LOWER SEMICONTINUITY OF QUASI-CONVEX FUNCTIONALS WITH NON-STANDARD GROWTH

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ABSTRACT. We study the lower semicontinuity properties of autonomous variational integrals whose energy densities are controlled by N-functions.

1. INTRODUCTION

In this paper we study the lower semicontinuity properties of a class of quasi-convex functionals of the Calculus of Variations. Consider the integral functional

$$F(u,\Omega)=\int_{\Omega} f(Du(x))\,dx \tag{1.1}$$

where $\Omega\subseteq\mathbb{R}^n$ is a bounded and open set, $u:\Omega\to\mathbb{R}^N$ is a measurable function sufficiently regular, and $f:\mathbb{R}^{Nn}\to\mathbb{R}$ is *quasi-convex* in Morrey' sense, see [Mo], i.e., f is continuous and for every $A\in\mathbb{R}^{Nn}$ and $\varphi\in C_c^\infty\left(\Omega,\mathbb{R}^N\right)$ there holds

$$f(A)\,\mathcal{L}^n(\Omega)\le\int_{\Omega} f(A+D\varphi(x))\,dx, \tag{1.2}$$

denoting with $\mathcal{L}^n(\Omega)$ the n dimensional Lebesgue's measure of Ω.

Assume that f satisfies the non-standard growth condition

$$-c\,(1+\Phi_1(|A|))\le f(A)\le c\,(1+\Phi(|A|)), \tag{1.3}$$

with c a positive constant, Φ_1 and Φ *N-functions* (see Section 2 for definitions) such that Φ_1 grows slower than Φ at infinity (see Remark 3.3).

When in (1.3) $\Phi_1(t)=t^{p_1}$ and $\Phi(t)=t^p$, with $1<p_1<p$ or $1=p_1\le p$, the functional $F(\cdot,\Omega)$ in (1.1) was proven to be sequentially lower semicontinuous in the weak topology of $W^{1,p}$ by Acerbi and Fusco [AFu] and by Marcellini [Ma1].

If, moreover, f is non negative then the lower semicontinuity inequality

$$\liminf_{r\to+\infty} F(u_r,\Omega)\ge F(u,\Omega) \tag{1.4}$$

has been established along sequences $(u_r)\in W^{1,p}$ converging in the weak topology of $W^{1,q}$ for $q\ge\frac{n}{n+1}p$ by Marcellini [Ma2] and recently for $q\ge\frac{n-1}{n}p$ by Fonseca and Malý [FoM] and Malý [M2]. See also Kristensen [Kr] for a refinement.

Under further structure assumptions on f, Fonseca and Marcellini [FoMa] proved the case $q>p-1$ and then Malý [M2],[M3], refined the result to $q\ge p-1$.

In the polyconvex case, i.e., $f(A)=g(T(A))$ where g is convex and $T(A)$ denotes the set of all minors of the matrix $A\in\mathcal{M}^{N\times n}$, Dacorogna and Marcellini [DMa] proved the lower semicontinuity inequality (1.4) for $q>n-1$, while the border case $q=n-1$ was stated by Acerbi and Dal Maso [ADM], Celada and Dal

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Maso [CDM] and Dal Maso and Sbordone [DMS]. An elementary approach was found by Fusco and Hutchinson [FuH], see also Malý [M1] for related results.

Notice that for functionals $F(\cdot,\Omega)$ defined as in (1.1) the weak sequential lower semicontinuity in $W^{1,p}$, $p>1$, can be rephrased as follows: for every sequence $(u_r)\in W^{1,1}$ such that

$$u_r \to u \text{ strongly } L^1_{loc} \text{ and } \liminf_{r\to+\infty}\int_\Omega |Du_r|^p\,dx < +\infty \tag{1.5}$$

then

$$\liminf_{r\to+\infty} F(u_r,\Omega) \geq F(u,\Omega).$$

With the general growth condition (1.3), the natural setting where to study lower semicontinuity properties for functionals defined by (1.1) is provided by the functional spaces generated by N-functions, called *Orlicz spaces.*

Ball [B] was the first to set some variational problems in the framework of *Orlicz-Sobolev spaces.* Recently, the first author has considered in [F] quasi-convex integrals with the non-standard growth conditions (1.3) obtaining lower semicontinuity in the weak $*$ topology of the Orlicz-Sobolev space W^1L^Φ (see Section 2 for references) provided Φ satisfies a sub-homogeneity property at infinity called $\triangle_2$-*condition*, i.e., there exist $m>1$ and $t_o\geq 0$ such that for every $\lambda>1$ and $t\geq t_o$ there holds

$$\Phi(\lambda t) \leq \lambda^m \Phi(t).$$

Those results are also applied to give existence theorems for Dirichlet's boundary value problems (see [F]).

The structure and properties of Orlicz spaces are close to the standard L^p case if $\Phi\in\triangle_2$, while if $\Phi\notin\triangle_2$ the theory is quite different. Indeed, let Φ be a N-function, set

$$K^\Phi = \left\{ u:\Omega\to\mathbb{R}^{\mathrm{N}} \text{ measurable: } \int_\Omega \Phi(|u|)\,dx < +\infty \right\},$$

denote with L^Φ the linear hull of K^Φ, which is a Banach space if endowed with the gauge norm, then $K^\Phi\equiv L^\Phi$ if and only if $\Phi\in\triangle_2$. This lack of linear structure has consequences in the study of semicontinuity for functionals like in (1.1) whose integrand satisfies the growth condition (1.3).
Indeed, if $\Phi\notin\triangle_2$ then $F(\cdot,\Omega)$ is not finite a priori on the whole W^1L^Φ, unlike the case $\Phi\in\triangle_2$, but just on the convex set

$$W^{1,\Phi,1} = \left\{ u\in W^{1,1} : \int_\Omega \Phi(|Du|)\,dx < +\infty \right\},$$

which is strictly contained in W^1L^Φ.
However, assuming the analogue condition of (1.5), i.e., $(u_r)\in W^{1,1}$ such that

$$u_r \to u \text{ strongly } L^1_{loc} \text{ and } \liminf_{r\to+\infty}\int_\Omega \Phi(|Du_r|)\,dx < +\infty\ , \tag{1.6}$$

we are able to prove the lower semicontinuity of $F(\cdot,\Omega)$ along such sequences.

The main result of the paper is the following (see Section 3 Theorem 3.2).

Let $\Omega\subseteq\mathbb{R}^n$ be a bounded and open set with Lipschitz boundary, let $F(\cdot,\Omega)$ be defined as in (1.1) *with $f:\mathbb{R}^{Nn}\to\mathbb{R}$ a quasi-convex function satisfying for every*

$A \in \mathbb{R}^{Nn}$

$$0 \leq f(A) \leq c(1 + \Phi(|A|)), \tag{1.7}$$

with c a positive constant and Φ a N-function.

Then for every $(u_r) \in W^{1,\Phi,1}(\Omega, \mathbb{R}^N)$ satisfying (1.6) there holds

$$\liminf_{r \to \infty} F(u_r, \Omega) \geq F(u, \Omega).$$

We remark that if $\Phi \notin \triangle_2$, the integral boundedness condition in (1.6) is not even implied by the norm convergence of $W^1 L^\Phi$, thus, unlike the case $\Phi \in \triangle_2$, it is not equivalent to weak $*$ convergence in $W^1 L^\Phi$ which is in turn implied by (1.6). However, (1.6) turns out to be a natural condition when dealing with minimizing sequences of coercive functionals in $W^1 L^\Phi$, i.e., with energy densities satisfying

$$c_1(\Phi(|A|) - 1) \leq f(A) \leq c(\Phi(|A|) + 1) \tag{1.8}$$

for every $A \in \mathbb{R}^{Nn}$ and for some positive constants c_1, c.

Moreover, in that case, take $u_o \in W^{1,\Phi,1}$ and consider the boundary value problem

$$\inf\left\{F(u, \Omega) : u \in u_o + W_o^{1,1}\right\},$$

we prove that the infimum is attained as it happens in the $W^1 L^\Phi$ setting when $\Phi \in \triangle_2$ (see [F] and Remark 3.8).

Eventually, it is possible to give explicit examples of non trivial applications of previous results constructing quasi-convex functions verifying the non-standard growth conditions (1.7), (1.8), in the latter case provided the dominating N-function Φ satisfies a sort of sub-additivity condition at infinity (see Section 4).

The plan of the paper is the following: in Section 2 we recall some definitions and prove some properties of N-functions and Orlicz spaces; in Section 3 we prove the semicontinuity result Theorem 3.2; in Section 4 we give some examples of quasi-convex functions with non-standard growth (1.7), (1.8).

2. N-Functions and Orlicz spaces.

In this section we recall some definitions and known properties of N-functions, Orlicz, Orlicz-Sobolev spaces (see for references [Ad],[KR],[RR]).

A continuous and convex function $\Phi : [0, +\infty) \to [0, +\infty)$ is called *N-function* if it satisfies

$$\Phi(0) = 0, \Phi(t) > 0 \ t > 0, \lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0, \lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty, \tag{2.1}$$

e.g. take $\Phi_{p,\alpha}(t) = t^p \log^\alpha(1 + t)$ for $p > 1$ and $\alpha \geq 0$ or $p = 1$ and $\alpha > 0$.

Actually, only the growth at infinity really matters in the definition of N-function. Indeed, given a continuous and convex function $Q : [0, +\infty) \to [0, +\infty)$ satisfying

$$\lim_{t \to +\infty} \frac{Q(t)}{t} = +\infty$$

there exist a N-function Φ and $t_o > 0$ such that for every $t \geq t_o$ there holds

$$\Phi(t) = Q(t).$$

Such a function Q is called *principal part* of the N-function Φ. Since this, we will not distinguish any longer the two concepts, e.g. we will refer as N-functions to the functions $\Gamma_0(t) = t^{\ln t}$, $\Gamma_\beta(t) = \exp\left(t^\beta\right) - 1$, $\beta > 0$, which have not super-linear growth in 0.

In the sequel we will often use the following convexity inequality: for every s, $t \in [0, +\infty)$ and $\lambda > 1$

$$\Phi(s+t) \le \tfrac{1}{\lambda}\Phi(\lambda s) + \left(1 - \tfrac{1}{\lambda}\right)\Phi\left(\tfrac{\lambda}{\lambda-1}t\right). \tag{2.2}$$

Let Φ be a N-function, let Ψ denote the Fenchel's conjugate of Φ, i.e.,

$$\Psi(t) = \sup\{st - \Phi(s) : s \ge 0\}, \tag{2.3}$$

Ψ is a N-function called the *complementary N-function* of Φ. By the very definition the pair Φ,Ψ satisfies *Young's inequality*, i.e., for every $s, t \in [0, +\infty)$ there holds

$$st \le \Phi(s) + \Psi(t).$$

A useful class of N-functions is provided by the following definition. We say that Φ belongs to class $\triangle_2$, denoted by $\Phi \in \triangle_2$, if there exist $m > 1$ and $t_o \ge 0$ such that for every $\lambda > 1$, $t \ge t_o$ there holds

$$\Phi(\lambda t) \le \lambda^m \Phi(t). \tag{2.4}$$

Take for instance $\Phi_{p,\alpha}(t) = t^p \log^\alpha(1+t)$ for $p > 1$ and $\alpha \ge 0$ or $p = 1$ and $\alpha > 0$, then $\Phi_{p,\alpha} \in \triangle_2$, while $\Gamma_0(t) = t^{\ln t} \notin \triangle_2$ and $\Gamma_\beta(t) = \exp\left(t^\beta\right) - 1 \notin \triangle_2$ for any $\beta > 0$.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and open set, the *Orlicz class* $K^\Phi\left(\Omega, \mathbb{R}^N\right)$ is the set of all (equivalence classes modulo equality $\mathcal{L}^n$ a.e. in Ω of) measurable functions $u : \Omega \to \mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(|u|)\, dx < +\infty, \tag{2.5}$$

where $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^N$.

The *Orlicz space* $L^\Phi\left(\Omega, \mathbb{R}^N\right)$ is defined to be the linear hull of $K^\Phi\left(\Omega, \mathbb{R}^N\right)$, thus it consists of all measurable functions u such that $\lambda u \in K^\Phi\left(\Omega, \mathbb{R}^N\right)$ for some $\lambda > 0$. Moreover, the equality $K^\Phi\left(\Omega, \mathbb{R}^N\right) \equiv L^\Phi\left(\Omega, \mathbb{R}^N\right)$ holds if and only if $\Phi \in \triangle_2$.

Define the functional $\|u\|_{\Phi,\Omega} : L^\Phi\left(\Omega, \mathbb{R}^N\right) \to [0, +\infty)$ by

$$\|u\|_{\Phi,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \le 1\right\}, \tag{2.6}$$

it is a norm, called the *gauge norm*, and $L^\Phi\left(\Omega, \mathbb{R}^N\right)$ is a Banach space if endowed with it. In the sequel we will denote $\|\cdot\|_{\Phi,\Omega}$ simply by $\|\cdot\|_\Phi$, and the norm convergence in $L^\Phi\left(\Omega, \mathbb{R}^N\right)$ by $s - L^\Phi\left(\Omega, \mathbb{R}^N\right)$. It easily follows the continuous immersion $L^\Phi\left(\Omega, \mathbb{R}^N\right) \to L^1\left(\Omega, \mathbb{R}^N\right)$ if both spaces are equipped with the gauge norm.

Notice that by the very definition of the norm for any $u \in L^\Phi\left(\Omega, \mathbb{R}^N\right)$ we have

$$\|u\|_\Phi \le 1 + \int_\Omega \Phi(|u|)\, dx. \tag{2.7}$$

Denote by $E^\Phi\left(\Omega, \mathbb{R}^N\right)$ the closure of $C_c^\infty\left(\Omega, \mathbb{R}^N\right)$ in $s - L^\Phi\left(\Omega, \mathbb{R}^N\right)$, the inclusions

$$E^\Phi\left(\Omega, \mathbb{R}^N\right) \subseteq K^\Phi\left(\Omega, \mathbb{R}^N\right) \subseteq L^\Phi\left(\Omega, \mathbb{R}^N\right)$$

are trivial with equalities holding if and only if $\Phi \in \triangle_2$.

A useful characterization of $E^\Phi\left(\Omega, \mathbb{R}^N\right)$ is given in the following lemma (see Proposition 4 [RR, p.52]).

Lemma 2.1. *Let* $u \in L^{\Phi}\left(\Omega, \mathbb{R}^N\right)$, *set* $k_{\Phi}^u = \sup\left\{\lambda \geq 0 : \lambda u \in K^{\Phi}\left(\Omega, \mathbb{R}^N\right)\right\}$, *define* $l_{\Phi}^u : [0, k_{\Phi}^u] \to [0, +\infty]$ *by*

$$l_{\Phi}^u(\lambda) = \int_{\Omega} \Phi(\lambda |u|)\, dx,$$

then l_{Φ}^u *is continuous, increasing and*

$$\lim_{\lambda \to \left(k_{\Phi}^u\right)^-} l_{\Phi}^u(\lambda) = l_{\Phi}^u(k_{\Phi}^u) \leq +\infty.$$

Moreover, $E^{\Phi}\left(\Omega, \mathbb{R}^N\right) = \left\{u \in L^{\Phi}\left(\Omega, \mathbb{R}^N\right) : k_{\Phi}^u = +\infty\right\}$.

We stress the attention on the fact that if $\Phi \notin \triangle_2$ the values of k_{Φ}^u and $l_{\Phi}^u(k_{\Phi}^u)$ can be independently assigned, i.e., given any $0 < \alpha, \beta < +\infty$ there exist u, $v \in L^{\Phi}\left(\Omega, \mathbb{R}^N\right)$ with $k_{\Phi}^u = k_{\Phi}^v = \alpha$ such that $l_{\Phi}^u(\alpha) = \beta$ and $l_{\Phi}^v(\alpha) = +\infty$ (see [RR, p.54]). This last remark gives a characterization of condition $\triangle_2$.

Lemma 2.2. *Let* Φ *be a N-function,* $\Phi \in \triangle_2$ *if and only if for every family* $(u_i)_{i \in I} \subseteq L^{\Phi}\left(\Omega, \mathbb{R}^N\right)$ *which is norm bounded there holds*

$$\sup_{i \in I} \int_{\Omega} \Phi(|u_i|)\, dx < +\infty.$$

Another consequence of the previous remark is that norm convergence does not imply convergence of integrals in the case $\Phi \notin \triangle_2$. Indeed, if $u_r \to u$ $s - L^{\Phi}\left(\Omega, \mathbb{R}^N\right)$ the convexity of Φ implies

$$\liminf_{r \to +\infty} \int_{\Omega} \Phi(|u_r|)\, dx \geq \int_{\Omega} \Phi(|u|)\, dx, \tag{2.8}$$

with the possibility of strict inequality holding in (2.8). However, the integral convergence holds for suitable sub-multiples of the limit.

Lemma 2.3. *Let* (u_r), $u \in L^{\Phi}\left(\Omega, \mathbb{R}^N\right)$ *be such that* $u_r \to u$ $s - L^{\Phi}\left(\Omega, \mathbb{R}^N\right)$, *if* $\lambda \in [0, k_{\Phi}^u)$ *then*

$$\lim_{r \to +\infty} \int_{\Omega} \Phi(\lambda |u_r|)\, dx = \int_{\Omega} \Phi(\lambda |u|)\, dx. \tag{2.9}$$

Proof. Fix $\lambda \in (0, k_{\Phi}^u)$, by (2.8) we have only to prove the inequality

$$\limsup_{r \to +\infty} \int_{\Omega} \Phi(\lambda |u_r|)\, dx \leq \int_{\Omega} \Phi(\lambda |u|)\, dx,$$

the case $\lambda = 0$ being trivial.

By the very definition of the norm and the convexity of Φ it follows

$$\|w\|_{\Phi} \leq 1 \Rightarrow \int_{\Omega} \Phi(|w|)\, dx \leq \|w\|_{\Phi},$$

hence for any $\sigma > 0$ there exists $r(\sigma)$ such that for every $r \geq r(\sigma)$

$$\int_{\Omega} \Phi(\sigma |u_r - u|)\, dx \leq \sigma \|u_r - u\|_{\Phi} \leq 1. \tag{2.10}$$

Fix $\sigma > 1$ such that $\lambda < \lambda\sigma < k_{\Phi}^u$, then by (2.2)

$$\int_{\Omega} \Phi(\lambda |u_r|)\, dx \leq \frac{1}{\sigma} \int_{\Omega} \Phi(\lambda\sigma |u|)\, dx + \left(1 - \frac{1}{\sigma}\right) \int_{\Omega} \Phi\left(\tfrac{\lambda\sigma}{\sigma - 1} |u_r - u|\right) dx, \tag{2.11}$$

hence passing to the superior limit for $r \to +\infty$ in (2.11) we get by (2.10)

$$\limsup_{r\to+\infty} \int_\Omega \Phi\left(\lambda\left|u_r\right|\right)dx \le \frac{1}{\sigma}\int_\Omega \Phi\left(\lambda\sigma\left|u\right|\right)dx,$$

and so Lemma 2.1 yields the conclusion by letting $\sigma \to 1^+$. □

The *Orlicz-Sobolev space* $W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$ consists of all (equivalence classes modulo equality $\mathcal{L}^n$ a.e. in Ω of) measurable functions $u \in L^\Phi\left(\Omega,\mathbb{R}^N\right)$ whose first order distributional derivatives belong to $L^\Phi\left(\Omega,\mathbb{R}^N\right)$. As in the case of ordinary Sobolev spaces, it is a Banach space if endowed with the norm

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \|Du\|_\Phi\,.$$

Denote by $W^1_oE^\Phi\left(\Omega,\mathbb{R}^N\right)$ the closure of $C^\infty_c\left(\Omega,\mathbb{R}^N\right)$ in the norm topology of $W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$, indicated by $s - W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$. Let us state a generalization of Rellich-Kondrakov's compact embedding theorem ([Ad], Lemma 7.1 [EOP]).

Theorem 2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a open bounded set with Lipschitz boundary, let Φ be a N-function, then the embedding $W^1L^\Phi\left(\Omega,\mathbb{R}^N\right) \to L^\Phi\left(\Omega,\mathbb{R}^N\right)$ is compact.*

Let $\lambda > 0$ and consider, similarly to Marcellini [M3], the convex functional sets

$$W^{1,\Phi,\lambda}\left(\Omega,\mathbb{R}^N\right) = \left\{u \in W^{1,1}\left(\Omega,\mathbb{R}^N\right) : \int_\Omega \Phi\left(\lambda\left|Du\right|\right)dx < +\infty\right\}.$$

The next lemma yields the set inclusion $W^{1,\Phi,\lambda}\left(\Omega,\mathbb{R}^N\right) \subseteq W^1_{loc}L^\Phi\left(\Omega,\mathbb{R}^N\right)$ (see Lemma 1 [BhL]).

Lemma 2.5. *Let $C \subseteq \mathbb{R}^n$ be a convex, bounded and open set, then for every $\lambda > 0$ and $u \in W^{1,\Phi,\lambda}\left(\Omega,\mathbb{R}^N\right)$ there holds*

$$\int_C \Phi\left(\tfrac{\lambda}{d}\left|u-u_C\right|\right)dx \le \left(\tfrac{\omega_n d^n}{\mathcal{L}^n(C)}\right)^{1-\frac{1}{n}} \int_C \Phi\left(\lambda\left|Du\right|\right)dx,$$

where $u_C = \frac{1}{\mathcal{L}^n(C)}\int_C u\,dx$, $d = diamC$, $\omega_n = \mathcal{L}^n\left(B_{(0,1)}\right)$ and $B_{(0,1)}$ is the unit ball of $\mathbb{R}^n$.

The set inclusion $W^{1,\Phi,\lambda}\left(\Omega,\mathbb{R}^N\right) \subseteq W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$ is related to the regularity of Ω, it is a consequence of Lemma 2.7 below for which we need the following result (see Lemma 1 [T]).

Lemma 2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and open set with Lipschitz boundary, then there exists a positive constant $c = c(n,\Omega)$ such that for every $u \in W^{1,1}\left(\Omega,\mathbb{R}^N\right)$*

$$\left|u(x)\right| \le c\left(\|u\|_{L^1(\Omega,\mathbb{R}^N)} + \int_\Omega \tfrac{\left|Du(y)\right|}{\left|x-y\right|^{n-1}}dy\right)$$

for $\mathcal{L}^n$ a.e. $x \in \Omega$.

Lemma 2.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and open set with Lipschitz boundary, then there exist positive constants $c_i = c_i(n,\Omega)$, $1 \le i \le 2$, such that for every $u \in W^{1,\Phi,1}\left(\Omega,\mathbb{R}^N\right)$ and $\lambda > 1$, there holds*

$$\int_\Omega \Phi\left(\tfrac{c_1}{\lambda}\left|u\right|\right)dx \le \Phi\left(\tfrac{c_2}{\lambda-1}\|u\|_{L^1(\Omega,\mathbb{R}^N)}\right)\mathcal{L}^n(\Omega) + \int_\Omega \Phi\left(\left|Du\right|\right)dx.$$

Proof. Let $r > diam\Omega$, consider the kernel $J : B_{(0,r)} \to [0,+\infty)$ defined by

$$J(x) = \begin{cases} k\,|x|^{1-n} & B_{(0,r)} \setminus \{0\} \\ 0 & \text{otherwise} \end{cases}$$

where k is chosen such that $\|J\|_{L^1(\mathbb{R}^n)} = 1$.

Define v to be the zero extension of $|Du|$ to $\mathbb{R}^n$, then applying Lemma 2.6 and (2.2) for $\mathcal{L}^n$ a.e. $x \in \Omega$ we have

$$\Phi\left(\tfrac{k}{c\lambda}\,|u(x)|\right) \le \Phi\left(\tfrac{k}{\lambda-1}\,\|u\|_{L^1(\Omega,\mathbb{R}^N)}\right) + \Phi\left(\int_{\mathbb{R}^n} J(y-x)\,v(y)\,dy\right)$$

thus by a suitable version of Jensen's inequality, i.e.,

$$\Phi\left(\int_{\mathbb{R}^n} J(y-x)\,v(y)\,dy\right) \le \int_{\mathbb{R}^n} J(y-x)\,\Phi(v(y))\,dy,$$

and integrating over Ω we get

$$\begin{aligned} &\int_\Omega \Phi\left(\tfrac{k}{c\lambda}\,|u|\right) dx \\ &\quad\le \Phi\left(\tfrac{k}{\lambda-1}\,\|u\|_{L^1(\Omega,\mathbb{R}^N)}\right)\mathcal{L}^n(\Omega) + \int_\Omega dx \int_{\mathbb{R}^n} J(y-x)\,\Phi(v(y))\,dy \\ &\quad\le \Phi\left(\tfrac{k}{\lambda-1}\,\|u\|_{L^1(\Omega,\mathbb{R}^N)}\right)\mathcal{L}^n(\Omega) + \int_\Omega \Phi(|Du(x)|)\,dx, \end{aligned}$$

and so we are done setting $c_1(n,\Omega) = \frac{k}{c}$ and $c_2(n,\Omega) = cc_1$. $\square$

Let $W_o^{1,\Phi,\lambda}(\Omega,\mathbb{R}^N) = W_o^{1,1} \cap W^{1,\Phi,\lambda}(\Omega,\mathbb{R}^N)$; for any bounded set Ω the inclusion $W_o^{1,\Phi,\lambda}(\Omega,\mathbb{R}^N) \subseteq W^1L^\Phi(\Omega,\mathbb{R}^N)$ holds by using the following lemma which generalizes to the vectorial case Lemma 3.2 [Ma3] (see [Mi]).

Lemma 2.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and open set, let $d = diam\Omega$ and $\lambda > 0$, if $u \in W_o^{1,\Phi,\lambda}(\Omega,\mathbb{R}^N)$ then*

$$\int_\Omega \Phi\left(\tfrac{2\lambda}{Nd}\,|u|\right) dx \le \int_\Omega \Phi(\lambda\,|Du|)\,dx.$$

As a consequence of Lemma 2.8 we deduce that the L^Φ norm of the gradient and the W^1L^Φ norm are equivalent on $W_o^{1,\Phi,\lambda}(\Omega,\mathbb{R}^N)$. More precisely if $u \in W_o^{1,\Phi,\lambda}(\Omega,\mathbb{R}^N)$ then

$$\|u\|_\Phi \le \tfrac{Nd}{2}\,\|Du\|_\Phi\,. \tag{2.12}$$

Next lemma states a density result in $W_o^{1,\Phi,\lambda}(\Omega,\mathbb{R}^N)$ (see [Go2],[Mi] for related results).

Lemma 2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and open set, let $u \in W_o^{1,\Phi,\lambda}(\Omega,\mathbb{R}^N)$ be such that $spt u \subset\subset \Omega$, then there exists a sequence $(u_r) \subset C_c^\infty(\Omega,\mathbb{R}^N)$ such that*

(1) $u_r \to u$ $s - W^{1,1}(\Omega,\mathbb{R}^N)$;
(2) $\int_\Omega \Phi(|u_r|)\,dx \to \int_\Omega \Phi(|u|)\,dx$;
(3) $\int_\Omega \Phi(|Du_r|)\,dx \to \int_\Omega \Phi(|Du|)\,dx$.

Proof. Let J_ε be a mollifier, let $u_r = J_{\frac{1}{r}} * u$, then standard convolution results yield $u_r \in C_c^\infty(\Omega,\mathbb{R}^N)$ if r is suitable and assertion (1) hence follows.

To prove (2) note that by Jensen's inequality for $\mathcal{L}^n$ a.e. $x \in \Omega$

$$0 \leq \Phi\left(\left|u_r\left(x\right)\right|\right) \leq \left(J_{\frac{1}{r}} * \Phi\left(\left|u\right|\right)\right)\left(x\right),$$

moreover, since

$$J_{\frac{1}{r}} * \Phi\left(\left|u\right|\right) \rightarrow \Phi\left(\left|u\right|\right) \ \ s - L^1\left(\Omega\right) \text{ and } \mathcal{L}^n \text{ a.e. } x \in \Omega,$$

(2) holds by the continuity of Φ and Lebesgue's Dominated Convergence theorem.

To prove (3) observe that since $sptu \subset\subset \Omega$, if $\frac{1}{r} < d\left(sptu, \partial\Omega\right)$ then

$$D_i\left(J_{\frac{1}{r}} * u\right)\left(x\right) = \left(J_{\frac{1}{r}} * D_i u\right)\left(x\right)$$

for $\mathcal{L}^n$ a.e. $x \in \Omega$ and for every $1 \leq i \leq n$, so that we can conclude analogously to (2). □

We now introduce the weak $*$ convergence in $L^\Phi\left(\Omega, \mathbb{R}^N\right)$, which we will denote by $*w - L^\Phi\left(\Omega, \mathbb{R}^N\right)$. Since the Orlicz space $L^\Phi\left(\Omega, \mathbb{R}^N\right)$ is isometrically isomorphic to the dual space of $E^\Psi\left(\Omega, \mathbb{R}^N\right)$ a sequence $u_r \rightarrow u \ *w - L^\Phi\left(\Omega, \mathbb{R}^N\right)$ if and only if for every $v \in E^\Psi\left(\Omega, \mathbb{R}^N\right)$ there holds

$$\lim_{r \rightarrow +\infty} \int_\Omega u_r v dx = \int_\Omega u v dx.$$

By means of the Hahn-Banach theorem we have that $u_r \rightarrow u \ *w - W^1 L^\Phi\left(\Omega, \mathbb{R}^N\right)$ if and only if (u_r), $(D_i u_r)$, $1 \leq i \leq n$, converge to u, $D_i u$ respectively. As a consequence of the previous statements we deduce that $L^\Phi\left(\Omega, \mathbb{R}^N\right)$ is reflexive if and only if both Φ and Ψ belong to class $\triangle_2$.

Eventually, $W_o^1 E^\Phi\left(\Omega, \mathbb{R}^N\right)$ is $*w - W^1 L^\Phi\left(\Omega, \mathbb{R}^N\right)$ closed if and only if $\Phi \in \triangle_2$ (see [Do],[Go1]), in the sequel we denote by $W_o^1 L^\Phi\left(\Omega, \mathbb{R}^N\right)$ its weak $*$ closure.

3. Semicontinuity.

Let f be quasi-convex, i.e., f is continuous and satisfies inequality (1.2), then f is separately convex in each variable (see [D]) and thus for every $\theta \in [0,1]$ and $z \in \mathbb{R}^{Nn}$ we get

$$f\left(\theta A\right) \leq \sum_{0 \leq k \leq Nn} \theta^{Nn-k}\left(1-\theta\right)^k \sum_{|\alpha|=k} f\left(\pi_k^\alpha\left(A\right)\right), \tag{3.1}$$

where α is a multi-index of components $\alpha_i \in \{1, \ldots, Nn\}$ and length $|\alpha| = \alpha_1 + \ldots + \alpha_{Nn}$, considering two multi-indices equal up to permutations, and where $\pi_k^\alpha : \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn}$ is the projection on the k-plane

$$\Pi_\alpha = \left\{y \in \mathbb{R}^{Nn} : y_{\alpha_1} = y_{\alpha_2} = \ldots = y_{\alpha_k} = 0\right\},$$

with the convention that $\pi_0^{(0,\ldots,0)} = Id_{\mathbb{R}^{Nn}}$ and $\Pi_{(0,\ldots,0)} = \mathbb{R}^{Nn}$ if $k = 0$.

Lemma 3.1. *Let Φ be an N-function and $f : \mathbb{R}^{Nn} \rightarrow \mathbb{R}$ be quasi-convex and satisfying*

$$f\left(A\right) \leq c\left(1 + \Phi\left(|A|\right)\right), \tag{3.2}$$

then there exists a positive constant $c_1 = c_1\left(Nn\right)$ such that for every $\theta \in [0,1]$ and $A \in \mathbb{R}^{Nn}$

$$f\left(\theta A\right) \leq \theta^{Nn} f\left(A\right) + c_1\left(1-\theta\right)\left(1 + \Phi\left(|A|\right)\right). \tag{3.3}$$

Proof. Since Φ is increasing, by (3.2) for every α and k we get

$$f\left(\pi_k^\alpha(A)\right) \le c\left(1+\Phi\left(\left|\pi_k^\alpha(A)\right|\right)\right) \le c\left(1+\Phi\left(|A|\right)\right),$$

then (3.3) follows by (3.1) setting $c_1 = c\sum_{1\le k\le Nn}\binom{Nn}{k}$. □

Let us recall our main result.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and open set with Lipschitz boundary, let $F(\cdot,\Omega)$ be defined as in (1.1) with $f:\mathbb{R}^{Nn}\to\mathbb{R}$ a quasi-convex function satisfying for every $A\in\mathbb{R}^{Nn}$*

$$0 \le f(A) \le c\left(1+\Phi\left(|A|\right)\right), \tag{3.4}$$

with c a positive constant and Φ a N-function.

Then for every $(u_r)\in W^{1,\Phi,1}\left(\Omega,\mathbb{R}^N\right)$ satisfying (1.6) there holds

$$\liminf_{r\to\infty} F\left(u_r,\Omega\right) \ge F\left(u,\Omega\right).$$

Remark 3.1. *By the sequential lower semicontinuity of the map $v\to\int_\Omega\Phi\left(|v|\right)dx$ in the $w-L^1\left(\Omega,\mathbb{R}^N\right)$ convergence and by (1.6) it follows $u\in W^{1,\Phi,1}\left(\Omega,\mathbb{R}^N\right)$.*

Remark 3.2. *The quasi-convexity inequality (1.2) can be extended also for test functions in $W_o^{1,\Phi,1}\left(\Omega,\mathbb{R}^N\right)$ under growth conditions (1.7).*

Indeed, given $\varphi\in W_o^{1,\Phi,1}\left(\Omega,\mathbb{R}^N\right)$ first assume that $spt\varphi\subset\subset\Omega$ and consider the sequence $(\varphi_r)\subset C_c^\infty\left(\Omega,\mathbb{R}^N\right)$ provided by Lemma 2.9. We may further suppose that $D\varphi_r\to D\varphi$ $\mathcal{L}^n$ a.e. in Ω, hence by Lebesgue's Dominated Convergence theorem

$$f(A)\,\mathcal{L}^n(\Omega) \le \lim_{r\to+\infty}\int_\Omega f\left(A+D\varphi_r(x)\right)dx = \int_\Omega f\left(A+D\varphi(x)\right)dx.$$

If $\varphi\in W_o^{1,\Phi,1}\left(\Omega,\mathbb{R}^N\right)$ is any, let Σ be a bounded and open set such that $\Sigma\supset\supset\Omega$, define φ_o to be the zero extension of φ to Σ, then $\varphi_o\in W_o^{1,\Phi,1}\left(\Sigma,\mathbb{R}^N\right)$ and $spt\varphi_o\subset\subset\Sigma$, thus by previous step, (1.2) holds for φ_o on Σ, i.e.,

$$f(A)\,\mathcal{L}^n(\Sigma) \le \int_\Sigma f\left(A+D\varphi_o(x)\right)dx = \int_\Omega f\left(A+D\varphi(x)\right)dx + f(A)\,\mathcal{L}^n\left(\Sigma\setminus\Omega\right),$$

and so (1.2) holds for φ on Ω.

Remark 3.3. *The statement of Theorem 3.2 holds more generally if the growth condition (1.7) is substituted by (1.3), i.e., for every $A\in\mathbb{R}^{Nn}$*

$$-c\left(1+\Phi_1\left(|A|\right)\right) \le f(A) \le c\left(1+\Phi\left(|A|\right)\right),$$

provided Φ_1 is a N-function such that for every $\lambda>0$

$$\lim_{t\to+\infty}\frac{\Phi(t)}{\Phi_1(\lambda t)} = +\infty. \tag{3.5}$$

Indeed, under assumption (3.5), if $(u_r)\in W^{1,\Phi,1}\left(\Omega,\mathbb{R}^N\right)$ satisfies the integral boundedness condition (1.6), the sequence $\left(\Phi_1\left(|Du_r|\right)\right)$ is equi-absolutely integrable by De la Vallée Poissin's criterion (see [KR, p.95]), then arguing like Kristensen (Theorem 3.1 Step1 [Kr]*) we reduce to the case $f\ge0$.*

Remark 3.4. *Following Marcellini* [Ma1] *(see also* [F]*) one can prove that quasi-convexity and (3.4) yield for every A, $B\in\mathbb{R}^{Nn}$*

$$\left|f(A)-f(B)\right| \le c\left(1+\frac{\Phi\left(2\left(1+|A|+|B|\right)\right)}{1+|A|+|B|}\right)|A-B|.$$

This kind of control on f is no longer utilizable in our setting when Φ is a N-function not in class $\triangle_2$.

First we prove a special case.

Lemma 3.3. *If in the statement of Theorem 3.2 the limit u is affine, i.e., $Du(x) \equiv A_o$ for some $A_o \in \mathbb{R}^{Nn}$ and $\mathcal{L}^n$ a.e. $x \in \Omega$, then*

$$\liminf_{r\to\infty} F(u_r,\Omega) \geq F(u,\Omega).$$

Proof. *Step 1*: Suppose u_r, u have the same boundary values, i.e., $(u-u_r) \in W_o^{1,\Phi,1}(\Omega,\mathbb{R}^N)$ for every r, then the result easily follows by quasi-convexity and Remark 3.2.

Step 2: Suppose that $(u_r) \in W^{1,\Phi,\lambda}(\Omega,\mathbb{R}^N)$ for some $\lambda > 1$ and that

$$\sup_r \int_\Omega \Phi(\lambda|Du_r|)\,dx < +\infty. \tag{3.6}$$

Proceeding as Marcellini [Ma1],[Ma2] we change the boundary value of u_r in a suitable way. Let $\Omega_o \subset\subset \Omega$ be an open set, fix $k = \frac{1}{2}dist\left(\overline{\Omega_o},\partial\Omega\right)$ and $h \in \mathbb{N}$, then for $1 \leq i \leq h$ define the open sets

$$\Omega_i = \left\{x \in \Omega : dist(x,\partial\Omega) < \tfrac{i}{h}k\right\}$$

and consider a family of cut-off functions $\varphi_i \in C_c^\infty(\Omega)$ such that

$$0 \leq \varphi_i \leq 1,\ \varphi_i \equiv 1 \text{ on } \Omega_{i-1},\ \varphi_i \equiv 0 \text{ on } \Omega\setminus\Omega_i,\ |D\varphi_i| \leq \tfrac{h+1}{k}.$$

For every r let $v_r = u_r - u$, notice that $v_r \to 0$ $s-L^1_{loc}(\Omega,\mathbb{R}^N)$, then define the functions

$$v_{i,r} = \varphi_i v_r,$$

thus $v_{i,r} \in W_o^{1,\Phi,1}(\Omega,\mathbb{R}^N)$ for every i provided r is big enough. Indeed, $v_{i,r} \in W_o^{1,1}(\Omega,\mathbb{R}^N)$ by the very definition, moreover applying twice (2.2) and by the choice of φ_i we get

$$\int_\Omega \Phi(|Dv_{i,r}|)\,dx \leq \int_\Omega \Phi(\lambda|Du_r|)\,dx$$
$$+\Phi\left(\tfrac{\lambda}{\sqrt{\lambda}-1}|A_o|\right)\mathcal{L}^n(\Omega) + \int_\Omega \Phi\left(\tfrac{h+1}{k}\tfrac{\sqrt{\lambda}}{\sqrt{\lambda}-1}|v_r|\right)dx.$$

The assertion follows from (3.6) and Theorem 2.4, since the compactness of the embedding $W^1L^\Phi(\Omega,\mathbb{R}^N) \to L^\Phi(\Omega,\mathbb{R}^N)$ implies $v_r \to 0$ $s-L^\Phi(\Omega,\mathbb{R}^N)$ and thus by Lemma 2.3 for every $\sigma > 0$ there holds

$$\lim_{r\to+\infty}\int_\Omega \Phi(\sigma|v_r|)\,dx = 0.$$

By Step 1 we deduce

$$\begin{aligned}
F(u,\Omega) &\leq F(u+v_{i,r},\Omega) = \int_\Omega f(A_o + Dv_{i,r})\,dx\\
&= \int_{\Omega_{i-1}} f(Du_r)\,dx + \int_{\Omega_i\setminus\Omega_{i-1}} f(A_o+Dv_{i,r})\,dx + \int_{\Omega\setminus\Omega_i} f(A_o)\,dx\\
&\leq \int_\Omega f(Du_r)\,dx + \int_{\Omega_i\setminus\Omega_{i-1}} f(A_o+Dv_{i,r})\,dx + f(A_o)\mathcal{L}^n(\Omega\setminus\Omega_o).
\end{aligned} \tag{3.7}$$

Choosing $1 < \theta < \lambda$, by (3.6) and (2.2) we have

$$\sup_r \int_\Omega \Phi\left(\theta\left|Dv_r\right|\right) dx$$
$$\leq \sup_r \int_\Omega \Phi\left(\lambda\left|Du_r\right|\right) dx + \Phi\left(\tfrac{\lambda\theta}{\lambda-\theta}\left|A_o\right|\right) \mathcal{L}^n\left(\Omega\right) \leq c_1 < +\infty,$$

therefore there exists $1 \leq j \leq h$ such that

$$\sup_r \int_{\Omega_j \backslash \Omega_{j-1}} \Phi\left(\theta\left|Dv_r\right|\right) dx \leq \frac{c_1}{h}. \tag{3.8}$$

Now we estimate the integrals in (3.7) for such j. By applying (2.2) and by (3.8) we get

$$\int_{\Omega_j \backslash \Omega_{j-1}} f\left(A_o + Dv_{j,r}\right) dx$$
$$\leq c \int_{\Omega_j \backslash \Omega_{j-1}} \left(1 + \Phi\left(\left|A_o\right| + \left|\varphi_j\right|\left|Dv_r\right| + \left|D\varphi_j\right|\left|v_r\right|\right)\right) dx$$
$$\leq c_2 \mathcal{L}^n\left(\Omega \backslash \Omega_o\right) + \frac{c_3}{h} + c_4 \int_\Omega \Phi\left(\tfrac{h+1}{k} \tfrac{\theta}{\sqrt{\theta}-1}\left|v_r\right|\right) dx. \tag{3.9}$$

So by (3.9), (3.7) becomes

$$F\left(u,\Omega\right) \leq F\left(u_r,\Omega\right) + \frac{c_3}{h} + c_4 \int_\Omega \Phi\left(\tfrac{h+1}{k} \tfrac{\theta}{\sqrt{\theta}-1}\left|v_r\right|\right) dx + c_5 \mathcal{L}^n\left(\Omega \backslash \Omega_o\right),$$

the assertion then follows passing to the limit for $r \to +\infty$, $\mathcal{L}^n\left(\Omega \backslash \Omega_o\right) \to 0$ and $h \to +\infty$.

Step 3: Let us remove assumption (3.6). Given $(u_r) \in W^{1,\Phi,1}\left(\Omega, \mathbb{R}^N\right)$ satisfying (1.6) consider a subsequence, not relabelled for convenience, such that

$$\lim_{r \to +\infty} \int_\Omega \Phi\left(\left|Du_r\right|\right) dx = \liminf_{r \to +\infty} \int_\Omega \Phi\left(\left|Du_r\right|\right) dx. \tag{3.10}$$

Fix $\lambda > 1$, then define

$$u_{r,\lambda} = \tfrac{1}{\lambda} u_r \text{ and } u_\lambda = \tfrac{1}{\lambda} u.$$

Notice that $(u_{r,\lambda})$, $u_\lambda \in W^{1,\Phi,\lambda}\left(\Omega, \mathbb{R}^N\right)$, $u_{r,\lambda} \to u_\lambda$ $s - L^1_{loc}\left(\Omega, \mathbb{R}^N\right)$ and $(Du_{r,\lambda})$ satisfies condition (3.6), hence by Step2 we get

$$F\left(u_\lambda,\Omega\right) \leq \liminf_{r \to +\infty} F\left(u_{r,\lambda},\Omega\right). \tag{3.11}$$

Since by (3.3) of Lemma 3.1 for every r and for $\mathcal{L}^n$ a.e. $x \in \Omega$ there holds

$$f\left(Du_{r,\lambda}\left(x\right)\right) \leq \tfrac{1}{\lambda^{Nn}} f\left(Du_r\left(x\right)\right) + c\left(1 - \tfrac{1}{\lambda^{Nn}}\right)\left(1 + \Phi\left(\left|Du_r\left(x\right)\right|\right)\right), \tag{3.12}$$

integrating the inequality above and setting $k = \sup_r \int_\Omega \Phi\left(\left|Du_r\right|\right) dx$, with $k < +\infty$ by (3.10), we get

$$F\left(u_{r,\lambda},\Omega\right) \leq \tfrac{1}{\lambda^{Nn}} F\left(u_r,\Omega\right) + c\left(1 - \tfrac{1}{\lambda^{Nn}}\right)\left(k + \mathcal{L}^n\left(\Omega\right)\right). \tag{3.13}$$

Then, by passing to the inferior limit in (3.13), we get by (3.11)

$$F\left(u_\lambda,\Omega\right) \leq \tfrac{1}{\lambda^{Nn}} \liminf_{r \to +\infty} F\left(u_r,\Omega\right) + c\left(1 - \tfrac{1}{\lambda^{Nn}}\right)\left(k + \mathcal{L}^n\left(\Omega\right)\right). \tag{3.14}$$

Eventually, since $u_\lambda \to u \ s - W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$ and since $F(\cdot,\Omega)$ is sequentially lower semicontinuous in that convergence by a simple application of Fatou's lemma, there holds

$$F(u,\Omega) \le \liminf_{\lambda\to 1^+} F(u_\lambda,\Omega) \le \liminf_{r\to+\infty} F(u_r,\Omega)$$

passing to the inferior limit for $\lambda \to 1^+$ on both sides of (3.14). □

The proof of Theorem 3.2 now follows using the Fonseca-Müller's blow-up technique [FoMu] (see also [FoMa],[FoM]).

Proof.(Theorem 3.2) Given $(u_r) \in W^{1,\Phi,1}L^\Phi\left(\Omega,\mathbb{R}^N\right)$ satisfying condition (1.6) we get

$$\liminf_{r\to+\infty} F(u_r,\Omega) < +\infty.$$

Moreover, condition (1.6), Theorem 2.4 and Theorem 2.7 assure that $u_r \to u \ s - L^\Phi\left(\Omega,\mathbb{R}^N\right)$, and by extracting subsequences, not relabelled for convenience, we have that

$$\liminf_{r\to+\infty} F(u_r,\Omega) = \lim_{r\to+\infty} F(u_r,\Omega).$$

Moreover, we can assume the existence of μ, ν positive and finite Radon measures such that

$$\mu = \lim_{r\to+\infty} \mathcal{L}^n\lfloor f(Du_r), \nu = \lim_{r\to+\infty} \mathcal{L}^n\lfloor \Phi(|Du_r|), \tag{3.15}$$

where, given any mesurable function $g : \Omega \to [0,+\infty)$ the measure $\mathcal{L}^n\lfloor g$ is defined on Borel sets of Ω by

$$(\mathcal{L}^n\lfloor g)(E) = \int_E g(x)\,dx,$$

and the limits in (3.15) are to be intended in the sense of measures, i.e., for every $\varphi \in C_c^0\left(\Omega,\mathbb{R}^N\right)$ there holds

$$\lim_{r\to+\infty}\int_\Omega \varphi f(Du_r)\,dx = \int_\Omega \varphi d\mu; \ \lim_{r\to+\infty}\int_\Omega \varphi\Phi(|Du_r|)\,dx = \int_\Omega \varphi d\nu.$$

We are going to show that for $\mathcal{L}^n$ a.e. $x \in \Omega$ there holds

$$\frac{d\mu}{d\mathcal{L}^n}(x) = \lim_{\varepsilon\to 0^+}\frac{\mu\left(B_{(x,\varepsilon)}\right)}{\mathcal{L}^n\left(B_{(x,\varepsilon)}\right)} \ge f(Du(x)). \tag{3.16}$$

Indeed, if (3.16) holds, we have that for any $\varphi \in C_c^0\left(\Omega,\mathbb{R}^N\right)$ such that $0 \le \varphi \le 1$

$$\lim_{r\to+\infty} F(u_r,\Omega) \ge \lim_{r\to+\infty}\int_\Omega \varphi f(Du_r)\,dx = \int_\Omega \varphi d\mu \ge \int_\Omega \varphi f(Du)\,dx,$$

thus the lower semicontinuity inequality follows letting φ increase to 1 and applying Levi's theorem.

To prove (3.16) we recall that there exists a set $\Omega_o \subset \Omega$ such that $\mathcal{L}^n(\Omega\backslash\Omega_o) = 0$, and that if $x \in \Omega_o$ the quantities

$$\frac{d\mu}{d\mathcal{L}^n}(x), \frac{d\nu}{d\mathcal{L}^n}(x) \text{ are finite} \tag{3.17}$$

and

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{n+1}}\int_{B_{(x,\varepsilon)}}|u(y)-u(x)-Du(x)(y-x)|\,dy=0. \tag{3.18}$$

Let $x_o\in\Omega_o$ and let $\varepsilon_k\to 0^+$ be such that $\mu\left(\partial B_{(x_o,\varepsilon_k)}\right)=0$, $\nu\left(\partial B_{(x_o,\varepsilon_k)}\right)=0$ for every k, then, setting $B=B_{(0,1)}$ and $\omega_n=\mathcal{L}^n(B)$, we get

$$\begin{aligned}\lim_{k\to+\infty}\frac{\mu\left(B_{(x_o,\varepsilon_k)}\right)}{\mathcal{L}^n\left(B_{(x_o,\varepsilon_k)}\right)}&=\lim_{k\to+\infty}\lim_{r\to+\infty}\int_{B_{(x_o,\varepsilon_k)}}f(Du_r)\,dx\\&=\lim_{k\to+\infty}\lim_{r\to+\infty}\frac{1}{\omega_n}\int_B f(Du_{r,k})\,dx,\end{aligned}$$

where for every $y\in B$

$$u_{r,k}(y)=\frac{1}{\varepsilon_k}\left(u_r(x_o+\varepsilon_k y)-u(x_o)\right).$$

Notice that $(u_{r,k})\in W^{1,\Phi,1}\left(B,\mathbb{R}^N\right)$ and $(\Phi(|Du_{r,k}|))$ is $L^1\left(B,\mathbb{R}^N\right)$ norm bounded. Indeed, by the choice of x_o we have

$$\begin{aligned}&\lim_{k\to+\infty}\lim_{r\to+\infty}\int_B\Phi(|Du_{r,k}|)\,dx\\&\quad=\lim_{k\to+\infty}\lim_{r\to+\infty}\frac{1}{\varepsilon_k^n}\int_{B_{(x_o,\varepsilon_k)}}\Phi(|Du_r|)\,dx=\omega_n\frac{d\nu}{d\mathcal{L}^n}(x_o)<+\infty.\end{aligned} \tag{3.19}$$

By taking into account the convergence $u_r\to u$ $s-L^\Phi\left(\Omega,\mathbb{R}^N\right)$ and (3.18) for $x=x_o$ and setting $u_o(x)=Du(x_o)\,x$, we get

$$\lim_{k\to+\infty}\lim_{r\to+\infty}\|u_{r,k}-u_o\|_{L^1(B,\mathbb{R}^N)}=0.$$

Thus $(u_{r,k})$ has a subsequence $v_k=u_{r_k,k}$ which is $s-L^1\left(B,\mathbb{R}^N\right)$ converging to the affine function u_o. Eventually, since by (3.19) (v_k) satisfies (1.6), by Lemma 3.3 inequality (3.16) follows, i.e.,

$$\frac{d\mu}{d\mathcal{L}^n}(x_o)=\lim_{k\to+\infty}\frac{1}{\omega_n}\int_B f(Dv_k)\,dx\geq f(Du(x_o)).\ \square$$

The previous theorem can be applied to solve Dirichlet's boundary value problems.

Corollary 3.4. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded and open set, let $f:\mathbb{R}^{Nn}\to\mathbb{R}$ be a quasi-convex function satisfying for every $A\in\mathbb{R}^{Nn}$*

$$c(\Phi(|A|)-1)\leq f(A)\leq c(1+\Phi(|A|)), \tag{3.20}$$

with c a positive constant and Φ a N-function. Let $F(\cdot,\Omega)$ be defined as in (1.1), $u_o\in W^{1,\Phi,1}\left(\Omega,\mathbb{R}^N\right)$, set $V=u_o+W_o^{1,1}\left(\Omega,\mathbb{R}^N\right)$, then the minimum problem

$$m=\inf_V F(\cdot,\Omega) \tag{3.21}$$

has solution.

Proof. Assumption $u_o\in W^{1,\Phi,1}\left(\Omega,\mathbb{R}^N\right)$ and the growth condition (3.20) assure that $-\infty<m<+\infty$. Let $(v_r)\subset V$ be a minimizing sequence for $F(\cdot,\Omega)$ on V, i.e.,

$$\lim_{r\to+\infty}F(v_r,\Omega)=m,$$

then (3.20) implies

$$\sup_r \int_\Omega \Phi\left(|Dv_r|\right) dx < +\infty. \tag{3.22}$$

Let $u_r = v_r - u_o$, then by (2.2), (3.22) implies $u_r \in W_o^{1,\Phi,\frac{1}{2}}\left(\Omega, \mathbb{R}^N\right)$ and

$$\sup_r \int_\Omega \Phi\left(\tfrac{1}{2}|Du_r|\right) dx \le \int_\Omega \Phi\left(|Du_o|\right) dx + \sup_r \int_\Omega \Phi\left(|Dv_r|\right) dx. \tag{3.23}$$

Poincaré inequality yields

$$\sup_r \|u_r\|_{W^{1,1}(\Omega,\mathbb{R}^N)} < +\infty,$$

thus, (3.23), Dunford-Pettis' theorem and Rellich-Kondrakov's theorem imply the existence of $u \in W^{1,1}\left(\Omega, \mathbb{R}^N\right)$ and a subsequence of (u_r), not relabelled for convenience, such that $u_r \to u$ $w - W^{1,1}\left(\Omega, \mathbb{R}^N\right)$ and $s - L^1\left(\Omega, \mathbb{R}^N\right)$.
Then $u \in W_o^{1,1}\left(\Omega, \mathbb{R}^N\right)$, and $(u_o + u) \in V \cap W^{1,\Phi,1}\left(\Omega, \mathbb{R}^N\right)$ since by (3.22)

$$\int_\Omega \Phi\left(|D(u_o + u)|\right) dx \le \underline{\lim}_{r\to+\infty} \int_\Omega \Phi\left(|Dv_r|\right) dx < +\infty.$$

Eventually, by applying Theorem 3.2, $(u_o + u)$ is a minimizer for $F(\cdot, \Omega)$ on V. □

Remark 3.5. *The assumption $u_o \in W^{1,\Phi,1}\left(\Omega, \mathbb{R}^N\right)$ is necessary for the problem to be well posed if we want u_o itself to be in the competing class V and the functional $F(\cdot, \Omega)$ to be finite a priori in at least one point.*

Remark 3.6. *We point out that since the convergence introduced in (1.6) implies $*w-\ W^1L^\Phi\left(\Omega, \mathbb{R}^N\right)$ convergence, and minimizing sequences for problem (3.24) below satisfy (1.6) because of (3.20), Theorem 3.2 applies also to solve*

$$\inf\left\{F(\cdot, \Omega) : u \in u_o + W_o^1 L^\Phi\left(\Omega, \mathbb{R}^N\right)\right\}. \tag{3.24}$$

Remark 3.7. *In our general setting we avoid to consider the minimum problem*

$$\inf\left\{F(\cdot, \Omega) : u \in u_o + W_o^{1,\Phi,1}\left(\Omega, \mathbb{R}^N\right)\right\}, \tag{3.25}$$

since, if $\Phi \notin \triangle_2$, condition (1.6) is not sufficient to ensure the weak $*$ closure of $W_o^{1,\Phi,1}\left(\Omega, \mathbb{R}^N\right)$. Indeed, from the proof of Corollary 3.4 we can only deduce that the minimizers belong to the class $u_o + W_o^{1,\Phi,\frac{1}{2}}\left(\Omega, \mathbb{R}^N\right)$.

Anyhow, we emphasize that the set where we consider the minimum problem is the domain of the functional.

Remark 3.8. *In case $\Phi \in \triangle_2$ all the minimum problems (3.21), (3.24), (3.25) reduce to the same since in that case $*w-\ W^1L^\Phi\left(\Omega, \mathbb{R}^N\right)$ convergence is equivalent to the convergence introduced in (1.6), cfr. Lemma 2.2, and $W_o^{1,\Phi,1}\left(\Omega, \mathbb{R}^N\right) \equiv W_o^1 L^\Phi\left(\Omega, \mathbb{R}^N\right) \equiv W_o^1 E^\Phi\left(\Omega, \mathbb{R}^N\right)$ (see* [Fog],[Go3]*).*

4. Quasi-convex functions with non-standard growth.

In this section we exhibit some quasi-convex functions satisfying conditions (1.7), (1.8) with the N-function Φ not necessarily belonging to $\triangle_2$. Actually, concerning condition (1.8), we are not able to deal with the general case but we produce such quasi-convex functions if the dominating N-function Φ satisfies a sort of sub-additivity condition at infinity, i.e., there exists $r_o > 0$ such that

$$C_{\Phi}(r_o) = \limsup_{t\to+\infty} \frac{\Phi(t+r_o)}{\Phi(t)+\Phi(r_o)} < +\infty. \tag{4.1}$$

When (4.1) holds, it is easy to prove that $C_\Phi(r) < +\infty$ for every $r > 0$ and that the map $C_\Phi : [0,+\infty) \to [0,+\infty)$ is non-decreasing and lower bounded by $C_\Phi(0) = 1$.

Notice that by (2.2) and (2.4) $\Phi \in \triangle_2$ implies $C_\Phi(r) \equiv 1$, but $\triangle_2$ N-functions are not the only ones satisfying (4.1). Indeed, consider the N-functions $\Gamma_0(t) = t^{\ln t}$ and $\Gamma_\beta(t) = \exp\left(t^\beta\right) - 1$, $0 < \beta \le 1$, then $\Gamma_0, \Gamma_\beta \notin \triangle_2$, but an easy computation yields $C_{\Gamma_0}(r) \equiv 1$, $C_{\Gamma_\beta}(r) \equiv 1$, $0 < \beta < 1$, and $C_{\Gamma_1}(r) = \exp(r)$.

Moreover, we remark that (4.1) is not fulfilled if the exponential growth is too fast, e.g. $C_{\Gamma_\beta}(r) \equiv +\infty$ for any $\beta > 1$.

We now construct a N-function satisfying (4.1) with polynomial growth and not belonging to class $\triangle_2$. A first example of this kind was produced by Krasnosel'skij and Rutickii (see [KR, p.29],[RR, p.27]).

Fix $a > 1$ and $1 < q < p$, define the function $\varphi_{q,p} : [0,+\infty) \to [0,+\infty)$ as

$$\varphi_{q,p}(s) = \begin{cases} qs^{q-1} & 0 \le s \le 1 \\ ps^{p-1} & 1 \le s \le a \\ \alpha_i & s \in [a_i, a_{i+1}] \end{cases} \tag{4.2}$$

where α_i and a_i are defined recursively by: $a_0 = a$ and for $i \ge 0$

$$\alpha_i = pa_i^{p-1} = qa_{i+1}^{q-1}. \tag{4.3}$$

Then define $\Phi_{q,p} : [0,+\infty) \to [0,+\infty)$ by

$$\Phi_{q,p}(t) = \int_0^t \varphi_{q,p}(s)\, ds, \tag{4.4}$$

we claim that $\Phi_{q,p}$ is a N-function satisfying the desired properties.

By their very definition the sequences (a_i), (α_i) and $\left(\frac{\alpha_i}{\alpha_{i-1}}\right)$ are increasingly diverging to $+\infty$. Moreover, by direct computation if i is large enough we have

$$\Phi_{q,p}(2a_i) \ge \left(1 + \tfrac{\alpha_i}{\alpha_{i-1}}\right) \Phi_{q,p}(a_i). \tag{4.5}$$

Indeed, since $2a_i \le a_{i+1}$ for i sufficiently large, by definition (4.4) we get

$$\Phi_{q,p}(2a_i) = \Phi_{q,p}(a_i) + a_i\alpha_i, \tag{4.6}$$

so that (4.5) holds if and only if

$$\tfrac{1}{\alpha_{i-1}} \Phi_{q,p}(a_i) \le a_i. \tag{4.7}$$

Notice that since (α_i) is increasing and diverging to $+\infty$, from (4.2) there follows

$$\Phi_{q,p}(a_i) \le \Phi_{q,p}(a_0) + \alpha_{i-1}(a_i - a_0), \tag{4.8}$$

and thus (4.7) follows for i sufficiently large.
A similar computation holds true for the complementary N-function $\Psi_{q,p}$ of $\Phi_{q,p}$, so that neither $\Phi_{q,p}$ nor $\Psi_{q,p}$ belong to class $\triangle_2$.

Notice that $\Phi_{q,p}$ has q,p growth, i.e., there exist $c_i > 0$, $1 \le i \le 4$, such that

$$c_1 t^q - c_2 \le \Phi_{q,p}(t) \le c_3 t^p + c_4.$$

Moreover, these are the best powers to estimate $\Phi_{q,p}$, i.e., if $r \in (q,p)$ then

$$\liminf_{t\to+\infty} \frac{\Phi_{q,p}(t)}{t^r} = 0, \ \limsup_{t\to+\infty} \frac{\Phi_{q,p}(t)}{t^r} = +\infty.$$

Indeed, by (4.8) there follows

$$\begin{aligned} 0 &\le \liminf_{t\to+\infty} \frac{\Phi_{q,p}(t)}{t^r} \le \liminf_{i\to+\infty} \frac{\Phi_{q,p}(a_i)}{a_i^r} \\ &\le \liminf_{i\to+\infty} \left(\frac{\Phi_{q,p}(a_0)}{a_i^r} + \frac{\alpha_{i-1}(a_i - a_0)}{a_i^r} \right) = q \liminf_{i\to+\infty} a_i^{q-r} = 0. \end{aligned}$$

Now let $b_i = \frac{r}{r-1} a_i$, then $b_i \in (a_i, a_{i+1})$ and

$$\begin{aligned} \limsup_{t\to+\infty} \frac{\Phi_{q,p}(t)}{t^r} &\ge \limsup_{i\to+\infty} \frac{\Phi_{q,p}(b_i)}{b_i^r} \\ &\ge \frac{1}{b_i^r} \int_{a_i}^{b_i} \varphi_{q,p}(s)\, ds = \tfrac{p(r-1)^{r-1}}{r^r} \limsup_{i\to+\infty} a_i^{p-r} = +\infty. \end{aligned}$$

Eventually, an easy computation shows that choosing $1 < q < p \le q+1$, $\Phi_{q,p}$ satisfies also (4.1).

In the sequel, given $f : \mathbb{R}^{Nn} \to \mathbb{R}$ we denote by Qf the quasi-convex envelope of f, i.e., the greatest quasi-convex function less or equal to f, which turns out to be defined by

$$Qf = \sup \{ g \le f : q \text{ quasi-convex} \}.$$

Following Zhang [Z], assume we are given a quasi-convex function f for which the sub-level set

$$K_\alpha = \left\{ A \in \mathcal{M}^{N\times n} : f(A) \le \alpha \right\}$$

is compact and non convex for some $\alpha \in \mathbb{R}$, then in Theorem 1.1 of the same paper it is proven that the quasi-convex envelope of the distance function from K_α, $Qd(\cdot, K_\alpha)$, satisfies

$$Qd(A, K_\alpha) = 0 \Leftrightarrow A \in K_\alpha.$$

Therefore, the function $f_q : \mathcal{M}^{N\times n} \to [0, +\infty)$ defined by

$$f_q(A) = \max \left\{ [d(A, coK_\alpha)]^q, Qd(A, K_\alpha) \right\},$$

where coK_α is the convex hull of K_α, is quasi-convex, non convex and satisfies

$$c_1 |A|^q - c_2 \le f_q(A) \le c_3 |A|^q + c_4$$

for some positive constants c_i, $1 \le i \le 4$, and for every $A \in \mathcal{M}^{N\times n}$.

We want to generalize that construction using N-functions as well as powers. First notice that given any N-function Φ, the function

$$g_\Phi(A) = \Phi(Qd(A, K_\alpha)) \tag{4.9}$$

is quasi-convex, non convex and it satisfies (1.7) provided $0 \in K_\alpha$.
Thus, as we will see in the sequel, assumption (4.1) on Φ plays a crucial role if we want to construct a quasi-convex function satisfying the more restrictive condition (1.8). Now let Φ be a N-function satisfying (4.1) and define

$$f_\Phi(A) = \max \left\{ \Phi(d(A, coK_\alpha)); Qd(A, K_\alpha) \right\}, \tag{4.10}$$

then f_Φ turns out to be quasi-convex and non convex since $f_\Phi(A) \le 0$ if and only if $A \in K_\alpha$.

Let us prove that there exist positive constants c_i, $1 \le i \le 4$, such that for every $A \in \mathcal{M}^{N\times n}$ there holds

$$c_1 \Phi(|A|) - c_2 \le f_\Phi(A) \le c_3 \Phi(|A|) + c_4. \tag{4.11}$$

Notice that (4.11) is equivalent to proving

$$0 < \liminf_{|A|\to+\infty} \frac{f_\Phi(A)}{\Phi(|A|)} \le \limsup_{|A|\to+\infty} \frac{f_\Phi(A)}{\Phi(|A|)} < +\infty. \tag{4.12}$$

Let $B(0,R) \supset K_\alpha$, then, by the very definition of f_Φ, we get

$$\begin{aligned}\liminf_{|A|\to+\infty} \frac{f_\Phi(A)}{\Phi(|A|)} &\ge \liminf_{|A|\to+\infty} \frac{\Phi(d(A, coK_\alpha))}{\Phi(|A|)} \\ &\ge \liminf_{|A|\to+\infty} \frac{\Phi(\max\{|A|-R;0\})}{\Phi(|A|)} = \frac{1}{C_\Phi(R)} > 0.\end{aligned}$$

Finally, to prove (4.12) notice that since K_α is bounded for every $A \in \mathcal{M}^{N\times n}$ there holds

$$Qd(A, K_\alpha) - \mathrm{diam} K_\alpha \le d(A, coK_\alpha) \le Qd(A, K_\alpha),$$

so that for $|A|$ sufficiently large we have

$$f_\Phi(A) = \Phi(d(A, coK_\alpha)).$$

Thus, since the map $d(\cdot, coK_\alpha)$ is Lipschitz continuous with Lipschitz constant 1, we get by condition (4.1)

$$\begin{aligned}&\limsup_{|A|\to+\infty} \frac{f_\Phi(A)}{\Phi(|A|)} \\ &\quad\le \limsup_{|A|\to+\infty} \frac{\Phi(|A| + d(0, coK_\alpha))}{\Phi(|A|)} = C_\Phi(d(0, \mathrm{co}K_\alpha)) < +\infty.\end{aligned}$$

In order to provide an explicit example of such a construction consider $A, B \in \mathcal{M}^{N\times n}$ such that $\mathrm{rank}(A-B) \ge 2$ and set $K = \{A, B\}$. Then K is compact and not convex. Moreover, it is well known (see [Z]) that there exists a non negative function with sub-quadratic growth whose zero set is K.

In the sequel we will construct quasi-convex functions with such a choice of K following the previous scheme. Let $g_{q,p}$ be defined by (4.9), where $\Phi_{q,p}$ is defined by (4.2) with $1 < q < p$, then $g_{q,p}$ is a quasi-convex, non convex function.

Consider the functional

$$G_{q,p}(u, \Omega) = \int_\Omega g_{q,p}(Du(x))\, dx,$$

then Theorem 3.2 assures the lower semicontinuity of $G_{q,p}(\cdot, \Omega)$ in a different topology with respect to all the results provided by classical Sobolev spaces (see all the references in the Introduction).

Now let f_{Γ_β} be defined by (4.10), where $\Gamma_\beta(t) = \exp\left(t^\beta\right) - 1$ for any $0 < \beta \le 1$, thus f_{Γ_β} is quasi-convex and non convex but we do not know whether it is polyconvex or not. Consider the functional

$$F_\beta(u, \Omega) = \int_\Omega f_{\Gamma_\beta}(Du(x))\, dx,$$

then Theorem 3.2 assures its lower semicontinuity with respect to convergence introduced in (1.6) and Corollary 3.4 applies to finding minimizers for an exponential growth type Dirichlet's boundary value problem.

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