjecture as follows: $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, where

$$\Sigma_1 := \{x \in \Sigma : \lim_{\rho \downarrow 0} \mathscr{D}(x, \rho) = 0\}, \quad \text{the subset of triple junctions or spiders}$$
$$\Sigma_2 := \{x \in \Sigma : \lim_{\rho \downarrow 0} \mathscr{A}(x, \rho) = 0\}, \quad \text{the subset of crack-tips}$$
$$\Sigma_3 := \{x \in \Sigma : \liminf_{\rho \downarrow 0} \mathscr{D}(x, \rho) > 0, \liminf_{\rho \downarrow 0} \mathscr{A}(x, \rho) > 0\}.$$

According to the Mumford & Shah conjecture we should expect $\Sigma_3 = \emptyset$.

In the paper [2], Ambrosio, Fusco & Hutchinson investigated the connection between the higher integrability of ∇u and the Mumford & Shah conjecture. If Conjecture 1.4 does hold, then $\nabla u \in L^p_{loc}$ for all $p < 4$ (cp. with [2, Proposition 6.3] under $C^{1,1}$ regularity assumptions on $\overline{J_u}$, see also Proposition 1.10 below). Viceversa, the higher integrability can be translated into an estimate for the size of the singular set Σ of $\overline{J_u}$ (see [2, Corollary 5.7]): in particular this set has Hausdorff dimension $2 - p/2$ under the apriori assumption that $\nabla u \in L^p_{loc}$ for some $p > 2$. In fact [2] proves also an higher-dimensional analog of this second result.

Theorem 1.6 (Ambrosio, Fusco & Hutchinson [2]). *If $u \in \mathcal{M}(\Omega)$ and $|\nabla u| \in L^p_{loc}(\Omega)$ for some $2 < p < 4$, then*

$$\dim_{\mathcal{H}} \Sigma \leq 2 - p/2 \in (0, 1).$$

Few remarks are in order:

(i) the limitation $p < 4$ is motivated not only because we need the rhs in the estimate above to be positive, but also because explicit examples show that it is the best exponent one can hope for (see the crack-tip example below);

(ii) If we were able to prove the higher integrability property for every $p < 4$ then we would infer that $\dim_{\mathcal{H}} \Sigma = 0$. Clearly, a big step towards the solution in positive of the Mumford & Shah conjecture. For further progress in this direction see Proposition 1.10 below.

Theorem 1.6 is a straightforward corollary of a much deeper and technically demanding result, that we report in the 2-dimensional case of interest here though it holds true with a similar statement in n-dimensions as well.

Theorem 1.7 (Ambrosio, Fusco & Hutchinson, [2]). *The subset of triple junctions Σ_1 has Hausdorff dimension zero.*

Given Theorem 1.7 for granted, Theorem 1.6 is a simple consequence of soft measure theoretic arguments. We shall comment further on Theorem 1.7 in Section 3. Instead, here we outline the proof of Theorem 1.6 to show the role of higher integrability.

Sketch of the Proof of Theorem 1.6. Suppose that $|\nabla u| \in L^p_{loc}$, then for all $s \in (2 - p/2, 1)$ the set

$$\Lambda_s := \left\{ x \in \Omega : \limsup_{\rho} \rho^{-s} \int_{B_\rho(x)} |\nabla u|^p dy > 0 \right\}$$

satisfies $\mathcal{H}^s(\Lambda_s) = 0$ by an elementary covering argument.

Hence, if we rewrite Σ as the disjoint union of $\Sigma \cap \Lambda_s$ and of $\Sigma \setminus \Lambda_s$, we deduce the estimate $\dim_{\mathcal{H}}(\Sigma \cap \Lambda_s) \leq s$.

Furthermore, it is easy to prove that $\Sigma\setminus\Lambda_s\subseteq\Sigma_1$, since if $x\in\Sigma\setminus\Lambda_s$ by the higher integrability it follows that

$$\mathscr{D}(x,\rho)=\rho^{-1}\int_{B_\rho(x)}|\nabla u|^2dy\leq\pi^{1-\frac{2}{p}}\,\rho^{1+\frac{2}{p}(s-2)}\left(\rho^{-s}\int_{B_\rho(x)}|\nabla u|^pdy\right)^{\frac{2}{p}}\xrightarrow{\rho\downarrow 0^+}0.$$

By taking into account Theorem 1.7 we have that $\dim_{\mathcal{H}}(\Sigma\setminus\Lambda_s)=0$.

In conclusion, we infer that for all $s\in(2-p/2,1)$

$$\dim_{\mathcal{H}}\Sigma=\dim_{\mathcal{H}}(\Sigma\cap\Lambda_s)\leq s,$$

by letting $s\downarrow(2-p/2)^+$ we are done. □

The estimate $\dim_{\mathcal{H}}\Sigma<1$ was already present in literature (see David [10], Maddalena & Solimini [21]), though not related to the higher integrability property of the gradient.

So far, in [17, Theorem 1.1] we have been able to prove the following statement that was conjectured by De Giorgi in all space dimensions (cp. with [13, Conjecture 1]).

Theorem 1.8 (De Lellis & Focardi [17]). *There is $p>2$ such that $\nabla u\in L^p_{\rm loc}(\Omega)$ for all $u\in\mathcal{M}(\Omega)$ and for all open sets $\Omega\subseteq\mathbb{R}^2$.*

For a hint of the proof and further comments see Section 3.

Let us now go back to the role of the exponent 4 in the higher integrability result. We consider crack-tip minimizers (Bonnet & David [6]), i.e. functions that up to a rigid motion can be written as

$$u(\rho,\theta)=C\pm\sqrt{\frac{2}{\pi}\rho}\cdot\sin(\theta/2)$$

for $\theta\in(-\pi,\pi)$ and $\rho>0$, and some constant $C\in\mathbb{R}$. Simple calculations imply that crack-tip minimizers satisfy

$$|\nabla u|\in L^p_{loc}(\mathbb{R}^2)\setminus L^4_{loc}(\mathbb{R}^2)\quad\text{for all } p<4.$$

Actually, beyond the scale of L^p space something better holds true: $|\nabla u|\in L^{4,\infty}_{loc}(\mathbb{R}^2)$. The latter is a weak-Lebesgue space, i.e. if $U\subseteq\mathbb{R}^2$ is open then $f\in L^{4,\infty}_{loc}(U)$ if and only if for all $U'\subset\subset U$ there exists $K=K(U')>0$ such that

$$|\{x\in U':\,|f(x)|>\lambda\}|\leq K\lambda^{-4}\qquad\text{for all }\lambda>0.$$

As a side effect of our considerations, we remark a small improvement of the result in [2] in the 2-dimensional case: a weaker form of the Mumford-Shah conjecture in 2d is equivalent to a sharp L^p estimate of the gradient of the minimizers.

Conjecture 1.9. *If $u \in \mathcal{M}(\Omega)$, then $\overline{J_u}$ is the union of (at most) countably many injective C^0 arcs $\gamma_i : [a_i, b_i] \to \Omega$ which are C^1 on $]a_i, b_i[$ and satisfy the two conditions of Conjecture 1.4.*

Our refinement of the result in [2] is in the following proposition (see [16, Proposition 1.5]).

Proposition 1.10 (De Lellis & Focardi [16])**.** *The Conjecture 1.9 is true for $u \in \mathcal{M}(\Omega)$ if and only if $\nabla u \in L^{4,\infty}_{loc}(\Omega)$.*

The if direction of Proposition 1.10 is achieved by first proving that $\overline{J_u}$ has locally finitely many connected components and then invoking the regularity theory developed by Bonnet [5]. In turn, the proof that the connected components are locally finite is a fairly simple application of David's ε-regularity theory [11]. The subtle difference between Conjecture 1.4 and Conjecture 1.9 is in the following point: assuming Conjecture 1.9 holds, if $p = \gamma_i(a_i)$ is a "loose end" of the arc γ_i, i.e. does not belong to any other arc, then the techniques in [5] show that any blow-up is a cracktip, but do not give the uniqueness. In particular, Bonnet is not able to exclude the possibility that γ_i "spirals" around p infinitely many times (compare with the discussion at the end of [5, Section 1]). As far as we know this point is still open.

We have concluded this long introduction to the motivations of our researches, in the rest of the paper we shall go into more details on the results we proved in [16] and [17] in Sections 2 and 3, respectively.

2. The Density Lower Bound Estimate

We first introduce some useful notation. Given $u \in \mathcal{M}(\Omega)$, $z \in \Omega$ and $r \in (0, \mathrm{dist}(z, \partial\Omega))$ let

$$e_z(r) := \int_{B_r(z)} |\nabla u|^2 dx, \quad \ell_z(r) := \mathcal{H}^1(S_u \cap B_r(z)), \quad \text{and} \quad m_z(r) := \mathrm{MS}(u, B_r(z)).$$

The quantity $m_z(\cdot)$ in Theorem 1.3 allows us to take advantage of a suitable monotonicity formula, discovered independently by David and Léger in [12] and Maddalena and Solimini in [21].

Lemma 2.1. *Let $u \in \mathcal{M}(\Omega)$, then for every $z \in \Omega$ and for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(z, \partial\Omega))$*

$$\int_{\partial B_r(z)} \left(\left(\frac{\partial u}{\partial \nu}\right)^2 - \left(\frac{\partial u}{\partial \tau}\right)^2 \right) d\mathcal{H}^1 + \frac{\ell_z(r)}{r} = \frac{1}{r} \int_{J_u \cap \partial B_r(z)} |\langle \nu_u^\perp(x), x\rangle| d\mathcal{H}^0(x),$$

$\frac{\partial u}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$ being the projections of ∇u in the normal and tangential directions to $\partial B_r(z)$, respectively.

In [16, Appendix A] we gave an alternative proof of Lemma 2.1 above, by exploiting directly the Euler-Lagrange equation tested on special radial inner variations.

A simple iteration of Theorem 1.3 gives a density lower bound as in (5) with an explicit constant θ_0 (see [16, Corollary 1.2]).

Corollary 2.1 (De Lellis & Focardi [16])**.** *If $u \in \mathcal{M}(\Omega)$, then $\mathcal{H}^1(\overline{S_u} \setminus J_u) = 0$ and*

$$\frac{\ell_z(r)}{2r} \geq \frac{\pi}{2^{24}} \qquad \text{for all } z \in \overline{S_u} \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{8}$$

Let us now skecth the proof of Theorem 1.3.

Sketch of the proof of Theorem 1.3. The proof is based upon a direct variational argument exploting the following geometrical fact: if $\ell_z(r) < r$ for some $r \in (0, \mathrm{dist}(z, \partial\Omega))$, then

$$\exists\, \rho \in (0, r) \quad \text{such that} \quad \mathcal{H}^0(S_u \cap \partial B_\rho(z)) = 0.$$

This argument has no direct analogue in dimension greater than 2 as simple examples show. In spite of this, Bucur & Luckhaus [7], independently from us, were able to improve remarkably this idea and carry on the proof without our dimensional limitation.

Testing the minimality of u with the harmonic competitor having the same boundary value on $\partial B_\rho(z)$ and taking into account that $m_z(r) < 1$, it is easy to infer that $m_z(\rho) < \rho$. To deduce this, we employ the monotonicity Lemma 2.1.

Actually, we need to propagate the estimates in a quantitative way: if

$$m_z(r) \leq (1 - \varepsilon) r \quad \text{for some } r \in (0, \mathrm{dist}(z, \partial\Omega)),\ \varepsilon \in (0, 1)$$

then

$$m_z(\rho) < (1-\varepsilon)\rho \quad \text{for some } \rho \in (0, r).$$

An iteration of the previous argument gives that

$$\theta^1_*(S_u, z) := \liminf_{\rho \downarrow 0^+} \frac{\ell_z(\rho)}{\rho} \leq (1-\varepsilon)/2.$$

From this, it turns out that $\Omega_u := \{x \in \Omega : m_x(r) < r\}$ is an open set satisfying

$$\Omega_u \cap \{x \in S_u : \theta^1(S_u, x) = 1\} = \emptyset.$$

The latter equality implies the inclusion $\overline{S_u} \subseteq \Omega \setminus \Omega_u$; actually by minimality of u it is elementary to check that

$$\Omega \setminus \Omega_u = \overline{J_u} = \overline{S_u},$$

and thus we are done. □

A natural question is the sharpness of the estimates (6) and (8). The analysis performed by Bonnet [5] suggests that $\pi/2^{24}$ in (8) should be replaced by 1/2 and 1 in (6) by 2. Note that the square root function $u(r, \theta) = \sqrt{\frac{2}{\pi} r} \cdot \sin(\theta/2)$ satisfies $\ell_0(r) = e_0(r) = r$ for all $r > 0$. Thus both the constants conjectured above would be sharp by [11, Section 62]. Unfortunately, we cannot prove any of them.

A similar result can be established for quasi-minimizers of the Mumford & Shah energy, the most prominent examples being minimizers of the functional in equation (1). More precisely, a quasi-minimizer is any function v in $SBV(\Omega)$ with $\mathrm{MS}(v) < +\infty$ and satisfying for some $\omega \geq 0$ and $\alpha > 0$ and for all balls $B_\rho(z) \subset \Omega$

$$\mathrm{MS}(v, B_\rho(z)) \leq \mathrm{MS}(w, B_\rho(z)) + \omega\, \rho^{1+\alpha} \qquad \text{whenever } \{w \neq v\} \subset\subset B_\rho(z).$$

We can then prove the ensuing infinitesimal version of (6) (cp. with [16, Corollary 1.3]).

Corollary 2.2 (De Lellis & Focardi [16])**.** *Let v be a quasi-minimizers of the Mumford & Shah energy, then*

$$\overline{S_u} = \overline{J_u} = \left\{ z \in \Omega : \liminf_{r \downarrow 0^+} \frac{m_z(r)}{r} \geq \frac{2}{3} \right\}. \tag{9}$$

The proof of this corollary, though, needs a blow-up analysis and a new SBV Poincaré-Wirtinger type inequality of independent interest, obtained by improving upon some ideas contained in [19] (cp. with [16, Theorem B.6]); it is, therefore, much more technical.

Let us finally remark that it is possible to improve slightly Theorem 1.3 by combining the ideas of its proof hinted to above with the SBV Poincaré-Wirtinger type inequality in [16, Theorem B.6], and show that actually

$$\Omega \setminus \overline{J_u} = \{x \in \Omega : \, m_x(r) \leq r \quad \text{for some } r \in \mathrm{dist}(x, \partial\Omega)\}$$

(see [16, Remark 2.3]).

3. The Higher Integrability Result

Following a classical path, the key ingredient to establish Theorem 1.8 is a reverse Hölder inequality for the gradient, which we state independently (see [17, Theorem 1.3]).

Theorem 3.1 (De Lellis & Focardi [17]). *For all $q \in (1,2)$ there exist $\rho \in (0,1)$ and $C > 0$ such that*

$$\|\nabla u\|_{L^2(B_\rho)} \leq C \|\nabla u\|_{L^q(B_1)} \qquad \textit{for any } u \in \mathcal{M}(B_1). \tag{10}$$

Using the obvious scaling invariance of (3), Theorem 3.1 yields a corresponding reverse Hölder inequality for balls of arbitrary radius: Theorem 1.8 is then a consequence of (by now) classical arguments (see for instance [20]). The exponent p could be explicitely estimated in terms of q, C and ρ. However, since our argument for Theorem 3.1 is indirect, we do not have any explicit estimate for C (ρ can instead be computed). Hence, combining Theorem 1.8 with [2] we can only conclude that the dimension of the singular set of $\overline{J_u}$ is strictly smaller than 1. This was already proved in [11] using different arguments and, though not stated there, Guy David pointed out to us that the corresponding dimension estimate could be made explicit. In fact, after discussing the present result, he suggested to C. De Lellis that also the constant C in Theorem 3.1 might be estimated: a viable strategy would combine the core argument of this paper with some ideas from [11] (the proof of Theorem 3.1 given here makes already a fundamental use of the paper [11], but depends only on the ε-regularity theorem for "spiders" and "segments"). However, the resulting estimate would give an extremely small number, whereas the proof would very likely become much more complicated. Since we do not see any way to make further progress, we have decided not to pursue this issue here.

In addition, our indirect proof has some interesting side results that we shall highligt in what follows. Indeed, in this section we shall give a rapid sketch of the proof of Theorem 3.1, and rather than discussing all the details we shall mainly focus on a compactness result, Theorem 3.2 below, that is one of the most important ingredients to establish Theorem 3.1, and on the related consequences. We strongly believe that Theorem 3.2 has some interest in its own.

Sketch of the proof of Theorem 3.1. We fix an exponent $q \in (1,2)$ and a suitable radius ρ (whose choice will be specified later) for which (10) is false, that is

$$\|\nabla u_k\|_{L^2(B_\rho)} \geq k\|\nabla u_k\|_{L^q(B_1)} \quad \text{for a sequence } (u_k)_{k\in\mathbb{N}} \in \mathcal{M}(B_1). \tag{11}$$

Since the Mumford & Shah energy of any $u \in \mathcal{M}(B_1)$ can be easily bounded apriori by 2π, we have $\|\nabla u_k\|_{L^q(B_1)} \to 0$. A suitable competitor argument then shows that:

(a) The L^2 energy of the gradients of u_k converge to 0;
(b) The jump set J_{u_k} of u_k converges in the local Hausdorff metric to a set J which is a (locally finite) union of minimal connections.

Though this last statement is, intuitively, quite clear, it is technically demanding, because we do not have any apriori control of the norms $\|u_k\|_{L^1}$, thus preventing the use of Ambrosio's $(G)SBV$ compactness theorem. We can not even employ De Giorgi's SBV Poincaré-Wirtinger inequality, since it holds true in a regime of small jumps rather than of small gradients as the current one.

A very similar issue is investigated in [2, Proposition 5.3, Theorem 5.4] under the stronger assumption that $\|\nabla u_k\|_{L^2}$ converges to 0. Such results hinge upon the notion of Almgren's area minimizing sets, and thus need a delicate study of the behaviour of the composition of SBV functions with Lipschitz deformations that are not necessarily one-to-one, and some specifications on the regularity theory for those sets. Instead, in [17, Proposition 5.1] (see Proposition 3.2 below) we set the analysis into the more natural framework of Caccioppoli partitions. Because of this, as pointed out in item (a) above, the fact that the Dirichlet energy of u_k is infinitesimal turns out to be a consequence of (11) and of the energy upper bound for functions in $\mathcal{M}(B_1)$.

Having established (a) and (b), an elementary argument shows the existence of a universal constant ρ such that the intersection of J with $B_{2\rho}$ is:

(i) either empty;
(ii) or a straight segment;
(iii) or a spider, i.e. three segments meeting at a common point with equal angles.

We use then the regularity theory developed by David (see [11]) to conclude that, if k is large enough, $\overline{J_{u_k}} \cap B_{2\rho}$ is diffeomorphic to (and a small perturbation of) one of these three cases. Finally a variational argument (based on a simple "Fubini and competitor" trick) shows the existence of a constant C (independent of k) with the property that

$$\|\nabla u_k\|_{L^2(B_\rho)} \le C\|\nabla u_k\|_{L^q(B_1)} \tag{12}$$

which contradicts (11). □

To state the compactness result we need to introduce the notion of Caccioppoli partition.

Definition 3.1. *A Caccioppoli partition of Ω is a countable partition $\mathscr{E} = \{E_i\}_{i=1}^{\infty}$ of Ω in sets of (positive Lebesgue measure and) finite perimeter with $\sum_{i=1}^{\infty} \mathrm{Per}(E_i, \Omega) < \infty$.*

For each Caccioppoli partition $\mathscr{E}$ we set $J_{\mathscr{E}} := \bigcup_i \partial^* E_i$. The partition $\mathscr{E}$ is said to be minimal if

$$\mathcal{H}^1(J_{\mathscr{E}}) \le \mathcal{H}^1(J_{\mathscr{F}})$$

for all Caccioppoli partitions $\mathscr{F}$ for which there exists an open subset $\Omega' \subset\subset \Omega$ with $\sum_{i=1}^{\infty} \mathcal{L}^2\left((F_i \triangle E_i) \cap (\Omega \setminus \Omega')\right) = 0$.

There is an important correspondance between Caccioppoli partitions and the subspace of "piecewise constant" SBV functions, in literature indicated as SBV_0 (see [4, Theorems 4.23, 4.25 and 4.39]), in such a way that minimizing the Mumford & Shah energy over SBV_0 corresponds exactly to the minimal area problem for Caccioppoli partitions.

We state Theorem 3.2 only in the 2-dimensional case of interest here. In spite of this, the analogous result in any dimension can be obtained only with straightforward notational changes in the statement below (and also in the corresponding proof). Nevertheless, dimension 2 enters dramatically in the proof of Theorem 3.1 as the structure of minimal Caccioppoli partitions in $\mathbb{R}^2$ can be described precisely via minimal connections as done in item (b) above (cp. with [17, Proposition 3.2]).

Theorem 3.2 (De Lellis & Focardi [17]). *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(B_1)$ be such that*

$$\lim_k \|\nabla u_k\|_{L^1(B_1)} = 0. \tag{13}$$

Then, (up to the extraction of a subsequence not relabeled for convenience) there exists a minimal Caccioppoli partition $\mathscr{E} = \{E_i\}_{i\in\mathbb{N}}$ such that $(\overline{J_{u_k}})_{k\in\mathbb{N}}$ converges locally in the Hausdorff distance to $\overline{J_{\mathscr{E}}}$ and

$$\lim_k \mathrm{MS}(u_k, A) = \lim_k \mathcal{H}^1(J_{u_k} \cap A) = \mathcal{H}^1(J_{\mathscr{E}} \cap A) \qquad \textit{for all open sets } A \subset B_1. \tag{14}$$

Let us finally discuss some interesting consequences of Theorem 3.2:

(i) Blow-up limits on singular points of a minimizer in the regime of small gradients, i.e. points $x \in \Sigma_1$, are minimal Caccioppoli partitions (in any dimension!).

In particular, thanks to the structure of minimal Caccioppoli partitions in 2-dimensions mentioned above, (locally) they are triple junctions. Finally, Theorem 3.2 provides an indirect proof of the local minimality of triple junctions alternative to that in [1];

(ii) A more elementary proof of the estimate (and of its analogue in any dimension!)

$$\dim_{\mathcal{H}}\Sigma_1 = 0 \qquad \text{(recall that } \Sigma_1 = \{x \in \Sigma : \lim_{\rho\downarrow 0} \mathscr{D}(x,\rho) = 0\}),$$

follows from Theorem 3.2, the regularity theory for minimal Caccioppoli partitions by Massari & Tamanini [22], and standard blow-up arguments (see [18]).

No use of Almgren's area minimazing sets and of the corresponding regularity theory is then needed.

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