Approximation results by difference schemes of fracture energies: the vectorial case

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Abstract

We provide a variational approximation, in the sense of De Giorgi's Γ-convergence, by finite-difference schemes of functionals of the type

$$\int_\Omega \psi(\nabla u)\,dx + \int_{J_u} g\left(u^+ - u^-, \nu_u\right)\,d\mathcal{H}^2$$

defined for $u \in SBV(\Omega;\mathbb{R}^N)$, where Ω is an open set in $\mathbb{R}^3$, ψ and g are assigned. More precisely, ψ is a quasi-convex function with p-growth, $p > 1$, and g satisfies standard lower semicontinuity conditions. The approximating functionals are of the form

$$\int_{\mathcal{T}_\varepsilon\cap\Omega} \psi_\varepsilon\left(\nabla u(x)\right)\,dx$$

where ψ_ε is an interaction potential taking into account a separation of scales, $\mathcal{T}_\varepsilon$ is a suitable regular triangulation of $\mathbb{R}^3$ and u is affine on each element of the assigned triangulation.

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1 Introduction

According to a recent trend of research, which deduce continuum theories starting from an atomistic description of the media (see [19],[7],[8],[9],[1]), we provide a variational approximation of energies of the type

$$\mathcal{E}(u) = \int_\Omega \psi\left(\nabla u\right)\,dx + \int_{J_u} g(u^+ - u^-)\varphi(\nu_u)\,d\mathcal{H}^2, \tag{1.1}$$

defined for $u \in SBV(\Omega;\mathbb{R}^N)$, where Ω is an open bounded set of $\mathbb{R}^3$, $\psi : \mathbb{R}^{N\times 3} \to [0,+\infty)$, $g : \mathbb{R}^N \to [0,+\infty)$ and $\varphi : \mathbb{S}^2 \to [0,+\infty)$ are assigned.

These models derive from the theory of brittle fracture for hyperelastic materials. For such materials the elastic deformation outside the fracture can be modeled by an elastic energy density independent of the crack. The usual assumptions required on ψ are quasi-convexity and superlinear growth (see [4]). For what the surface term is corcerned, it is not completely clear which are the proper assumptions consistent with finite elasticity; here we choose to follow the theory of Griffith's materials (see [16],[3]), i.e., the energy density is represented by a function g, depending on the jump of u, and by a convex function of the normal to the crack site, φ. As already stated, the natural class of deformations to be considered is the space $SBV\left(\Omega;\mathbb{R}^N\right)$ of special functions with bounded variation.

Our approximation relies on finite-differences discretization schemes, following the approach proposed by De Giorgi to treat the Mumford-Shah problem in Computer Vision (see [15]), and applied to Fracture Mechanics firstly by Braides, Dal Maso and Garroni [7] and then by Braides and Gelli [8],[9] (see also [1],[19]).

While the previous results mainly study the scalar case, here we deal with the vectorial one. In this paper we prove the Γ-convergence to energies of type (1.1) of the family of approximating functionals defined as

$$\int_{\mathcal{T}_\varepsilon\cap\Omega} \psi_\varepsilon\left(\nabla u(x)\right)\, dx \tag{1.2}$$

where $\mathcal{T}_\varepsilon$ is a regular triangulation of $\mathbb{R}^3$, ψ_ε is a suitable non-convex interaction potential and $u:\mathbb{R}^3\to\mathbb{R}^N$ is continuous and affine on each element of $\mathcal{T}_\varepsilon$.

The main problem in the vectorial case is to give a definition of discrete schemes that is coherent with the discrete method, i.e., find a suitable 'ε-discretization of the gradient', $D_\varepsilon u$, by finite-differences, and, simultaneously, find proper potentials ψ_ε, in order to obtain, by means of a separation of scales, the assigned bulk density, ψ, and the corresponding surface one.

Since we are interested in non-isotropic bulk energy densities, a quite natural choice for $D_\varepsilon u$, in order to recover the global behaviour of the gradient matrix, is the finite-differences matrix below

$$D_\varepsilon u = \frac{1}{\varepsilon}\left(\langle u(\alpha+\varepsilon \mathrm{e}_\ell)-u(\alpha), e_k\rangle\right)_{\substack{\ell=1,2,3\\ k=1,\dots,N}}. \tag{1.3}$$

Let us remark that the gradient ∇u, in (1.2), coincides exactly with the matrix $D_\varepsilon u$ defined above, since we choose to identify a 'discrete function' u (i.e., defined on the nodes of the triangulation $\mathcal{T}_\varepsilon$) with its continuous piecewise affine interpolation, still denoted by u.

Notice that the scalar models considered so far are based on discretizations, $D^\ell_\varepsilon u$, accounting for increments only along given integer directions, i.e., $D_\varepsilon u$ in (1.3) has to be replaced by

$$\frac{1}{\varepsilon}(u(\alpha+\varepsilon \mathrm{e}_\ell)-u(\alpha)).$$

In addition, in the case of linear elasticity [1], $D_\varepsilon u$ is chosen to be the projection along a fixed direction of the increment of u in the same direction, i.e.,

$$\frac{1}{\varepsilon}\langle u(\alpha+\varepsilon \mathrm{e}_\ell)-u(\alpha), \mathrm{e}_\ell\rangle.$$

Both these approaches allow to get a complete characterization of the limit by studying the asymptotic behaviour of one-dimensional functionals. On the other hand, the only possible bulk energy densities obtained as limit are those determined by summing up all the contribution on fixed directions.

We overcome this drawback defining ψ_ε as ψ for values of $D_\varepsilon u$ smaller than a given threshold and a suitable function taking into account the contributions of $D_\varepsilon u$ along the coordinate directions otherwise (see (3.3)).

Thus we obtain as limit energies of type (1.1), with the surface term of the form

$$\sum_{\ell=1,2,3} \int_{J_u} g_\ell \left(u^+ - u^-\right) |\langle \nu_u, \mathrm{e}_\ell \rangle| \, d\mathcal{H}^2$$

with $g_\ell : \mathbb{R}^N \setminus \{0\} \to [0, +\infty)$ subadditive, continuous functions, superlinear at 0 (for more details see Section 3).

Eventually, in Section 4 we consider the two-dimensional case for which we exhibit two approximation results related to different definitions of ψ_ε. The first one is the two-dimensional formulation of the model introduced above and can be still treated by an integral geometric reduction technique. The second model requires more sophisticated tools (see Proposition 1.16 [5]), indeed, a new construction must be performed in order to have an estimate along direction $\mathrm{e}_2 - \mathrm{e}_1$, that is, a direction in which difference quotients are not involved. This difficulty can be bypassed by considering the lattice generated by vectors $\mathrm{e}_1, \mathrm{e}_2 - \mathrm{e}_1$ and by constructing a one-dimensional profiled function, affine on the slanted unitary cell P_ε (see Proposition 4.3). The result of this model yields an anisotropic surface term that penalizes in different way crack sites, according to their orientation with respect to the basis $\{\mathrm{e}_1, \mathrm{e}_2\}$.

The plan of the paper is the following: Section 2 is a short summary concerning some results on (generalized) functions with bounded variation; Section 3 contains the main result in case $n = 3$; Section 4 is devoted to study the case $n = 2$.

2 Notations and Preliminaries

If $a, b \in \mathbb{R}$ we write $a \wedge b$ and $a \vee b$ for the minimum and maximum between a and b, respectively. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$ and with $| \cdot |$ the usual euclidean norm, without specifying the dimension n when there is no risk of confusion. Moreover, let $\{\mathrm{e}_\ell\}_{1 \le \ell \le n}$ be the canonical base of $\mathbb{R}^n$, $n = 2, 3$. Since we deal with vector-valued functions, in order to make no confusion arise, we will always denote with $\mathbb{R}^N$ the codomain and with $\{e_k\}_{1 \le k \le N}$ its canonical base.
For every $x, y \in \mathbb{R}^n$, $[x, y]$ denotes the segment between x and y.

If Ω is a bounded open subset of $\mathbb{R}^n$, for every $\eta > 0$ set $\Omega_\eta := \{x \in \Omega : d(x, \partial\Omega) > \eta\}$.

Let $\mathcal{A}(\Omega)$ be the family of open subsets of Ω. If μ is a Borel measure and B is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, $k \geq 0$.

The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure, unless otherwise specified. We use standard notations for Lebesgue and Sobolev spaces.

2.1 BV functions

We recall some definitions and properties concerning functions with bounded variation. The main reference is the book [4] (see also [13]). Let Ω be a bounded open set of $\mathbb{R}^n$ and let $S = \mathbb{R}^N \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^N$.

Definition 2.1 *Let $B \subset \Omega$ be a Borel set such that $\mathcal{L}^n(B_\rho(x) \cap B) > 0$ for every $\rho > 0$ where $B_\rho(x) := \{y \in \Omega : |x-y| \leq \rho\}$. We say that $z \in S$ is the* approximate limit *in $x \in \Omega$ of $u \in L^1(\Omega; \mathbb{R}^N)$* in the domain *$B$, and we write $z = \mathrm{ap} - \lim_{\substack{y \to x \\ y \in B}} u(y)$, if for every neighbourhood U of z in S there holds*

$$\lim_{\rho \to 0} \frac{\mathcal{L}^n(\{y \in B_\rho(x) \cap B : u(y) \notin U\})}{\mathcal{L}^n(B_\rho(x) \cap B)} = 0.$$

Denote by S_u the complement of the set of points where the approximate limit of u exists, it is well known that $\mathcal{L}^n(S_u) = 0$. Define the function $\tilde{u} : \Omega \setminus S_u \to S$ by

$$\tilde{u}(x) = \mathrm{ap} - \lim_{\substack{y \to x \\ y \in \Omega}} u(y),$$

thus u is equal a.e. on Ω to $\tilde{u}$. Notice that $\tilde{u}$ is allowed to take the value ∞ but $\mathcal{L}^n(\{\tilde{u} = \infty\}) = 0$.

Definition 2.2 *We say that $x \in \Omega$ is a* jump point *of u, and we write $x \in J_u$, if there exist $a, b \in S$, and a vector $\nu \in \mathbb{S}^{n-1}$ such that $a \neq b$ and*

$$a = \mathrm{ap} - \lim_{\substack{y \to x \\ y \in \pi^-(x,\nu) \cap \Omega}} u(y), \quad b = \mathrm{ap} - \lim_{\substack{y \to x \\ y \in \pi^+(x,\nu) \cap \Omega}} u(y), \tag{2.4}$$

where $\pi^\pm(x, \nu) = \{y \in \mathbb{R}^n : \pm\langle y - x, \nu\rangle > 0\}$.

The triplet (a, b, ν), uniquely determined by (2.4) up to a permutation of (a, b) and a change of sign of ν, will be denoted by $(u^+(x), u^-(x), \nu_u(x))$.

Definition 2.3 *We say that u is* approximately differentiable *at a point $x \in \Omega \setminus S_u$ such that $\tilde{u}(x) \neq \infty$, if there exists a matrix $L \in \mathbb{R}^{N \times n}$ such that*

$$\mathrm{ap} - \lim_{\substack{y \to x \\ y \in \Omega}} \frac{|u(y) - \tilde{u}(x) - L(y - x)|}{|y - x|} = 0. \tag{2.5}$$

If u is approximately differentiable at a point x, the matrix L uniquely determined by (2.5), will be denoted by $\nabla u(x)$ and will be called the approximate gradient *of u at x.*

Definition 2.4 *Let $u \in L^1\left(\Omega;\mathbb{R}^N\right)$, we say that u is a* function of Bounded Variation *in Ω, we write $u \in BV\left(\Omega;\mathbb{R}^N\right)$, if the distributional derivative Du of u is representable by a $N \times n$ matrix valued Radon measure on Ω.*

Let us consider the Lebesgue decomposition of Du with respect to $\mathcal{L}^n$, i.e., $Du = D^a u + D^s u$, where $D^a u$ is the absolutely continuous part and $D^s u$ is the singular one. The density of $D^a u$ with respect to $\mathcal{L}^n$ coincides a.e. with the approximate gradient ∇u of u. Define the *jump part* of Du, $D^j u$, to be the restriction of $D^s u$ to S_u, and the *Cantor part*, $D^c u$, to be the restriction of $D^s u$ to $\Omega \setminus S_u$, thus we have

$$Du = D^a u + D^j u + D^c u.$$

The set J_u turns out to be $\mathcal{H}^{n-1}$ rectifiable; moreover, $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. The following representation formula holds true for the jump part of Du

$$D^j u = \left(u^+ - u^-\right) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

where for $a, b \in \mathbb{R}^N$ $a \otimes b$ denotes the $N \times n$ matrix whose entries are $a_i b_j$, $1 \le i \le N$ and $1 \le j \le n$. In the sequel we denote $[u](x) = (u^+ - u^-)(x)$ for $x \in J_u$.

Definition 2.5 *Given $u \in BV(\Omega;\mathbb{R}^N)$, we say that u is a* special function with bounded variation *in Ω, and we write $u \in SBV(\Omega;\mathbb{R}^N)$, if $D^c u = 0$.*

Functionals involved in free-discontinuity problems are often not coercive in SBV, then it is useful to consider the following wider class (see [4]).

Definition 2.6 *Given $u \in L^1(\Omega)$, we say that u is a* generalized special function with bounded variation *in Ω, and we write $u \in GSBV(\Omega)$, if $((-T) \vee u \wedge T) \in SBV(\Omega)$ for every $T \in \mathbb{N}$.*

Moreover, let $u \in L^1\left(\Omega;\mathbb{R}^N\right)$, $u \in (GSBV(\Omega))^N$ if $\langle u, e_k\rangle \in GSBV(\Omega)$ for every $1 \le k \le N$.

Define for any $p > 1$

$$(GSBV^p(\Omega))^N := \{u \in (GSBV(\Omega))^N : \nabla u \in L^p\left(\Omega;\mathbb{R}^{N\times n}\right),\ \mathcal{H}^{n-1}(J_u) < +\infty\}. \tag{2.6}$$

2.1.1 Lower semicontinuity in GSBV

In this subsection we state some semicontinuity results for variational integrals in $GSBV$. The first one, proved by Kristensen in [17] in a more general version (see also [4]), deals with the lower semicontinuity of volume energies. Let us first recall the notion of quasi-convexity.

Definition 2.7 *We say that a continuous integrand $\psi : \mathbb{R}^{N\times n} \to [0, +\infty)$ is* quasi-convex *if*

$$\psi(X)\,\mathcal{L}^n(\Omega) \le \int_\Omega \psi(X + D\varphi(x))\,dx \tag{2.7}$$

for every $X \in \mathbb{R}^{N\times n}$ and $\varphi \in C_c^1\left(\Omega;\mathbb{R}^N\right)$.

Theorem 2.8 *Let* $(u_j) \subset (GSBV(\Omega))^N$ *be satisfying*

$$\sup_j \left\{ \int_\Omega |\nabla u_j|^p \, dx + \int_{S_{u_j}} \theta \left(\left| u_j^+ - u_j^- \right| \right) \, d\mathcal{H}^{n-1} \right\} < +\infty, \tag{2.8}$$

where $\theta : [0, +\infty) \to [0, +\infty]$ *is a concave function such that* $\frac{\theta(t)}{t} \to +\infty$ *as* $t \to 0^+$.

If $u_j \to u$ *in measure, then* $u \in (GSBV(\Omega))^N$.

Moreover, for every quasi-convex integrand $\psi : \mathbb{R}^{N \times n} \to [0, +\infty)$ *such that*

$$|\psi(X)| \leq C (1 + |X|^p)$$

for every $X \in \mathbb{R}^{N \times n}$ *with* $p > 1$ *and* C *a positive constant, there holds*

$$\int_\Omega \psi(\nabla u) \, dx \leq \liminf_j \int_\Omega \psi(\nabla u_j) \, dx.$$

In order to state an analogous result for surface energies we need to introduce the notion of subadditivity.

Definition 2.9 *Let* $\triangle := \{(z, z) : z \in \mathbb{R}^N\}$. *We say that a function* $\vartheta : \mathbb{R}^N \times \mathbb{R}^N \setminus \triangle \to [0, +\infty]$ *is* subadditive *if for all distinct* $z_i \in \mathbb{R}^N$, $i = 1, 2, 3$, *we have*

$$\vartheta(z_1, z_2) \leq \vartheta(z_1, z_3) + \vartheta(z_3, z_2).$$

We extend ϑ to the whole $\mathbb{R}^N \times \mathbb{R}^N$ setting $\vartheta \equiv 0$ on $\triangle$. The following result is an easy generalization to the vector-valued case of Theorem 4.3 of [6].

Theorem 2.10 *Let* $\Omega \subset \mathbb{R}$ *be a bounded open set, let* $\vartheta : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty]$ *be a symmetric, subadditive and lower semicontinuous function.*

Let $(u_j) \subset SBV\left(\Omega; \mathbb{R}^N\right)$ *be satisfying (2.8); if* $u_j \to u$ *in measure then* $u \in SBV\left(\Omega; \mathbb{R}^N\right)$ *and*

$$\int_{J_u} \vartheta\left(u^+, u^-\right) \, d\mathcal{H}^0 \leq \liminf_j \int_{J_{u_j}} \vartheta\left(u_j^+, u_j^-\right) \, d\mathcal{H}^0.$$

Let us point out that the results stated in Theorem 2.10 heavily depend on the one-dimensional setting. Indeed, being $(u_j) \subset SBV\left(\Omega; \mathbb{R}^N\right)$ not equi-bounded in L^∞ a priori, by Theorem 2.8 we can only infer that $u \in (GSBV(\Omega))^N$. On the other hand, the superlinearity and the monotonicity assumptions on the function ϑ of (2.8) and the choice $n = 1$ imply $u \in SBV\left(\Omega; \mathbb{R}^N\right)$.

2.2 Γ-convergence

We recall the notion of Γ-convergence (see [12]). Let (X, d) be a metric space. A family (F_ε) of functionals $F_\varepsilon : X \to [0, +\infty]$ is said to Γ-*converge* to a functional $F : X \to [0, +\infty]$ at $u \in X$, and we write $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$, if for every sequence (ε_j) of positive numbers decreasing to 0 the following two conditions hold:

(i) (*lower semicontinuity inequality*) for all sequences (u_j) converging to u in X we have $F(u) \leq \liminf_j F_{\varepsilon_j}(u_j)$;

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u in X such that $F(u) \geq \limsup_j F_{\varepsilon_j}(u_j)$.

We say that F_ε Γ-converges to F if $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$ at all points $u \in X$ and that F is the Γ-*limit* of F_ε.

In the sequel we will denote by $\Gamma(meas)$-lim inf, $\Gamma(meas)$-lim sup and $\Gamma(L^1)$-lim inf, $\Gamma(L^1)$-lim sup, the lower and upper Γ-limits on the space L^1 endowed with the metric of the L^1 strong convergence and the convergence in measure, respectively.

3 Main result

Let $Q = [0, 1]^3$ and consider the triangulation given by the six congruent simplices T_r, $r = 1, \ldots, 6$, defined by

$$
\begin{aligned}
&T_1 = co\{0, \mathrm{e}_1, \mathrm{e}_3, \mathrm{e}_2 + \mathrm{e}_3\}, \quad T_4 = co\{\mathrm{e}_1, \mathrm{e}_1 + \mathrm{e}_2, \mathrm{e}_2 + \mathrm{e}_3, \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3\},\\
&T_2 = co\{\mathrm{e}_1, \mathrm{e}_3, \mathrm{e}_1 + \mathrm{e}_3, \mathrm{e}_2 + \mathrm{e}_3\}, \quad T_5 = co\{\mathrm{e}_1, \mathrm{e}_1 + \mathrm{e}_3, \mathrm{e}_2 + \mathrm{e}_3, \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3\},\\
&T_3 = co\{0, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_2 + \mathrm{e}_3\}, \quad T_6 = co\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_2, \mathrm{e}_2 + \mathrm{e}_3\},
\end{aligned}
$$

(see Figure 1 below).

Figure 1: the partition $(T_i)_{r=1,\ldots,6}$ of the unitary cube.

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set, for every subset $A \subseteq \Omega$ and for $r = 1, \ldots, 6$ define the sets of tetrahedra

$$
\begin{aligned}
\mathcal{T}_\varepsilon^r(A) &:= \{\alpha + \varepsilon T_r : \alpha + \varepsilon T_r \subseteq A,\ \alpha \in \varepsilon\mathbb{Z}^3\},\\
T_\varepsilon^r(A) &:= \{\alpha + \varepsilon T_r : (\alpha + \varepsilon T_r) \cap A \neq \emptyset,\ \alpha \in \varepsilon\mathbb{Z}^3\},
\end{aligned}
$$

which identify the simplices properly contained in A and those intersecting A, respectively. In case $A = \Omega$ we will drop the dependence on Ω in the definitions above.

Moreover, denote by $\mathcal{A}_\varepsilon\left(\Omega; \mathbb{R}^N\right)$ the set of functions $u : \Omega \to \mathbb{R}^N$ such that u is continuous on Ω and affine on each simplex belonging to $\cup_{r=1}^6 \mathcal{T}_\varepsilon^r \supseteq \Omega$.

Let $\psi : \mathbb{R}^{N\times 3} \to [0, +\infty)$ and $g_\ell : \mathbb{R}^N \setminus \{0\} \to [0, +\infty)$, $\ell = 1, 2, 3$, satisfy (h1) ψ is a quasi-convex function with superlinear growth, i.e., there exist $p > 1$ and $C_1, C_2 > 0$ such that for every $X \in \mathbb{R}^{N\times 3}$

$$C_1 |X|^p \leq \psi(X) \leq C_2 (1 + |X|^p); \tag{3.1}$$

(h2) g_ℓ is a symmetric, subadditive, lower semicontinuous function such that $\inf_{\mathbb{R}^N\setminus\{0\}} g_\ell > 0$;

(h3) g_ℓ is an upper semicontinuous function bounded in a neighbourhood of $z = 0$.

Notice that the subadditivity and the local boundedness assumptions on g_ℓ imply the existence of a positive constant c_2 such that for every $z \in \mathbb{R}^N \setminus \{0\}$ and for $\ell = 1, 2, 3$ there holds

$$c_1 \leq g_\ell(z) \leq c_2(1 + |z|), \tag{3.2}$$

where $c_1 = \min_\ell\{\inf_{\mathbb{R}^N\setminus\{0\}} g_\ell\}$.

Let $u_T := ((-T) \wedge \langle u, e_k\rangle \vee T)_{1\leq k\leq N}$, and assume that

(h4) for every $u \in (GSBV^p(\Omega))^N$ there exists a sequence $(T_j) \subseteq [0, +\infty)$ with $T_j \to +\infty$ such that

$$\limsup_j \int_{J_u} g_\ell\left(\left[u_{T_j}\right]\right) |\langle \nu_u, \mathrm{e}_\ell\rangle|\, d\mathcal{H}^2 = \int_{J_u} g_\ell([u]) |\langle \nu_u, \mathrm{e}_\ell\rangle|\, d\mathcal{H}^2,$$

Notice that this technical condition is fulfilled in case all the g_ℓ are bounded on $\mathbb{R}^N \setminus \{0\}$. We will make further comments on this assumption in Remark 3.2 and Remark 3.6.

Let us introduce the approximating functionals. First, extend g_ℓ to $\mathbb{R}^N$ setting $g_\ell(0) = 0$, thus preserving its lower semicontinuity property, then define $\psi^g_\varepsilon : \mathbb{R}^{N\times 3} \to [0, +\infty)$ as

$$\psi^g_\varepsilon(X) := \begin{cases} \psi(X) & \text{if } |X| \leq \lambda_\varepsilon \\ \dfrac{1}{\varepsilon}\displaystyle\sum_{\ell=1}^{3} g_\ell(\varepsilon X \mathrm{e}_\ell) & \text{otherwise,} \end{cases} \tag{3.3}$$

where $(\lambda_\varepsilon) \subset [0, +\infty)$ is such that $\lambda_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$ and $\sup_{\varepsilon>0}(\varepsilon\lambda^p_\varepsilon) < +\infty$. Consider the family of functionals $\mathcal{F}^g_\varepsilon : L^1(\Omega; \mathbb{R}^N) \to [0, +\infty]$ given by

$$\mathcal{F}^g_\varepsilon(u) := \begin{cases} \displaystyle\int_{\cup_{r=1}^6 \mathcal{T}^r_\varepsilon} \psi^g_\varepsilon(\nabla u(x))\, dx & \text{if } u \in \mathcal{A}_\varepsilon(\Omega; \mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

Then the following result holds.

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary and assume* (h1)-(h4). *Then,* $(\mathcal{F}^g_\varepsilon)$ Γ*-converges with respect to both the convergence in measure and strong* $L^1\left(\Omega;\mathbb{R}^N\right)$ *to the functional* $\mathcal{F}^g : L^1\left(\Omega;\mathbb{R}^N\right) \to [0,+\infty]$ *defined by*

$$\mathcal{F}^g(u) := \begin{cases} \displaystyle\int_\Omega \psi(\nabla u)dx + \int_{J_u} g\left(u^+ - u^-, \nu_u\right) \, d\mathcal{H}^2 & \text{if } u \in \left(GSBV^p(\Omega)\right)^N \\ +\infty & \text{otherwise,} \end{cases}$$

where $g : \mathbb{R}^N \times \mathbb{S}^1 \to [0,+\infty)$ *is defined by*

$$g(z,\nu) := \sum_{\ell=1}^{3} g_\ell\,(z)\, |\langle \nu, \mathrm{e}_\ell\rangle|.$$

Remark 3.2 *For instance, if we assume that*

$$h(|z|) \vee c_1 \leq g_\ell(z) \leq c_2\,(1 + h(|z|)) \tag{3.4}$$

with $h : [0,+\infty) \to [0,+\infty)$ *an increasing function and* c_1*,* c_2 *positive constants, then* (h4) *is satisfied.*

In case $h(t) = t$ *the control from above in* (3.4) *is automatically satisfied as noticed in (3.2). Moreover, the additional control from below implies that the domain of the limit functional* $\mathcal{F}^g$ *is contained in* $\left(GSBV^p(\Omega)\right)^N \cap SBV\left(\Omega;\mathbb{R}^N\right)$.

3.1 Γ-liminf inequality

In this subsection we prove the lower bound inequality for Theorem 3.1. It will be deduced by a more general result, proved in Proposition 3.3 below, holding true in case the functions g_ℓ, used in the definition of the functionals $\mathcal{F}^g_\varepsilon$, all satisfy the following milder growth condition:

(h5) g_ℓ, $\ell = 1,2,3$, is a lower semicontinuous, symmetric and subadditive function such that for every $z \in \mathbb{R}^N \setminus \{0\}$

$$g_\ell(z) \geq \gamma(|z|), \tag{3.5}$$

where $\gamma : (0,+\infty) \to (0,+\infty]$ is a lower semicontinuous, increasing and subadditive function satisfying

$$\lim_{t\to 0^+} \frac{\gamma(t)}{t} = +\infty. \tag{3.6}$$

Notice that we recover (h2) choosing γ in (h5) to be constant.

Let us then prove a lower bound inequality in case the functions g_ℓ satisfy (h5) instead of (h2).

Proposition 3.3 *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set, assume* (h1) *and* (h5)*. Then, for any $u \in L^1\left(\Omega;\mathbb{R}^N\right)$*

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}^g_\varepsilon(u) \geq \mathcal{F}^g(u).$$

Proof. Let $(u_j) \subset \mathcal{A}_{\varepsilon_j}\left(\Omega;\mathbb{R}^N\right)$ and $u \in L^1\left(\Omega;\mathbb{R}^N\right)$ be such that $u_j \to u$ in measure. Moreover, assume that $\liminf_j \mathcal{F}^g_{\varepsilon_j}(u_j) = \lim_j \mathcal{F}^g_{\varepsilon_j}(u_j) < +\infty$. Consider the sets $\mathcal{N}^r_{\varepsilon_j} \subseteq \mathcal{T}^r_{\varepsilon_j}$ defined by

$$\mathcal{N}^r_{\varepsilon_j} := \{(\alpha+\varepsilon_j T_r) \in \mathcal{T}^r_{\varepsilon_j}:\ |\nabla u_j||_{(\alpha+\varepsilon_j T_r)} > \lambda_{\varepsilon_j}\}; \tag{3.7}$$

then, by taking into account the growth condition $g_\ell \geq \gamma$, $\ell = 1,2,3$, the subadditivity and the monotonicity of γ we get

$$\sum_{r=1}^{6} \sup_j \left(\varepsilon_j^2 \gamma\left(\varepsilon_j \lambda_{\varepsilon_j}\right) \#\mathcal{N}^r_{\varepsilon_j}\right) < +\infty. \tag{3.8}$$

In order to prove the Γ-liminf inequality we will show that the sets $\mathcal{N}^r_{\varepsilon_j}$ in (3.7) detect the jump set of u. Thus, we will divide $\mathcal{F}^g_{\varepsilon_j}$ into two terms contributing separately to the bulk and surface energies of the limit functional.

Step 1.(Bulk energy inequality) According to the scheme stated above we construct a sequence $(v_j) \subset SBV\left(\Omega;\mathbb{R}^N\right)$ such that $v_j \to u$ in measure, (v_j) satisfies locally all the assumptions of Theorem 2.8 and, with fixed $\eta > 0$, we have

$$\int_{\cup_{r=1}^6 (\mathcal{T}^r_{\varepsilon_j} \setminus \mathcal{N}^r_{\varepsilon_j})} \psi^g_{\varepsilon_j}\left(\nabla u(x)\right) dx \geq \int_{\Omega_\eta} \psi(\nabla v_j)\, dx + o(1) \tag{3.9}$$

for j sufficiently large. Indeed, let $v_j : \Omega \to \mathbb{R}^N$ be the function whose components are piecewise affine, uniquely determined by

$$v_j(x) := \begin{cases} u_j(x) & x \notin \cup_{r=1}^6 \mathcal{N}^r_{\varepsilon_j} \\ \\ u_j(\alpha) & x \in \alpha + \varepsilon_j T_r, \\ & (\alpha+\varepsilon_j T_r) \in \cup_{r=1}^6 \mathcal{N}^r_{\varepsilon_j} \end{cases} \tag{3.10}$$

It is easy to check that $v_j \to u$ in measure and there holds

$$\int_{\cup_{r=1}^6 (\mathcal{T}^r_{\varepsilon_j} \setminus \mathcal{N}^r_{\varepsilon_j})} \psi^g_{\varepsilon_j}\left(\nabla u(x)\right) dx = \int_{\cup_{r=1}^6 \mathcal{T}^r_{\varepsilon_j}} \psi\left(\nabla v_j\right) dx - \frac{\varepsilon_j^3}{6}\psi(0)\#(\cup_{r=1}^6 \mathcal{N}^r_{\varepsilon_j}).$$

Since, for j sufficiently large $\Omega_\eta \subseteq \cup_{r=1}^6 \mathcal{T}^r_{\varepsilon_j}$, by taking into account (3.8), the superlinearity of γ in 0 and the choice of λ_{ε_j}, we get (3.9).

Conditions (3.1) and (3.9) yield

$$\int_{\Omega_\eta} |\nabla v_j|^p\, dx \leq c\mathcal{F}^g_{\varepsilon_j}(u_j), \tag{3.11}$$

for some positive constant c. Moreover, notice that

$$J_{v_j} \subseteq \cup_{r=1}^{6} \cup_{\mathcal{N}_{\varepsilon_j}^r} \partial(\alpha + \varepsilon_j T_r)$$

and that $\left(v_j^+ - v_j^-\right)|_{(\alpha+\varepsilon_j T_r)}$ is a convex combination of the finite-differences computed in the nodes of the tetrahedron $(\alpha + \varepsilon_j T_r)$ belonging to $\mathcal{N}_{\varepsilon_j}^r$.

Therefore by using the subadditivity and the monotonicity of γ it is easy to check that

$$\int_{\Omega_\eta \cap J_{v_j}} \gamma\left(|[v_j]|\right) \, d\mathcal{H}^2 \leq c\mathcal{F}_{\varepsilon_j}^g(u_j), \tag{3.12}$$

for some positive constant c. Hence, the sequences $(\langle v_j, e_k \rangle)$, $k = 1, \ldots, N$, satisfy all the assumptions of Theorem 2.8 on Ω_η, so that $u \in (GSBV(\Omega))^N$ for every $\eta > 0$ and there holds

$$\int_{\Omega_\eta} |\nabla u|^p \, dx \leq \liminf_j \int_{\Omega_\eta} |\nabla v_j|^p \, dx,$$

$$\int_{\Omega_\eta \cap J_u} \gamma\left(|[u]|\right) \, d\mathcal{H}^2 \leq \liminf_j \int_{\Omega_\eta \cap J_{v_j}} \gamma\left(|[v_j]|\right) \, d\mathcal{H}^2.$$

The last two inequalities together with conditions (3.11) and (3.12) yield $u \in (GSBV(\Omega))^N$.

Eventually, by applying the lower semicontinuity result of Theorem 2.8 in (3.9), and then by passing to the limit on $\eta \to 0^+$, we get

$$\liminf_j \int_{\cup_{r=1}^6 (\mathcal{T}_{\varepsilon_j}^r \setminus \mathcal{N}_{\varepsilon_j}^r)} \psi_{\varepsilon_j}^g \left(\nabla u(x)\right) \, dx \geq \int_\Omega \psi\left(\nabla u\right) \, dx. \tag{3.13}$$

Step 2.(Surface energy inequality) With fixed $\ell = 1, 2, 3$ we will prove the following inequality

$$\liminf_j \frac{1}{\varepsilon_j} \int_{\cup_{r=1}^6 \mathcal{N}_{\varepsilon_j}^r} g_\ell\left(\varepsilon_j \nabla u(x) \mathrm{e}_\ell\right) \, dx \geq \int_{J_u} g_\ell\left([u]\right) |\langle \nu_u, \mathrm{e}_\ell \rangle| \, d\mathcal{H}^2. \tag{3.14}$$

To this aim, for any $r = 1, \ldots, 6$, we construct a sequence $(v_j^{\ell,r}) \subset SBV\left(\Omega; \mathbb{R}^N\right)$ with one-dimensional profile along e_ℓ, which is locally pre-compact in SBV in this given direction, but in general not globally in $GSBV$. More precisely, let $p_{\ell,r}$ be the unique vertex in T_r such that $p_{\ell,r} + \mathrm{e}_\ell \in T_r$ and define

$$v_j^{\ell,r}(x) := \begin{cases} u_j(\alpha + \varepsilon_j p_{\ell,r}) & \begin{array}{l} x \in \left(\alpha + \varepsilon_j [0,1)^3\right) \cap \Omega \\ (\alpha + \varepsilon_j T_r) \in \mathcal{N}_{\varepsilon_j}^r \end{array} \\ u_j(\alpha + \varepsilon_j p_{\ell,r}) + \nabla u_j(x) \mathrm{e}_\ell \langle x - \alpha - \varepsilon_j p_{\ell,r}, \mathrm{e}_\ell \rangle & \begin{array}{l} x \in \left(\alpha + \varepsilon_j [0,1)^3\right) \cap \Omega \\ (\alpha + \varepsilon_j T_r) \notin \mathcal{N}_{\varepsilon_j}^r \end{array} \end{cases},$$

then $\left(v_j^{\ell,r}\right) \subset SBV\left(\Omega;\mathbb{R}^N\right)$ and $v_j^{\ell,r} \to u$ in measure. Notice that, with fixed $\eta > 0$, by (3.1) we get

$$\int_{\Omega_\eta} \left| \frac{\partial v_j^{\ell,r}}{\partial \mathrm{e}_\ell} \right|^p dx \leq c\mathcal{F}_{\varepsilon_j}^g(u_j), \tag{3.15}$$

for some positive constant c. Moreover, since $\nu_{v_j^{\ell,r}} \in \{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ $\mathcal{H}^2$ a.e. in $J_{v_j^{\ell,r}}$ there holds

$$\begin{aligned}\frac{1}{\varepsilon_j}\int_{\mathcal{N}_{\varepsilon_j}^r} g_\ell\left(\varepsilon_j \nabla u_j(x)\mathrm{e}_\ell\right) dx &\geq \frac{1}{6}\int_{\Omega_\eta \cap J_{v_j^{\ell,r}}} g_\ell\left(\left[v_j^{\ell,r}\right]\right) \left|\langle \nu_{v_j^{\ell,r}}, \mathrm{e}_\ell\rangle\right| d\mathcal{H}^2 \\ &= \frac{1}{6}\int_{\Pi^{\mathrm{e}_\ell}} d\mathcal{H}^2 \int_{\Omega_\eta \cap (J_{v_j^{\ell,r}})_y^{\mathrm{e}_\ell}} g_\ell\left(\left[\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell,y}\right]\right) d\mathcal{H}^0,\end{aligned} \tag{3.16}$$

where the last equality follows by using the characterization of BV functions through their one-dimensional sections (see Section 3.11 [4]) and the generalized coarea formula for rectifiable sets (see Section 3.2.22 [14]). By passing to the limit on $j \to +\infty$ in (3.16) and by applying Fatou's lemma we have

$$\begin{aligned}&\liminf_j \frac{1}{\varepsilon_j}\int_{\mathcal{N}_{\varepsilon_j}^r} g_\ell\left(\varepsilon_j \nabla u_j(x)\mathrm{e}_\ell\right) dx \\ &\quad\geq \frac{1}{6}\int_{\Pi^{\mathrm{e}_\ell}} d\mathcal{H}^2 \liminf_j \int_{\Omega_\eta \cap (J_{v_j^{\ell,r}})_y^{\mathrm{e}_\ell}} g_\ell\left(\left[\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell,y}\right]\right) d\mathcal{H}^0,\end{aligned}$$

hence we deduce that for $\mathcal{H}^2$ a.e. $y \in \Pi^{\mathrm{e}_\ell}$ there holds

$$\liminf_j \int_{\Omega_\eta \cap (J_{v_j^{\ell,r}})_y^{\mathrm{e}_\ell}} g_\ell\left(\left[\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell,y}\right]\right) d\mathcal{H}^0 < +\infty. \tag{3.17}$$

Thus, for $\mathcal{H}^2$ a.e. $y \in \Pi^{\mathrm{e}_\ell}$, up to extracting subsequences depending on such a fixed y, we may assume $\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell,y} \to u^{\mathrm{e}_\ell,y}$ in measure on $(\Omega_\eta)_y^{\mathrm{e}_\ell}$, the inferior limit in (3.17) to be a limit and, by taking into account (3.15), also that

$$\sup_j \int_{(\Omega_\eta)_y^{\mathrm{e}_\ell}} \left|\left(\dot{v}_j^i\right)^{\mathrm{e}_\ell,y}\right|^p dt < +\infty.$$

Hence, the slices $\left(\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell,y}\right)$ satisfy on $(\Omega_\eta)_y^{\mathrm{e}_\ell}$ all the assumptions of Theorem 2.8, so that, by Theorem 2.10, we have

$$\begin{aligned}&\liminf_j \frac{1}{\varepsilon_j}\int_{\cup_{r=1}^6 \mathcal{N}_{\varepsilon_j}^r} g_\ell\left(\varepsilon_j \nabla u(x)\mathrm{e}_\ell\right) dx \\ &\quad\geq \frac{1}{6}\int_{\Pi^{\mathrm{e}_\ell}} d\mathcal{H}^2 \liminf_j \int_{\Omega_\eta \cap (J_{v_j^{\ell,r}})_y^{\mathrm{e}_\ell}} g_\ell\left(\left[\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell,y}\right]\right) d\mathcal{H}^0\end{aligned}$$

$$\geq \frac{1}{6}\int_{\Pi^{\mathrm{e}_\ell}} d\mathcal{H}^2 \int_{\Omega_\eta\cap (J_u)_y^{\mathrm{e}_\ell}} g_\ell\left([u^{\mathrm{e}_\ell,y}]\right) d\mathcal{H}^0$$
$$= \frac{1}{6}\int_{\Omega_\eta\cap J_u} g_\ell\left([u]\right) |\langle \nu_u, \mathrm{e}_\ell\rangle| \, d\mathcal{H}^2.$$

We deduce (3.14) passing to the limit on $\eta \to 0^+$ and using the subadditivity of the inferior limit.

To conclude it suffices to collect (3.13) and (3.14). □

Remark 3.4 *We claim that, by proceeding as in Step 1 of the proof of Proposition 3.3, one can prove that the families of functions* $(u_\varepsilon) \subseteq \mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right)$ *such that*

$$\sup_{\varepsilon>0}\{\mathcal{F}^g_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_{1,\Omega}\} < +\infty$$

are pre-compact in $L^1\left(\Omega;\mathbb{R}^N\right)$*. Indeed, to get the result it suffices to apply the GSBV Compactness theorem (see Theorem 2.2 [2]) to the family* (v_ε) *constructed in (3.10).*

3.2 Γ-limsup inequality

In this subsection we prove the upper bound inequality for Theorem 3.1.

Proposition 3.5 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded open set with Lipschitz boundary, assume* (h1)-(h4)*. Then, for any* $u \in L^1\left(\Omega;\mathbb{R}^N\right)$*,*

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon\to 0^+} \mathcal{F}^g_\varepsilon(u) \leq \mathcal{F}^g(u).$$

Proof. It suffices to prove the inequality above for $u \in (GSBV^p(\Omega))^N$ such that $\mathcal{F}^g(u) < +\infty$. We will reduce ourselves to prove the inequality first for a class of more regular functions.

Step 1. Let Ω' be an open set such that $\Omega' \supset\supset \Omega$ and suppose u is such that

(i) $\mathcal{H}^{n-1}\left(\overline{S_u}\setminus S_u\right) = 0$;

(ii) there exists a finite number of polyhedral sets K^r such that

$$\overline{S_u} = \Omega' \cap \cup_{h=1}^M K^h,$$

and for every $1 \leq h \leq M$ the set K^h is contained in a hyperplane Π^h with normal ν_h and $\Pi^h \neq \Pi^s$ for $h \neq s$;

(iii) $u \in W^{k,\infty}\left(\Omega\setminus\overline{S_u};\mathbb{R}^N\right)$ for every $k \in \mathbb{N}$.

Let us first fix some notations. With fixed $m \in \mathbb{N} \setminus \{0\}$, let

$$J_u^m := \left\{ x \in J_u : \left| u^+(x) - u^-(x) \right| \geq \frac{1}{m} \right\},$$

$\{J_u^m\}$ is an increasing family of sets such that $J_u = \cup_{m \in \mathbb{N} \setminus \{0\}} J_u^m$ and so

$$\lim_{m \to +\infty} \mathcal{H}^2(J_u^m) = \mathcal{H}^2(J_u).$$

Moreover, let $J := \overline{J_u}$ and define the sets

$$\mathcal{J}_\varepsilon^r := \cup_{\ell=1,2,3} \left\{ \alpha + \varepsilon T_r : \alpha \in \varepsilon\mathbb{Z}^3,\ \alpha + \varepsilon[p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell] \cap J \neq \emptyset \right\},$$

and

$$\mathcal{J}_{m,\varepsilon}^r := \cup_{\ell=1,2,3} \left\{ \alpha + \varepsilon T_r : \alpha \in \varepsilon\mathbb{Z}^3,\ \alpha + \varepsilon[p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell] \cap J_u^m \neq \emptyset \right\},$$

for $m \in \mathbb{N} \setminus \{0\}$, where the points $p_{\ell,r}$ hsve been defined in the proof of Proposition 3.3. Up to infinitesimal traslations we may assume that $J \cap \varepsilon\mathbb{Z}^3 = \emptyset$ for every $\varepsilon > 0$, then let u_ε be the continuous piecewise affine interpolation of the values $u(\alpha)$ with $\alpha \in \varepsilon\mathbb{Z}^3 \cap \Omega'$. Notice that $u_\varepsilon \in \mathcal{A}_\varepsilon\left(\Omega; \mathbb{R}^N\right)$ and $u_\varepsilon \to u$ strongly in $L^1\left(\Omega; \mathbb{R}^N\right)$. Denote as usual

$$\mathcal{N}_\varepsilon^r := \left\{ (\alpha + \varepsilon T_r) \in \mathcal{T}_\varepsilon^r : |\nabla u_\varepsilon||_{(\alpha+\varepsilon T_r)} > \lambda_\varepsilon \right\}.$$

By taking into account the BV "slicing theorem" (see Section 3.11 [4]) we have for $\ell = 1, 2, 3$ and for $x \in (\alpha + \varepsilon T_r)$

$$\begin{aligned} &\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell = u(\alpha + \varepsilon p_{\ell,r} + \varepsilon \mathrm{e}_\ell) - u(\alpha + \varepsilon p_{\ell,r}) \\ &= \int_0^\varepsilon \nabla u\left(\alpha + \varepsilon p_{\ell,r} + t\mathrm{e}_\ell\right) \mathrm{e}_\ell \, dt + \sum_{y \in J_u \cap (\alpha + \varepsilon[p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell))} [u](y) sgn\left(\langle \nu_u(y), \mathrm{e}_\ell \rangle\right). \end{aligned} \tag{3.18}$$

Thus, if $(\alpha + \varepsilon T_r) \in \mathcal{T}_\varepsilon^r \setminus \mathcal{J}_\varepsilon^r$, for any $x \in (\alpha + \varepsilon T_r)$, we have

$$\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell = \int_0^\varepsilon \nabla u\left(\alpha + \varepsilon p_{\ell,r} + t\mathrm{e}_\ell\right) \mathrm{e}_\ell \, dt, \tag{3.19}$$

for $\ell = 1, 2, 3$, from which it follows $|\nabla u_\varepsilon||_{(\alpha+\varepsilon T_r)} \leq \|\nabla u\|_{\infty,\Omega'}$. Define the vector fields $W_\varepsilon : \Omega \to \mathbb{R}^{N \times 3}$ by

$$W_\varepsilon(x) := \left(\frac{1}{\varepsilon} \int_0^\varepsilon \nabla u\left(\alpha + \varepsilon p_{\ell,r} + t\mathrm{e}_\ell\right) \mathrm{e}_\ell \, dt \right)_{\ell=1,2,3},$$

for $x \in \cup_{r=1}^6 T_\varepsilon^r \supseteq \Omega$. Then, $W_\varepsilon \to \nabla u$ strongly in $L^p\left(\Omega; \mathbb{R}^{N\times 3}\right)$ and by (3.19) it follows

$$\begin{aligned} &\limsup_{\varepsilon \to 0^+} \int_{\cup_{r=1}^6 (\mathcal{T}_\varepsilon^r \setminus \mathcal{J}_\varepsilon^r)} \psi_\varepsilon^g\left(\nabla u_\varepsilon(x)\right) \, dx = \limsup_{\varepsilon \to 0^+} \int_{\cup_{r=1}^6 (\mathcal{T}_\varepsilon^r \setminus \mathcal{J}_\varepsilon^r)} \psi\left(W_\varepsilon(x)\right) \, dx \\ &\leq \lim_{\varepsilon \to 0^+} \int_\Omega \psi\left(W_\varepsilon(x)\right) \, dx = \int_\Omega \psi\left(\nabla u(x)\right) \, dx, \end{aligned} \tag{3.20}$$

the last equality holding by the growth condition (3.1).

Consider the decomposition $\mathcal{J}_\varepsilon^r = (\mathcal{J}_\varepsilon^r \cap \mathcal{N}_\varepsilon^r) \cup (\mathcal{J}_\varepsilon^r \setminus \mathcal{N}_\varepsilon^r)$, then it follows

$$\int_{\cup_{r=1}^6 \mathcal{J}_\varepsilon^r} \psi_\varepsilon^g (\nabla u_\varepsilon(x))\, dx = \sum_{\ell=1}^3 \frac{1}{\varepsilon} \int_{\cup_{r=1}^6 (\mathcal{J}_\varepsilon^r \cap \mathcal{N}_\varepsilon^r)} g_\ell (\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell)\, dx$$

$$+ \int_{\cup_{r=1}^6 (\mathcal{J}_\varepsilon^r \setminus \mathcal{N}_\varepsilon^r)} \psi (\nabla u_\varepsilon(x))\, dx. \tag{3.21}$$

Let us estimate separately the two terms in (3.21) above.

Let $\overline{B} := \overline{B}_{(0, \|\nabla u\|_{\infty,\Omega} + 2M\|u\|_{\infty,\Omega})}$, then, since $\sup_{\overline{B}} g_\ell < +\infty$, for every $m \in \mathbb{N} \setminus \{0\}$ it follows

$$\sum_{\ell=1}^3 \frac{1}{\varepsilon} \int_{\cup_{r=1}^6 ((\mathcal{J}_\varepsilon^r \setminus \mathcal{J}_{m,\varepsilon}^r) \cap \mathcal{N}_\varepsilon^r)} g_\ell (\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell)\, dx \tag{3.22}$$

$$\leq c \frac{1}{\varepsilon} \mathcal{L}^3 \left(\{ x : \mathrm{dist}\,(x, J \setminus J_u^m) \leq \sqrt{3}\varepsilon \} \right) \leq c\mathcal{H}^2 \left(\overline{J_u \setminus J_u^m} \right) + o(1),$$

the last term being infinitesimal as $m \to +\infty$.

Moreover, let $\omega_m : [0, +\infty) \to [0, +\infty)$ be the maximum of the moduli of continuity of g_ℓ, $\ell = 1, 2, 3$, on the compact set $\overline{B} \setminus B_{(0, \frac{1}{m})}$, then for ε small enough we get by (3.18)

$$\frac{1}{\varepsilon} \int_{\cup_{r=1}^6 (\mathcal{J}_{m,\varepsilon}^r \cap \mathcal{N}_\varepsilon^r)} g_\ell (\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell)\, dx$$

$$\leq \frac{1}{\varepsilon} \sum_{r=1}^6 \sum_{\mathcal{J}_{m,\varepsilon}^r} \int_{\alpha + \varepsilon T_r} \left(\sum_{y \in J_u \cap (\alpha + \varepsilon [p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell))} g_\ell ([u](y)) \right) dx$$

$$+ \omega_m (\varepsilon \|\nabla u\|_{\infty,\Omega'}) \frac{\varepsilon^2}{6} \# \left(\cup_{r=1}^6 \mathcal{J}_{m,\varepsilon}^r \right), \tag{3.23}$$

the last inequality holding by the subadditivity and the symmetry of g_ℓ, $\ell = 1, 2, 3$. It can be proved that, by the regularity assumptions (i)-(iii) on u and the continuity of g_ℓ on $\mathbb{R}^N \setminus \{0\}$, we have

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\mathcal{J}_{m,\varepsilon}^r} \left(\sum_{y \in J_u \cap (\alpha + \varepsilon [p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell))} g_\ell ([u](y)) \right) dx$$

$$\leq \frac{1}{6} \int_J g_\ell ([u]) |\langle \nu_u, \mathrm{e}_\ell \rangle| \, d\mathcal{H}^2. \tag{3.24}$$

Hence, by (3.23) and (3.24) we infer

$$\limsup_{\varepsilon \to 0^+} \sum_{\ell=1}^3 \frac{1}{\varepsilon} \int_{\cup_{r=1}^6 (\mathcal{J}_{m,\varepsilon}^r \cap \mathcal{N}_\varepsilon^r)} g_\ell (\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell)\, dx \leq \int_{J_u} g([u], \nu_u)\, d\mathcal{H}^2. \tag{3.25}$$

By collecting (3.22), (3.25) and since $\mathcal{H}^2\left(\overline{J_u}\setminus J_u\right)=0$ by passing to the limit on $m\to+\infty$ we get

$$\limsup_{\varepsilon\to0^+}\sum_{\ell=1}^{3}\frac{1}{\varepsilon}\int_{\cup_{r=1}^{6}(\mathcal{J}_\varepsilon^r\cap\mathcal{N}_\varepsilon^r)} g_\ell\left(\varepsilon\nabla u_\varepsilon(x)\mathrm{e}_\ell\right)\,dx\le\int_{J_u} g\left([u],\nu_u\right)\,d\mathcal{H}^2. \tag{3.26}$$

In order to estimate the second term in (3.21), notice that by (3.1) there holds

$$\int_{\cup_{r=1}^{6}(\mathcal{J}_\varepsilon^r\setminus\mathcal{N}_\varepsilon^r)}\psi\left(\nabla u_\varepsilon(x)\right)\,dx\le C_2\frac{\varepsilon^3}{6}\left(1+\lambda_\varepsilon^p\right)\#\left(\mathcal{J}_\varepsilon^r\setminus\mathcal{N}_\varepsilon^r\right),$$

and the term on the right hand side above is infinitesimal as $\varepsilon\to0^+$. Indeed, with fixed $m\in\mathbb{N}\setminus\{0\}$, arguing as in (3.22) we deduce

$$\limsup_{\varepsilon\to0^+}\varepsilon^2\#\left(\mathcal{J}_\varepsilon^r\setminus\mathcal{N}_\varepsilon^r\right)\le\limsup_{\varepsilon\to0^+}\varepsilon^2\#\left(\mathcal{J}_\varepsilon^r\setminus\mathcal{J}_{m,\varepsilon}^r\right)\le c\mathcal{H}^2\left(\overline{J_u\setminus J_u^m}\right). \tag{3.27}$$

Hence, by assumption $\sup_{\varepsilon>0}\left(\varepsilon\lambda_\varepsilon^p\right)<+\infty$, (3.27) and by letting $m\to+\infty$, we have that

$$\limsup_{\varepsilon\to0^+}\int_{\cup_{r=1}^{6}(\mathcal{J}_\varepsilon^r\setminus\mathcal{N}_\varepsilon^r)}\psi\left(\nabla u_\varepsilon(x)\right)\,dx=0 \tag{3.28}$$

Eventually, by collecting (3.20), (3.26) and (3.28) we get the conclusion, i.e.,

$$\limsup_{\varepsilon\to0^+}\mathcal{F}_\varepsilon^g(u_\varepsilon)\le\int_\Omega\psi\left(\nabla u(x)\right)\,dx+\int_{J_u} g\left([u],\nu_u\right)\,d\mathcal{H}^2.$$

Step 2. Assume $u\in\left(GSBV^p(\Omega)\right)^N\cap L^\infty\left(\Omega;\mathbb{R}^N\right)$. By taking into account the SBV Extension Theorem (see Lemma 4.11 [5]), with fixed an open and bounded set $\Omega'\supset\supset\Omega$, there exists a function $\tilde{u}\in SBV\cap L^\infty\left(\Omega';\mathbb{R}^N\right)$ such that $\tilde{u}|_\Omega\equiv u$, $\nabla\tilde{u}\in L^p\left(\Omega';\mathbb{R}^{N\times2}\right)$, $\mathcal{H}^2\left(J_{\tilde{u}}\right)<+\infty$ and $\mathcal{H}^2\left(\partial\Omega\cap J_{\tilde{u}}\right)=0$.

By Theorem 3.1 [11] there exists a sequence (u_j) satisfying condition (i)-(iii) of *Step 1* such that $u_j\to u$ in $L^1\left(\Omega;\mathbb{R}^N\right)$, and, since the continuity and local boundedness of g, there holds

$$\lim_{j\to+\infty}\int_{\Omega\cap J_{u_j}} g\left([u_j],\nu_{u_j}\right)\,d\mathcal{H}^2=\int_{J_u} g\left([u],\nu_u\right)\,d\mathcal{H}^2.$$

Hence, by *Step 1* and the lower semicontinuity of the upper Γ-limit we conclude.

Step 3. Let $u\in\left(GSBV^p(\Omega)\right)^N$, recall that $u_T=\left((-T)\wedge\langle u,e_k\rangle\vee T\right)_{k=1,\dots,N}$ for any $T>0$. Then, $u_T\in\left(GSBV^p(\Omega)\right)^N\cap L^\infty\left(\Omega;\mathbb{R}^N\right)$ and $J_{u_T}\subseteq J_u$; moreover, by Theorem 4.40 [4] there holds

$$\mathcal{H}^2\left(\left\{x\in J_u:|u^\pm(x)|=+\infty\right\}\right)=0.$$

Hence, $\lim_{T\to+\infty}g_\ell\left([u_T](x)\right)=g_\ell\left([u](x)\right)$ for $\mathcal{H}^2$ a.e. $x\in J_u$, $\ell=1,2,3$. Then by assumption (h4) we may apply the Dominated Convergence Theorem and conclude. □

Remark 3.6 *Let us point out that the assumption* (h4) *is technical and needed only to recover the limsup estimate on* $(GSBV^p(\Omega))^N \setminus SBV(\Omega;\mathbb{R}^N)$*. Indeed, assume* $u \in SBV(\Omega;\mathbb{R}^N)$ *to be such that* $\mathcal{F}^g(u) < +\infty$*, by following the notations of Step 3 in Proposition 3.5 above and by taking into account (3.2) we get*

$$c_1 \leq g\left([u_T], \nu_{u_T}\right) \leq 2c_2\left(1 + |[u]|\right).$$

Moreover, since $u \in SBV(\Omega;\mathbb{R}^N)$ *implies* $|[u]| \in L^1(J_u;\mathcal{H}^2)$ *we have*

$$\lim_{T\to+\infty} \int_{J_u} g\left([u_T], \nu_{u_T}\right) \, d\mathcal{H}^2 = \int_{J_u} g\left([u], \nu_u\right) \, d\mathcal{H}^2.$$

Hence, in this case (h4) *is automatically satisfied.*

4 Discrete approximations in dimension 2

In this section we treat the two dimensional case. We provide two different approximation results. The first one is the transposition of Theorem 3.1 in dimension $n = 2$ for a fixed regular partition of the square $[0,1]^2$. The proof works using the same techniques of Theorem 3.1. Actually, the result is independent on the choice of the regular triangulation, in the sense that one may assign on each square $\alpha + \varepsilon[0,1]^2$, $\alpha \in \varepsilon\mathbb{Z}^2$, one among the two possible partitions (see Figure 2 below).

Figure 2: a random triangulation of $\mathbb{R}^2$

The second result is a slight variant of Theorem 3.1, but the surface term depends heavily on the assigned triangulation (for simplicity we choose the one in Figure 3).

Figure 3: regular partition of the square

The anisotropy induced by the model can be computed by means of the function φ of Lemma 4.4. To deal with this model more sophisticated tools need to be used.

Let $S_1 = co\{0, \mathrm{e}_1, \mathrm{e}_2\}$, $S_2 = co\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_2\}$ and define for $r = 1, 2$

$$\mathcal{T}_\varepsilon^r(A) := \{\alpha + \varepsilon S_r : \alpha + \varepsilon S_r \subseteq A,\, \alpha \in \varepsilon\mathbb{Z}^2\},$$

for $A \in \mathcal{A}(\Omega)$ and Ω a bounded open subset of $\mathbb{R}^2$. In case $A = \Omega$ we will drop the dependence on Ω in the definition above.

In the following we will use the same notations and assumptions (h1)-(h4) of Section 3 suitably changed according to the two dimensional setting.

Consider the family of functionals $\mathcal{F}_\varepsilon^g : L^1\left(\Omega;\mathbb{R}^N\right) \to [0,+\infty]$ given by

$$\mathcal{F}_\varepsilon^g(u) := \begin{cases} \displaystyle\int_{\mathcal{T}_\varepsilon^1\cup\mathcal{T}_\varepsilon^2} \psi_\varepsilon^g\left(\nabla u(x)\right) dx & \text{if } u \in \mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right) \\ +\infty & \text{otherwise} \end{cases}.$$

Then the following result holds.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and assume* (h1)-(h4)*. Then $(\mathcal{F}_\varepsilon^g)$ Γ-converges with respect to both the convergence in measure and strong $L^1\left(\Omega;\mathbb{R}^N\right)$ to the functional $\mathcal{F}^g : L^1\left(\Omega;\mathbb{R}^N\right) \to [0,+\infty]$ defined by*

$$\mathcal{F}^g(u) := \begin{cases} \displaystyle\int_\Omega \psi(\nabla u)dx + \int_{J_u} g\left(u^+ - u^-, \nu_u\right) d\mathcal{H}^1 & \textit{if } u \in \left(GSBV^p(\Omega)\right)^N \\ +\infty & \textit{otherwise,} \end{cases}$$

where $g : \mathbb{R}^N \times \mathbb{S}^1 \to [0,+\infty)$ is defined by

$$g(z,\nu) := \sum_{\ell=1}^{2} g_\ell\left(z\right)|\langle \nu, \mathrm{e}_\ell\rangle|.$$

Let us consider now the function $\psi_\varepsilon^\beta : \mathbb{R}^{N\times 2} \to [0,+\infty)$ defined by

$$\psi_\varepsilon^\beta\left(X\right) := \begin{cases} \psi\left(X\right) & \text{if } |X| \le \lambda_\varepsilon \\ \frac{1}{\varepsilon}\beta & \text{otherwise,} \end{cases}$$

where β is a positive constant. Note that even in case $g_1 = g_2 = \frac{\beta}{2}$ the functions ψ_ε^g, ψ_ε^β are different, since ψ_ε^g takes into account the values $X\mathrm{e}_1$, $X\mathrm{e}_2$ separately. If in the definition of the family $\mathcal{F}_\varepsilon^g$, ψ_ε^g is substituted by ψ_ε^β, we prove the following result for the corresponding family of functionals $(\mathcal{F}_\varepsilon^\beta)$.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and assume* (h1)*. Then $(\mathcal{F}_\varepsilon^\beta)$ Γ-converges with respect to both the convergence in measure and strong $L^1\left(\Omega;\mathbb{R}^N\right)$ to the functional $\mathcal{F}^\beta : L^1\left(\Omega;\mathbb{R}^N\right) \to [0,+\infty]$ defined by*

$$\mathcal{F}^\beta(u) := \begin{cases} \displaystyle\int_\Omega \psi(\nabla u)dx + \beta\int_{J_u} \varphi(\nu_u)\, d\mathcal{H}^1 & \textit{if } u \in \left(GSBV^p(\Omega)\right)^N \\ +\infty & \textit{otherwise} \end{cases}, \tag{4.1}$$

where $\varphi : \mathbb{S}^1 \to [0,+\infty)$ is given by

$$\varphi(\nu) := \begin{cases} |\langle \nu, \mathrm{e}_1\rangle| \vee |\langle \nu, \mathrm{e}_2\rangle| & \textit{if } \langle \nu, \mathrm{e}_1\rangle\langle \nu, \mathrm{e}_2\rangle \ge 0 \\ |\langle \nu, \mathrm{e}_1\rangle| + |\langle \nu, \mathrm{e}_2\rangle| & \textit{if } \langle \nu, \mathrm{e}_1\rangle\langle \nu, \mathrm{e}_2\rangle < 0 \end{cases}. \tag{4.2}$$

We now prove the lower semicontinuity inequality for the family of functionals $\left(\mathcal{F}_\varepsilon^\beta\right)$. To this aim we need to 'localize' the functionals $\mathcal{F}_\varepsilon^\beta$. For every $A \in \mathcal{A}(\Omega)$ and $u \in \mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right)$ let

$$\mathcal{F}_\varepsilon^\beta(u,A) := \begin{cases} \displaystyle\int_{\mathcal{T}_\varepsilon^1(A)\cup\mathcal{T}_\varepsilon^2(A)} \psi_\varepsilon^\beta\left(\nabla u(x)\right)dx & \text{if } u \in \mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right) \\ +\infty & \text{otherwise} \end{cases}.$$

We obtain separate estimates on the bulk and on the surface terms which we 'glue' together by means of Proposition 1.16 [5]. Besides using the same techniques applied in the proof of Proposition 3.3 in the two dimensional case, we will perform an additional construction with profile along the diagonal direction $\mathrm{e}_2 - \mathrm{e}_1$.

Proposition 4.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, assume* (h1)*. Then, for any $u \in L^1\left(\Omega;\mathbb{R}^N\right)$,*

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon^\beta(u) \geq \mathcal{F}^\beta(u).$$

Proof. Let $(u_j) \subset \mathcal{A}_{\varepsilon_j}\left(\Omega;\mathbb{R}^N\right)$ and $u \in L^1\left(\Omega;\mathbb{R}^N\right)$ be such that $u_j \to u$ in measure. By using analogous arguments of those of Proposition 3.3 it is possible to show that for every $A \in \mathcal{A}(\Omega)$ the following inequalities hold true:

$$\liminf_j \int_{\cup_{r=1}^2(\mathcal{T}_\varepsilon^r(A)\setminus\mathcal{N}_{\varepsilon_j}^r)} \psi_\varepsilon^\beta\left(\nabla u(x)\right)\,dx \geq \int_A \psi(\nabla u(x))\,dx \tag{4.3}$$

$$\liminf_j \beta\varepsilon_j \#\left(\mathcal{T}_\varepsilon^r(A)\cap\mathcal{N}_{\varepsilon_j}^r\right) \geq \beta\int_{A\cap J_u} |\langle\nu_u, \mathrm{e}_\ell\rangle|\,d\mathcal{H}^1, \tag{4.4}$$

for $r, \ell = 1, 2$. Thus to conclude it suffices to show that there holds for $r = 1, 2$

$$\liminf_j \beta\varepsilon_j \#\left(\mathcal{T}_\varepsilon^r(A)\cap\mathcal{N}_{\varepsilon_j}^r\right) \geq \beta\int_{A\cap J_u} |\langle\nu_u, \mathrm{e}_2 - \mathrm{e}_1\rangle|\,d\mathcal{H}^1. \tag{4.5}$$

Indeed, for $\ell = 1, 2$, let $\left(\delta_h^\ell\right)_h = \mathbb{Q}\cap[0,1]$, $\delta_h^1 + \delta_h^2 \leq 1$, then by using Proposition 1.16 [5] with,

$$\mu(A) := \liminf\nolimits_j \mathcal{F}_{\varepsilon_j}^\beta(u_j, A),$$
$$\lambda := \mathcal{L}^2 \llcorner (\Omega\setminus J_u) + \mathcal{H}^1 \llcorner J_u,$$
$$\phi_h(x) := \begin{cases} \psi(\nabla u) & \text{on } \Omega\setminus J_u \\ \beta\big(\,\delta_h^1|\langle\nu_u,\mathrm{e}_1\rangle| + \delta_h^2|\langle\nu_u,\mathrm{e}_2\rangle| \\ +\left(1-\delta_h^1-\delta_h^2\right)|\langle\nu_u,\mathrm{e}_2-\mathrm{e}_1\rangle|\big) & \text{on } J_u \end{cases},$$

the statement follows by noticing that if $x \in J_u$ is such that $\langle\nu_u(x),\mathrm{e}_1\rangle\langle\nu_u(x),\mathrm{e}_2\rangle \geq 0$, then

$$|\langle\nu_u(x),\mathrm{e}_2-\mathrm{e}_1\rangle| \leq |\langle\nu_u,\mathrm{e}_1\rangle| \vee |\langle\nu_u,\mathrm{e}_2\rangle|,$$

and if $x \in J_u$ is such that $\langle \nu_u(x), \mathrm{e}_1\rangle\langle \nu_u(x), \mathrm{e}_2\rangle < 0$, then

$$|\langle \nu_u(x), \mathrm{e}_2 - \mathrm{e}_1\rangle| = |\langle \nu_u(x), \mathrm{e}_1\rangle| + |\langle \nu_u(x), \mathrm{e}_2\rangle| \geq |\langle \nu_u, \mathrm{e}_1\rangle| \vee |\langle \nu_u, \mathrm{e}_2\rangle|.$$

To prove inequality (4.5) we will construct, for $r = 1, 2$, a sequence $(w_j^r) \subset SBV\left(\Omega; \mathbb{R}^N\right)$ with one-dimensional profile in $\mathrm{e}_2 - \mathrm{e}_1$ which is locally pre-compact in SBV in this given direction, but not in general globally in $GSBV$. Suppose $\liminf_j \mathcal{F}^\beta_{\varepsilon_j}(u_j) = \lim_j \mathcal{F}^\beta_{\varepsilon_j}(u_j) < +\infty$, consider the sets of triangles $\mathcal{N}^r_{\varepsilon_j} := \{(\alpha + \varepsilon_j S_r) \in \mathcal{T}^r_{\varepsilon_j} : \ |\nabla u_j||_{(\alpha+\varepsilon_j S_r)} > \lambda_{\varepsilon_j}\}$, for $r = 1, 2$, then we get

$$\sup_j \varepsilon_j \# \left(\mathcal{N}^1_{\varepsilon_j} \cup \mathcal{N}^2_{\varepsilon_j}\right) < +\infty. \tag{4.6}$$

Let

$$P_{\varepsilon_j} := \varepsilon_j \left\{x \in \mathbb{R}^2 : x = \lambda(-\mathrm{e}_1) + \mu(\mathrm{e}_2 - \mathrm{e}_1),\ \lambda, \mu \in [0, 1)\right\},$$

and define for $r = 1, 2$ the sequence

$$w_j^r(x) := \begin{cases} u_j(\alpha) & \begin{array}{l} x \in \left(\alpha + P_{\varepsilon_j}\right) \cap \Omega \\ (\alpha + \varepsilon_j S_r) \in \mathcal{N}^r_{\varepsilon_j} \end{array} \\ u_j(\alpha) + \frac{1}{\sqrt{2}} \nabla u_j(x)(\mathrm{e}_2 - \mathrm{e}_1)\langle x - \alpha, \mathrm{e}_2 - \mathrm{e}_1\rangle & \begin{array}{l} x \in \left(\alpha + P_{\varepsilon_j}\right) \cap \Omega \\ (\alpha + \varepsilon_j S_r) \notin \mathcal{N}^r_{\varepsilon_j} \end{array} \end{cases}.$$

Notice the analogy with the construction of $v_j^{\ell,r}$ in Proposition 3.3: in this case the cubic cell $\varepsilon_j[0, 1)^2$ is replaced by the slanted one P_{ε_j}.
We have that $(w_j^r) \subset SBV\left(\Omega; \mathbb{R}^N\right)$, $w_j^r \to u$ in measure and, for every $\eta > 0$ and $A \in \mathcal{A}(\Omega)$, by the growth condition of ψ

$$\int_{A_\eta} \left|\frac{\partial w_j^r}{\partial(\mathrm{e}_2 - \mathrm{e}_1)}\right|^p dx \leq c\mathcal{F}^\beta_{\varepsilon_j}(u_j, A), \tag{4.7}$$

for some positive constant c. Moreover, since $\nu_{w_j^r} \in \{\mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_2\}$ $\mathcal{H}^1$ a.e. in $J_{w_j^r}$ we have

$$\int_{A_\eta \cap J_{w_j^r}} \left|\langle \nu_{w_j^r}, \mathrm{e}_2 - \mathrm{e}_1\rangle\right| d\mathcal{H}^1 \leq \varepsilon_j \# \left(\mathcal{T}^r_{\varepsilon_j}(A) \cap \mathcal{N}^r_{\varepsilon_j}\right). \tag{4.8}$$

Notice that (4.6) together with (4.7), (4.8) for $A = \Omega$ assure that for $\mathcal{H}^1$ a.e. $y \in \Pi^{\mathrm{e}_2-\mathrm{e}_1}$, up to subsequences depending on such a fixed y, the slices $\left(\left(w_j^r\right)^{\mathrm{e}_2-\mathrm{e}_1,y}\right)$ satisfy on $(\Omega_\eta)^{\mathrm{e}_2-\mathrm{e}_1}_y$ assumption (2.8) of Theorem 2.8. Thus, by taking into account Fatou's lemma and Theorem 2.10, by passing to the inferior limit on $j \to +\infty$ in (4.8), we get

$$\frac{\beta}{2} \liminf_j \varepsilon_j \# \left(\mathcal{T}^r_{\varepsilon_j}(A) \cap \mathcal{N}^r_{\varepsilon_j}\right) \geq \frac{\beta}{2} \int_{A_\eta \cap J_u} |\langle \nu_u, \mathrm{e}_2 - \mathrm{e}_1\rangle|\, d\mathcal{H}^1. \tag{4.9}$$

□

The following result will be used in the proof of the limsup inequality. Notice that the ideas and strategy used in the proof are strongly related to the regularity assumptions on the set J.

Lemma 4.4 *Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ and denote by $\xi^\perp := (-\xi_2, \xi_1)$. Let $\nu \in \mathbb{S}^1$ and let $J \subseteq \Pi^\nu$ be a closed set with $\mathcal{H}^1(J) < +\infty$. Define*

$$J_\varepsilon^{\xi,r} := \left\{ \alpha + \varepsilon S_r^\xi : \alpha \in \varepsilon\mathbb{Z}^2,\ \left(\alpha + \varepsilon S_r^\xi\right) \cap J \neq \emptyset \right\},$$

where $S_1^\xi := co\{0, \xi, \xi^\perp\}$ and $S_2^\xi := co\{\xi, \xi^\perp, \xi + \xi^\perp\}$. Then, for $r = 1, 2$, we get

$$\limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left(J_\varepsilon^{\xi,r}\right)}{\varepsilon} \leq \frac{1}{2} \int_J \varphi\left(\nu\right)\, d\mathcal{H}^1,$$

where $\varphi : \mathbb{S}^1 \to [0, +\infty)$ is defined as in (4.2).

Proof. Let $J^\eta := \{x \in \Pi^\nu : d(x, J) < \eta\}$, then there exists a sequence $(\eta_j) \subseteq (0,1)$ such that $\eta_j \to 0^+$ and $\mathcal{H}^1(J^{\eta_j}) = \mathcal{H}^1\left(\overline{J^{\eta_j}}\right) \to \mathcal{H}^1(J)$. It suffices then to prove the assertion for an open set $A \subseteq \Pi^\nu$ essentially closed, i.e., $\mathcal{H}^1(A) = \mathcal{H}^1\left(\overline{A}\right) < +\infty$.

Let $A = \cup_{s \geq 1} A_s$, where A_s are the connected components of A in Π^ν; since for every $M \in \mathbb{N}$

$$\mathcal{L}^2\left(A_\varepsilon^{\xi,r}\right) \leq \sum_{s=1}^{M} \mathcal{L}^2\left((A_s)_\varepsilon^{\xi,r}\right) + \mathcal{L}^2\left(\left(\cup_{s \geq M} A_s\right)_\varepsilon^{\xi,r}\right), \tag{4.10}$$

we have that

$$\limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left(A_\varepsilon^{\xi,r}\right)}{\varepsilon} \leq \sum_{s=1}^{M} \limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left((A_s)_\varepsilon^{\xi,r}\right)}{\varepsilon} + 2\sqrt{2}|\xi|\mathcal{H}^1\left(\overline{\cup_{s \geq M} A_s}\right), \tag{4.11}$$

being the estimate on the second term in (4.10) due to a Minkowski's content argument (see [4]). Since A is supposed to be essentially closed there follows

$$\mathcal{H}^1\left(\cup_{s \geq M} A_s\right) = \mathcal{H}^1\left(\overline{\cup_{s \geq M} A_s}\right).$$

Hence,

$$\sup_{s \geq M} \mathcal{H}^1\left(\overline{\cup_{s \geq M} A_s}\right) = 0,$$

and, by passing to the supremum on M in (4.11), we get

$$\limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left(A_\varepsilon^{\xi,r}\right)}{\varepsilon} \leq \sum_{s \geq 1} \limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left((A_s)_\varepsilon^{\xi,r}\right)}{\varepsilon}.$$

Thus, we may assume A to be an open interval in Π^ν and, without loss of generality, we may also assume $\xi = \mathrm{e}_1$. For $\ell = 1, 2$ let us define

$$\mathcal{J}_\varepsilon^{\ell,r}(A) := \left\{ \alpha + \varepsilon S_r^\xi : \alpha \in \varepsilon\mathbb{Z}^2,\ \alpha + \varepsilon[p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell] \cap A \neq \emptyset \right\},$$

where the points $p_{\ell,r}$ are defined as in the proof of Proposition 3.3. Notice that in case $n = 2$ $p_{\ell,1} = 0$ for $\ell = 1, 2$, $p_{1,2} = \mathrm{e}_2$ and $p_{2,2} = \mathrm{e}_1$.

Then it can be easily proved that (see Figure 3 (i),(ii))

$$\varepsilon \# \mathcal{J}_\varepsilon^{\ell,r}(A) \leq \mathcal{H}^1(A) |\langle \nu, \mathrm{e}_\ell^\perp \rangle| + 2\varepsilon. \tag{4.12}$$

Figure 4: (i) case $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle \leq 0$ (ii) case $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle > 0$

Note that if $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle \leq 0$, then $\mathcal{J}_\varepsilon^{1,r} \cap \mathcal{J}_\varepsilon^{2,r} = \emptyset$, while if $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle > 0$, then either $\mathcal{J}_\varepsilon^{1,r} \subseteq \mathcal{J}_\varepsilon^{2,r}$ or $\mathcal{J}_\varepsilon^{2,r} \subseteq \mathcal{J}_\varepsilon^{1,r}$, according to the cases $|\langle \nu, \mathrm{e}_2 \rangle| \geq |\langle \nu, \mathrm{e}_1 \rangle|$, $|\langle \nu, \mathrm{e}_1 \rangle| \geq |\langle \nu, \mathrm{e}_2 \rangle|$.

Hence, we will treat separately the two cases. Assume first that $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle \leq 0$, then by (4.12)

$$\frac{\mathcal{L}^2\left(A_\varepsilon^{\mathrm{e}_1,r}\right)}{\varepsilon} = \frac{\varepsilon}{2}\left(\#\mathcal{J}^{1,r}(A) + \#\mathcal{J}_\varepsilon^{2,r}(A)\right) \leq \frac{1}{2}\mathcal{H}^1(A)\varphi(\nu) + o(1)$$

and the thesis follows.

If, instead, $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle > 0$ and $|\langle \nu, \mathrm{e}_2 \rangle| \geq |\langle \nu, \mathrm{e}_1 \rangle|$, then $\mathcal{J}_\varepsilon^{1,r} \subseteq \mathcal{J}_\varepsilon^{2,r}$ and by (4.12)

$$\frac{\mathcal{L}^2\left(A_\varepsilon^{\mathrm{e}_1,r}\right)}{\varepsilon} = \frac{\varepsilon}{2}\#\mathcal{J}^{2,r}(A) \leq \frac{1}{2}\mathcal{H}^1(A)\varphi(\nu) + o(1)$$

and the thesis follows. Analogously, we infer the thesis in case $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle > 0$ and $|\langle \nu, \mathrm{e}_1 \rangle| \geq |\langle \nu, \mathrm{e}_2 \rangle|$. □

Proposition 4.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, assume* (h1)*. Then, for any $u \in L^1\left(\Omega; \mathbb{R}^N\right)$,*

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^\beta(u) \leq \mathcal{F}^\beta(u).$$

Proof. Let $u \in \left(GSBV^p(\Omega)\right)^N$ be such that $\mathcal{F}^\beta(u) < +\infty$. Let us first prove the inequality for regular functions. Let Ω' be an open set such that $\Omega' \supset\supset \Omega$ and suppose u regular as in *Step 1* of Proposition 3.5. By using analogous notation, the set $\mathcal{J}_\varepsilon^r$ now equals to

$$\cup_{\ell=1,2}\left\{\alpha + \varepsilon S_r : \alpha \in \varepsilon\mathbb{Z}^2,\ \alpha + \varepsilon[p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell] \cap J \neq \emptyset\right\},$$

and the points $p_{\ell,r}$, in this case satisfy $p_{\ell,1}=0$ for $\ell=1,2$, $p_{1,2}=\mathrm{e}_2$ and $p_{2,2}=\mathrm{e}_1$.

Hence, we get

$$\begin{aligned}\limsup_{\varepsilon\to0^+}&\int_{\cup_{r=1}^2(\mathcal{T}_\varepsilon^r\setminus\mathcal{J}_\varepsilon^r)}\psi_\varepsilon^\beta\left(\nabla u_\varepsilon(x)\right)\,dx\\&=\limsup_{\varepsilon\to0^+}\int_{\cup_{r=1}^2(\mathcal{T}_\varepsilon^r\setminus\mathcal{J}_\varepsilon^r)}\psi_\varepsilon^\beta\left(W_\varepsilon(x)\right)\,dx\\&\leq\lim_{\varepsilon\to0^+}\int_\Omega\psi\left(W_\varepsilon(x)\right)\,dx=\int_\Omega\psi\left(\nabla u(x)\right)\,dx.\end{aligned}\tag{4.13}$$

By the very definition of ψ_ε^β and $\mathcal{N}_\varepsilon^r$ we have

$$\begin{aligned}\int_{\cup_{r=1}^2(\mathcal{J}_\varepsilon^r\cap\mathcal{N}_\varepsilon^r)}&\psi_\varepsilon^\beta\left(\nabla u_\varepsilon(x)\right)\,dx\\&=\beta\frac{\varepsilon}{2}\left(\#\left(\mathcal{J}_\varepsilon^1\cap\mathcal{N}_\varepsilon^1\right)+\#\left(\mathcal{J}_\varepsilon^2\cap\mathcal{N}_\varepsilon^2\right)\right)\\&\leq\frac{\beta}{\varepsilon}\left(\mathcal{L}^2\left(\left(\overline{\Omega\cap J_u}\right)_\varepsilon^{\mathrm{e}_1,1}\right)+\mathcal{L}^2\left(\left(\overline{\Omega\cap J_u}\right)_\varepsilon^{\mathrm{e}_1,2}\right)\right),\end{aligned}$$

and then by Lemma 4.4

$$\limsup_{\varepsilon\to0^+}\int_{\cup_{r=1}^2(\mathcal{J}_\varepsilon^r\cap\mathcal{N}_\varepsilon^r)}\psi_\varepsilon^\beta\left(\nabla u_\varepsilon(x)\right)\,dx\leq\beta\int_{\overline{\Omega\cap J_u}}\varphi(\nu_u)\,d\mathcal{H}^1.\tag{4.14}$$

Moreover, by taking into account (3.1), we get

$$\int_{\cup_{r=1}^2(\mathcal{J}_\varepsilon^r\setminus\mathcal{N}_\varepsilon^r)}\psi_\varepsilon^\beta\left(\nabla u_\varepsilon(x)\right)\,dx\leq C_2\frac{\varepsilon^2}{2}\left(1+\lambda_\varepsilon^p\right)\#\left(\cup_{r=1}^2(\mathcal{J}_\varepsilon^r\setminus\mathcal{N}_\varepsilon^r)\right),\tag{4.15}$$

the term on the right hand side above being infinitesimal as proved in (3.27) of Proposition 3.5. Eventually, by collecting (4.13), (4.14) and (4.15) we get

$$\limsup_{\varepsilon\to0^+}\mathcal{F}_\varepsilon^\beta(u_\varepsilon)\leq\int_\Omega\psi\left(\nabla u\right)\,dx+\beta\int_{J_u}\varphi(\nu_u)\,d\mathcal{H}^1.$$

To infer the result for every $u\in L^1\left(\Omega;\mathbb{R}^N\right)$ it suffices to argue like in *Step 2* and *Step 3* of the proof of Proposition 3.5. □

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