Variational Approximation of Free-Discontinuity Energies with Linear Growth

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Abstract

We provide a variational approximation for quasiconvex energies with linear growth, defined on vector valued generalized functions with bounded variation, in the framework of free-discontinuity problems.

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1 Introduction

Many problems arising in Mathematical Physics, Computer Vision Theory and Fracture Mechanics can be modelled as minimum problems of energies involving competing bulk and surface terms. A mathematical theory to prove existence and regularity results for this type of variational problems has been developed in the framework of BV, SBV functions, where the energies to be studied have the following general form

$$\int_\Omega f\left(x,u,\nabla u\right)\,dx+\int_\Omega h\left(x,u,dD^cu\right)+\int_{J_u}K\left(x,u^+,u^-,\nu_u\right)\,d\mathcal{H}^{n-1}, \tag{1.1}$$

(see the Preliminaries for the definition of all the quantities above). According to a terminology introduced by De Giorgi these problems are usually labelled as free-discontinuity problems. The difficulty arising in the numerical approximation of the solutions of these problems can be overcome by performing a preliminary variational approximation, in the sense of De Giorgi's Γ-convergence [16], via simpler functionals defined on spaces of smooth functions. In this framework we mention the model of Ambrosio and Tortorelli [7], [8]. By introducing an auxiliary variable v which asymptotically approaches $1-\mathcal{X}_{J_u}$, they consider approximating functionals defined for $u\in W^{1,p}\left(\Omega\right)$, $v\in W^{1,2}\left(\Omega\right)$ by

$$\int_\Omega v^2\left|\nabla u\right|^p\,dx+\int_\Omega\left(\frac{1}{\varepsilon}(1-v)^2+\varepsilon\left|\nabla v\right|^2\right)dx \tag{1.2}$$

for $p>1$, obtaining in the limit energies defined on $SBV\left(\Omega\right)$ by

$$\int_\Omega\left|\nabla u\right|^p\,dx+c\,\mathcal{H}^{n-1}\left(J_u\right).$$

This construction has been extended to the anisotropic vectorial case in [23], where the limit functionals take the form on $SBV\left(\Omega;\mathbb{R}^N\right)$

$$\int_\Omega f\left(x,u,\nabla u\right)\,dx+\int_{J_u}\varphi\left(\nu_u\right)\,d\mathcal{H}^{n-1} \tag{1.3}$$

with f a quasiconvex function in the gradient variable satisfying superlinear growth conditions and φ a norm. Energies of type (1.3) arise in particular in the Griffith's theory of Fracture Mechanics [3], [29], where u denotes the deformation of an hyperelastic and brittle body and J_u the crack surface.

To approximate more complex surface energies depending also on the traces $u^{\pm}$, which arise for instance in fracture models of Barenblatt's type [3], [9], in [1] they study a variant of the Ambrosio and Tortorelli construction, by replacing in (1.2) $|\nabla u|^p$ with $f\left(|\nabla u|\right)$ where f is convex and with linear growth. Indeed, this weaker penalization of ∇u enables a stronger interaction between the two competing terms in (1.2).

An obvious consequence of the linear growth assumption is the presence of a term accounting for the Cantor part of Du in the limit energy, which has the form on $BV\left(\Omega\right)$

$$\int_\Omega f\left(|\nabla u|\right)\,dx+\|D^c u\|\left(\Omega\right)+\int_{J_u} g\left(\left|u^+-u^-\right|\right)\,d\mathcal{H}^{n-1},$$

where g is defined by a suitable minimization formula highlighting the contribute of the two terms of (1.2). Their analysis is restricted to the scalar isotropic case where the use of an integral-geometric argument allows to reduce the n-dimensional problem to the 1-dimensional case.

In this paper we consider the full vectorial problem by studying the Γ-limit of the family of functionals defined for $u\in W^{1,1}\left(\Omega;\mathbb{R}^N\right)$, $v\in W^{1,2}\left(\Omega\right)$ by

$$F_\varepsilon(u,v)=\int_\Omega \psi(v)f\left(x,u,\nabla u\right)\,dx+\int_\Omega\left(\frac{1}{\varepsilon}W(v)+\varepsilon\left|\nabla v\right|^2\right)\,dx, \tag{1.4}$$

where f is a quasiconvex function in the gradient variable and satisfies linear growth conditions (for the set of assumptions on ψ, f and W see Section 3). We obtain in the limit functionals as in (1.1) with $h=f^\infty$, the recession function of f, and K defined suitably (see Section 3).

Due to the generality of the problems the mentioned integral-geometric approach does not longer apply and different arguments have to be exploited. The main tool of our analysis is the blow-up technique of Fonseca and Müller [25], [26] which has been intensively used for the study of the relaxation and lower semicontinuity properties of functional with linear growth ([24], [25], [26]) and for the study of anisotropic singular perturbations of nonconvex functionals in the vector-valued case [10]. We point out that in order to get more information on the interaction between the two terms in (1.4) we treat the two variables u, v as a single vector-valued one.

The proofs of the estimates on the diffuse and jump part of the limit functional rely on different arguments. The analysis of the diffuse part is reduced to the identification of relaxed functionals with linear growth as considered in [24]. In fact, one can note that for every (u,v) and $\varepsilon > 0$ we have

$$F_\varepsilon(u,v) \geq \int_\Omega \psi(v) f\left(x, u, \nabla u\right)\, dx + 2\int_\Omega \sqrt{W(v)}\, |\nabla v|\, dx,$$

and the diffuse part of the relaxation of the functional on the right-hand side above turns out to be the corresponding part of the limit functional.

For what concerns the surface part a non trivial use of blow-up techniques and De Giorgi's type averaging-slicing lemma (see Subsection 4.2) are needed to show that the surface energy density K can be written in terms of Dirichlet's boundary value problems, in the spirit of [10], that is

$$\begin{aligned} K\left(x_o, a, b, \nu\right) := \inf \Bigg\{ \int_{Q_\nu} \left(\psi(v) f^\infty\left(x_o, u, \nabla u\right) + L\, W(v) + \frac{1}{L}\left|\nabla v\right|^2\right) dx : \\ L > 0,\ (u,v) \in W^{1,1}\left(Q_\nu; \mathbb{R}^{N+1}\right), \\ (u,v) = (a,1)\ on\ \left(\partial Q_\nu\right)^-,\ (u,v) = (b,1)\ on\ \left(\partial Q_\nu\right)^+ \Bigg\}, \end{aligned}$$

where Q_ν is an open unit cube with two faces orthogonal to the direction ν and $(\partial Q_\nu)^\pm = \partial Q_\nu \cap \{\pm\langle x, \nu\rangle > 0\}$.

We point out that, even for scalar valued functions u, the minimization problems above are of vectorial type. This fact places some difficulty in order to give an explicit expression to K in the general case, while this can be done under isotropy assumptions on f^∞, as we show in Subsection 3.1. In such a case we prove that K can be calculated by restricting the infimum to functions (u,v) with one-dimensional profile. By virtue of this characterization, we provide an extension to the isotropic vector-valued case of the result of [1] (see Remarks 3.5, 3.9).

The paper is organized as follows: in Section 2 we recall some basic properties of Γ-convergence, BV and GBV functions and prove some preliminaty results; in Section 3 we state and discuss the main result of the paper (Theorem 3.2); Sections 4 and 5 are devoted to the proof of Theorem 3.2 in the BV case; in Section 6 we prove the full result; Section 7 is devoted to a convergence result for the minimizers of the approximating functionals; in Section 8 we discuss a generalization of the model.

2 Preliminaries and Notations

We denote by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^n$ and with $|\cdot|$ the usual euclidean norm, without specifying the dimension n when there is no risk of confusion. For every $t \in \mathbb{R}$, $[t]$ denotes its integer part.

If Ω is a bounded open subset of $\mathbb{R}^n$, $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ are the families of open and Borel subsets of Ω, respectively. We denote by $\mathcal{X}_B$ the characteristic function of the set $B \in \mathcal{B}(\Omega)$.

If μ is a Borel measure and B is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, $k \geq 0$. The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure, unless otherwise specified.

We use standard notations for Lebesgue and Sobolev spaces.

If $\nu \in \mathbb{S}^{n-1}$, we denote by $\Pi^\nu \subset \mathbb{R}^n$ the orthogonal space to ν, i.e., $\Pi^\nu = \{x \in \mathbb{R}^n : \langle \nu, x \rangle = 0\}$. With fixed $\{\nu_i\}_{1 \leq i \leq n-1}$ an orthogonal bases of Π^ν, set $Q'_\nu := \left\{ \sum_{1 \leq i \leq n-1} \lambda_i \nu_i : |\lambda_i| < \frac{1}{2} \right\}$. In case $\nu = e_n$ we take $\nu_i = e_i$, drop the subscript and use the notation $Q_{e_n} = Q$.

2.1 Relaxation and Γ-convergence

Let (X, d) be a metric space. We first recall the notion of *relaxed functional*. Let $F : X \to [0, +\infty]$. Then the relaxed functional $\overline{F}$ of F, or *relaxation* of F, is the greatest d-lower semicontinuous functional less than or equal to F and can be characterized as follows

$$\overline{F}(u) = \inf\{\liminf_j F(u_j) : \ u_j \to u\}.$$

A family $(F_\varepsilon)_{\varepsilon>0}$ of functionals $F_\varepsilon : X \to [0, +\infty]$ is said to Γ-*converge* to a functional $F : X \to [0, +\infty]$ at $u \in X$, and we write $F(u) = \Gamma\text{-}\lim_{\varepsilon \to 0^+} F_\varepsilon(u)$, if for every sequence (ε_j) of positive numbers decreasing to 0 the following two conditions hold:

(i) (*lower semicontinuity inequality*) for all sequences (u_j) converging to u in X we have $F(u) \leq \liminf_j F_{\varepsilon_j}(u_j)$;

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u in X such that $F(u) \geq \limsup_j F_{\varepsilon_j}(u_j)$.

We say that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to F if $F(u) = \Gamma\text{-}\lim_{\varepsilon \to 0^+} F_\varepsilon(u)$ at all points $u \in X$ and that F is the Γ-*limit* of $(F_\varepsilon)_{\varepsilon>0}$. If we define the *lower and upper* Γ-*limits* by

$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon \to 0+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

$$F'(u) = \Gamma\text{-}\liminf_{\varepsilon \to 0+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

respectively, then conditions (i) and (ii) are equivalent to $F'(u) = F''(u) = F(u)$. Note that the functions F' and F'' are lower semicontinuous.

The following theorem explains why the notion of Γ-convergence is convenient in the study of the asymptotic analysis of variational problems.

Theorem 2.1 *Let $F = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon$, and let $K \subset X$ be a compact set such that $\inf_X F_\varepsilon = \inf_K F_\varepsilon$ for all $\varepsilon > 0$. Then*

$$\exists \min_X F = \lim_{\varepsilon\to 0^+} \inf_X F_\varepsilon. \tag{2.1}$$

Moreover, if (u_j) is a converging sequence such that $\lim_j F_{\varepsilon_j}(u_j) = \lim_j \inf_X F_{\varepsilon_j}$ then its limit is a minimum point for F.

We also recall the notion of $\overline{\Gamma}$-*convergence*, which is useful when dealing with the integral representation of the Γ-limit of a family of integral functionals.

Definition 2.2 *Given $\Omega \subset \mathbb{R}^n$ an open set, let $F_\varepsilon : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ be such that the set function $F_\varepsilon(u, \cdot)$ is increasing on $\mathcal{A}(\Omega)$ and set*

$$F'(\cdot, A) := \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(\cdot, A), \quad F''(\cdot, A) := \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(\cdot, A)$$

for every $A \in \mathcal{A}(\Omega)$. We say that $(F_\varepsilon)_{\varepsilon>0}$ $\overline{\Gamma}$ -converges to $F : X \times \mathcal{A}(\Omega) \to [0, +\infty]$, if F is the inner regular envelope of both functionals F' and F'', i.e.,

$$\begin{aligned} F(u, A) &= \sup\{F'(u, A') : A' \in \mathcal{A}(\Omega), A' \subset\subset A\} \\ &= \sup\{F''(u, A') : A' \in \mathcal{A}(\Omega), A' \subset\subset A\}, \end{aligned}$$

for every $(u, A) \in X \times \mathcal{A}(\Omega)$.

The following theorem shows that $\overline{\Gamma}$-convergence enjoys useful compactness properties.

Theorem 2.3 *Every sequence $F_j : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ has a $\overline{\Gamma}$-convergent subsequence.*

The following results give us a criterion to establish when the $\overline{\Gamma}$-limit, as a set function, is a Borel measure. We recall that, according to the De Giorgi-Letta's criterion (see Theorem 1.53 [5]), an increasing set function $\lambda : \mathcal{A}(\Omega) \to [0, +\infty]$ is a measure if and only if it is superadditive, subadditive and inner regular.

Proposition 2.4 *Let $F_j : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ be such that $F_j(u, \cdot)$ is increasing and superadditive. Then both $F'(u, \cdot)$ and its inner regular envelope are superadditive. In particular, if (F_j) $\overline{\Gamma}$-converges to F, then $F(u, \cdot)$ is superadditive.*

Proposition 2.5 *Let $F_j : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ be such that*

$$F''(u, A' \cup B) \le F''(u, A) + F''(u, B)$$

for every $u \in X$ and for every A', A, $B \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$. Then the inner regular envelope of $F''(u, \cdot)$ is subadditive. In particular, if (F_j) $\overline{\Gamma}$-converge to F, then $F(u, \cdot)$ is subadditive.

If in addition there exists $G : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that $G(u, \cdot)$ is a measure and $F'' \leq G$, then $F''(u, \cdot)$ coincide with its inner regular envelope for every $A \in \mathcal{A}(\Omega)$ for wich $G(u, A) < +\infty$.

In particular if (F_j) $\overline{\Gamma}$-converge to F, then

$$F(u, A) = \Gamma\text{-}\lim_{j} F_j(u, A)$$

for every $A \in \mathcal{A}(\Omega)$ such that $G(u, A) < +\infty$.

We refer to [16] for an exposition of the main properties of Γ-convergence (see also [14]).

2.2 BV, GBV functions

Let $u : \Omega \to \mathbb{R}^N$ be a measurable function, let $S = \mathbb{R}^N \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^N$ and fix $x \in \Omega$. We say that $z \in S$ is the *approximate limit of* u at x *with respect to* Ω, we write $z = \mathrm{ap} - \lim_{\substack{y \to x \\ y \in \Omega}} u(y)$, if for every neighbourhood U of z in S there holds

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \mathcal{L}^n \left(\{ y \in \Omega : |y - x| < \rho, u(y) \notin U \} \right) = 0.$$

Denote by S_u the complement of the set of points where the approximate limit of u exists; it is well known that $\mathcal{L}^n(S_u) = 0$. Define the function $\tilde{u} : \Omega \setminus S_u \to S$ by

$$\tilde{u}(x) = \mathrm{ap} - \lim_{\substack{y \to x \\ y \in \Omega}} u(y),$$

thus u is equal a.e. to $\tilde{u}$. Notice that $\tilde{u}$ is allowed to take the values ∞ but $\mathcal{L}^n(\{\tilde{u} = \infty\}) = 0$.

Moreover, we say that u is *approximately differentiable* at a point $x \in \Omega \setminus S_u$ such that $\tilde{u}(x) \neq \infty$, if there exists a matrix $L \in \mathbb{R}^{N \times n}$ such that

$$\mathrm{ap} - \lim_{\substack{y \to x \\ y \in \Omega}} \frac{|u(y) - \tilde{u}(x) - L(y - x)|}{|y - x|} = 0. \tag{2.2}$$

If u is approximately differentiable at a point x, the matrix L uniquely determined by (2.2), will be denoted by $\nabla u(x)$ and will be called the *approximate gradient* of u at x.

2.2.1 Functions of Bounded Variation

We recall some definitions and basic results on functions with bounded variation. Our main reference is the book [5] (see also [21], [28]).

Definition 2.6 *Let $u \in L^1\left(\Omega;\mathbb{R}^N\right)$, we say that u is a function with* Bounded Variation *in Ω, we write $u \in BV\left(\Omega;\mathbb{R}^N\right)$, if the distributional derivative Du of u is representable by a $N \times n$ matrix valued measure on Ω with finite total variation $\|Du\|(\Omega)$ whose entries are denoted by $D_i u^\alpha$, i.e., if $\varphi \in C_c^1\left(\Omega;\mathbb{R}^N\right)$ then*

$$\sum_{\alpha=1}^{N} \int_\Omega u^\alpha div\varphi^\alpha \, dx = -\sum_{\alpha=1}^{N}\sum_{i=1}^{n} \int_\Omega \varphi_i^\alpha dD_i u^\alpha.$$

If $u \in BV\left(\Omega;\mathbb{R}^N\right)$, then u is approximately differentiable a.e. and S_u turns out to be countably $\left(\mathcal{H}^{n-1}, n-1\right)$ rectifiable, i.e.,

$$S_u = N \cup \bigcup_{i\geq 1} K_i,$$

where $\mathcal{H}^{n-1}(N) = 0$ and each K_i is a compact subset of a C^1 manifold. Hence, for $\mathcal{H}^{n-1}$ a.e. $y \in S_u$, we can define an *exterior unit normal* ν_u to S_u as well as *inner* and *outer traces* of u on S_u by

$$u^\pm(x) = \mathrm{ap} - \lim_{\substack{y\to x \\ y\in\pi^\pm(x,\nu_u(x))}} u(y) \tag{2.3}$$

where $\pi^\pm(x,\nu_u(x)) = \{y \in \mathbb{R}^n : \pm\langle y - x, \nu_u(x)\rangle > 0\}$. In such a case we write $x \in J_u$.

Let us point out that, in case $u \in BV\left(\Omega;\mathbb{R}^N\right)$, the definitions of ∇u, S_u, J_u, $u^\pm$ given above are essentially equivalent to those classically given by means of integral averages. We need those measure theoretic definitions since they make sense also in the more general framework of GBV functions as we will see below.

Let us consider the Lebesgue's decomposition of Du with respect to $\mathcal{L}^n$, then $Du = D^a u + D^s u$, where $D^a u$ is the absolutely continuous part and $D^s u$ is the singular one. The density of $D^a u$ with respect to $\mathcal{L}^n$ coincides a.e. with the approximate gradient ∇u of u. Define the *jump part* of Du, $D^j u$, to be the restriction of $D^s u$ to S_u and the *Cantor part*, $D^c u$, to be the restriction of $D^s u$ to $\Omega \setminus S_u$, thus we have

$$Du = D^a u + D^j u + D^c u.$$

We will denote by C_u the support of the measure $D^c u$. The representation $D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner S_u$ holds true, where, given $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$, $a \otimes b$ is the matrix with entries equal to $a_i b^j$, $1 \leq i \leq N$ and $1 \leq j \leq n$. Moreover, the $(n-1)$-dimensional density of the measure $\|D^j u\|$ is identified in the following lemma (see Lemma 2.6 [26]).

Lemma 2.7 *For $\mathcal{H}^{n-1}$ a.e. $x_o \in S_u$*

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{n-1}} \int_{S_u \cap \left(x_o + \delta Q_{\nu_u(x_o)}\right)} \left|u^+(x) - u^-(x)\right| \, d\mathcal{H}^{n-1}(x) = \left|u^+(x_o) - u^-(x_o)\right|.$$

Eventually we recall a locality property of Du we need in the sequel (see Proposition 3.92, Remark 3.93 [5]).

Proposition 2.8 *Let u_1, $u_2 \in BV\left(\Omega;\mathbb{R}^N\right)$ and define*

$$L = \{x \in \Omega \setminus (S_{u_1} \cup S_{u_2}) : \tilde{u}_1(x) = \tilde{u}_2(x)\},$$

then $Du_1 \llcorner L = Du_2 \llcorner L$.

2.2.2 Generalized functions of Bounded Variation

Functionals involved in free-discontinuity problems are often not coercive in the space $BV\left(\Omega;\mathbb{R}^N\right)$, then it is useful to consider the following wider class (see [19], Chapter 4 [5]).

Definition 2.9 *Given a Borel function $u : \Omega \to \mathbb{R}^N$, we say that u is a* Generalized Function with Bounded Variation *in Ω, we write $u \in GBV\left(\Omega;\mathbb{R}^N\right)$, if $g(u) \in BV(\Omega)$ for every $g \in C^1\left(\mathbb{R}^N\right)$ such that ∇g has compact support.*

Notice that $GBV \cap L^\infty\left(\Omega;\mathbb{R}^N\right) = BV \cap L^\infty\left(\Omega;\mathbb{R}^N\right)$.

Functions $u \in GBV\left(\Omega;\mathbb{R}^N\right)$ are approximately differentiable a.e. in Ω, and the set S_u turns out to be $\left(\mathcal{H}^{n-1}, n-1\right)$ rectifiable. Moreover, there exist a subset J_u of S_u, with $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$, and a Borel function $\nu_u : J_u \to \mathbb{S}^{n-1}$ such that the approximate limits (2.3) exist on J_u (see Proposition 1.3 [2]).

To give a rigorous mathematical sense to functionals involved in our problem we need to associate to a particular class of GBV functions a vector measure which can be regarded as the Cantor part of the generalized distributional derivative.
Let us first recall that if $u \in (GBV(\Omega))^N$ then a positive measure $\|D^c u\|$ is associated to u by setting $\|D^c u\|(B) := \sup_i \|D^c u^i\|(B)$ for every $B \in \mathcal{B}(\Omega)$, where $u^i := \Psi_i(u) \in BV\left(\Omega;\mathbb{R}^N\right)$, with Ψ_i defined as

$$\Psi_i(u) := \begin{cases} u & \text{if } |u| \leq a_i \\ \\ 0 & \text{if } |u| \geq a_{i+1} \end{cases}, \tag{2.4}$$

where $(a_i) \subset (0, +\infty)$ is a strictly increasing and diverging sequence, and for every $i \in \mathbb{N}$ $\Psi_i \in C^1\left(\mathbb{R}^N;\mathbb{R}^N\right)$ and $\|\nabla\Psi_i\|_\infty \leq 1$.
Actually, the sup above is independent of the truncation performed on u and it is also the pointwise limit and the least upper bound measure of the family $\left(\|D^c u^i\|\right)_{i\in\mathbb{N}}$.

For a GBV function u for which $\|D^c u\|$ is a finite measure, we define a vector measure whose total variation is exactly $\|D^c u\|$.

Lemma 2.10 *Let $u \in (GBV(\Omega))^N$ be such that $\|D^c u\|$ is a finite measure, then the sequence $\left(D^c u^i\right)$ pointwise converges to a vector measure $\lambda \in \mathcal{M}\left(\Omega; \mathbb{R}^{N\times n}\right)$ such that for every $B \in \mathcal{B}(\Omega)$*

$$\|\lambda\|(B) = \|D^c u\|(B).$$

Moreover, λ does not depend on the particular truncations chosen.

Definition 2.11 *Let $u \in (GBV(\Omega))^N$ be such that $\|D^c u\|$ is a finite measure, then we define $D^c u := \lambda$.*

Proof. Set

$$\Omega_\infty := \{x \in \Omega \setminus S_u : \tilde{u}(x) = \infty\} \tag{2.5}$$

and note that $\|D^c u\|(\Omega_\infty) = 0$. Indeed, for every $i \in \mathbb{N}$, $\Omega_\infty \subseteq \{x \in \Omega \setminus S_u : \tilde{u}^i(x) = 0\}$, thus, by Proposition 2.8, $\|D^c u^i\|(\Omega_\infty) = 0$. If we set $\Omega_j := \{x \in \Omega \setminus S_u : |\tilde{u}(x)| < a_j\}$, we then have that $\Omega = (\cup_{j\geq 1}\Omega_j) \cup N$, with $\|D^c u\|(N) = 0$. Let $i \geq j$, notice that $\tilde{u}^i \equiv \tilde{u}^j$ on Ω_j, and so, again by Proposition 2.8,

$$D^c u^i \llcorner \Omega_j = D^c u^j \llcorner \Omega_j. \tag{2.6}$$

Let us remark that since $\|D^c u\|$ is a finite measure then $(\|D^c u\|(\Omega \setminus \Omega_j))$ is infinitesimal.

Consider the set function $\lambda : \mathcal{B}(\Omega) \to \mathbb{R}^{N\times n}$ defined as

$$\lambda(B) := \lim_i D^c u^i(B).$$

Let us first notice that the limit above exists since

$$\begin{aligned}&\|D^c u^j(B) - D^c u^i(B)\| \\ &\qquad \leq \|D^c u^j\|(B \setminus \Omega_j) + \|D^c u^i\|(B \setminus \Omega_j) \leq 2\|D^c u\|(B \setminus \Omega_j),\end{aligned}$$

and one can easily check that $\lambda \in \mathcal{M}\left(\Omega; \mathbb{R}^{N\times n}\right)$.

In particular, this imply that $\left(D^c u^i\right)$ is weakly* convergent to the vector measure λ.

We claim that $\|D^c u\| \equiv \|\lambda\|$. First notice that since $\left(\|D^c u^i\|\right)$ converges to $\|D^c u\|$ weakly* in the sense of measures then $\|D^c u\|(A) \geq \|\lambda\|(A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover, with fixed $j \in \mathbb{N}$ for every $i \geq j$ by (2.6)

$$\|\lambda\| \llcorner \Omega_j = \|\lambda \llcorner \Omega_j\| = \|D^c u^i \llcorner \Omega_j\| = \|D^c u^i\| \llcorner \Omega_j = \|D^c u\| \llcorner \Omega_j,$$

from which there follows $\|\lambda\|(\Omega) = \|D^c u\|(\Omega)$ by passing to the limit on $j \to +\infty$. Hence, $\left(\|D^c u^i\|\right)$ converges weakly* in the sense of measures to $\|\lambda\|$ and so the conclusion follows.

Eventually, it is easy to check that the argument used does not depend on the particular family of truncating functions chosen. □

Eventually, consider the set

$$J_u^\infty := \{x \in J_u : u^+(x) = \infty \text{ or } u^-(x) = \infty\}.$$

In the following theorem we show that for GBV functions satisfying suitable a priori bounds, which for instance occours in our case, J_u^∞ is $\mathcal{H}^{n-1}$ negligible (see also Theorem 4.40 [5]).

Theorem 2.12 *Let $u \in GBV(\Omega)$ be such that*

$$\int_\Omega |\nabla u|\, dx + \int_{J_u} \theta\left(\left|u^+ - u^-\right|\right)\, d\mathcal{H}^{n-1} + \|D^c u\|(\Omega) < +\infty, \tag{2.7}$$

where $\theta : [0, +\infty) \to [0, +\infty)$ satisfies

$$\delta > 0 \Rightarrow \inf_{|t|>\delta} \theta > 0; \quad \liminf_{t\to 0^+} \frac{\theta(t)}{t} > 0. \tag{2.8}$$

Then

$$\mathcal{H}^{n-1}\left(J_u^\infty\right) = 0.$$

Proof. Assume first $n = 1$, in such a case we prove that $u \in BV(\Omega)$, and so the conclusion is a well known property of such functions.

Indeed, let $J_u^\delta = \{t \in J_u : |u^+(t) - u^-(t)| \le \delta\}$, then (2.7) and $(2.8)_1$ yield

$$\left(\inf_{|t|>\delta} \theta\right) \mathcal{H}^0\left(J_u \setminus J_u^\delta\right) \le \sum_{t \in J_u \setminus J_u^\delta} \theta\left(\left|u^+ - u^-\right|\right) < +\infty,$$

hence $M_\delta = \sup_{J_u \setminus (J_u^\delta \cup J_u^\infty)} |u^+ - u^-|$ is finite and actually it is a maximum. Thus, by $(2.8)_1$, we get

$$\sum_{t \in J_u \setminus J_u^\infty} \left|u^+ - u^-\right| \le c \sum_{t \in J_u^\delta} \theta\left(\left|u^+ - u^-\right|\right) + M_\delta \mathcal{H}^0\left(J_u \setminus J_u^\delta\right) < +\infty.$$

By (2.7) $\mathcal{H}^0\left(J_u^\infty\right) < +\infty$ and let $J_u^\infty = \{t_i\}_{1 \le i \le r}$ with $t_i < t_{i+1}$. Then, with fixed i, for every $x, y \in (t_i, t_{i+1})$ we get

$$|u_k(x) - u_k(y)| \le \left|\int_x^y |\nabla u|\, dt\right| + \sum_{t \in J_u \setminus J_u^\infty} \left|u^+ - u^-\right| + \|D^c u\|((x, y)),$$

where $u_k = (u \vee (-k)) \wedge k \in BV(\Omega)$, $k \in \mathbb{N}$. By choosing $y \in (t_i, t_{i+1}) \setminus \Omega_\infty$, where Ω_∞ is defined in (2.5), it follows that there exists a positive constant λ_i such that $|u_k(x)| \le \lambda_i$, and so by passing to the supremum on k we get

$$\sup_{(t_i, t_{i+1})} |u(x)| \le \lambda_i.$$

Hence, $J_u^\infty = \emptyset$ and $u \in L^\infty(\Omega)$, so that $u \in BV(\Omega)$.

In case $n > 1$ we can proceed analogously to Theorem 4.40 [5]. Indeed, by an integral-geometric technique one reduces the proof of $\mathcal{H}^{n-1}(J_u^\infty) = 0$ to the one dimensional setting for which the result follows by the discussion above. $\square$

Remark 2.13 *If $u \in (GBV(\Omega))^N$, one can show that $J_u^\infty = \cup_{i=1}^N J_{u_i}^\infty \cup N$, with $\mathcal{H}^{n-1}(N) = 0$. Then, from Theorem 2.12 we deduce that, if u_i satisfies (2.7) for each $i \in \{1, \dots, N\}$, then $\mathcal{H}^{n-1}(J_u^\infty) = 0$.*

2.3 Lower semicontinuity and integral representation in BV

In this section we will recall some results, we will use in the proof of Theorem 3.2, regarding lower semicontinuity and relaxation properties of linear integral functionals in BV and the integral representation of variational functionals in BV.

Let Ω be a bounded open set of $\mathbb{R}^n$ and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to [0, +\infty)$ be a Borel function. We say that f is *quasiconvex* in z if for a.e. $x \in \Omega$ and for every $u \in \mathbb{R}^N$

$$f(x, u, z)\, \mathcal{L}^n(\Omega) \leq \int_\Omega f(x, u, z + D\varphi(y))\, dy \tag{2.9}$$

for every $\varphi \in C_c^1(\Omega; \mathbb{R}^N)$.

For any $(x, u, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ define the *recession function* of f by

$$f^\infty(x, u, z) = \limsup_{t\to+\infty} \frac{f(x, u, tz)}{t}.$$

Consider the functional $F : L^1(\Omega; \mathbb{R}^N) \times \mathcal{A}(\Omega) \to [0, +\infty]$ defined by

$$F(u, A) := \begin{cases} \displaystyle\int_A f(x, u, \nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega; \mathbb{R}^N) \\ +\infty & \text{if } u \in L^1(\Omega; \mathbb{R}^N) \setminus W^{1,1}(\Omega; \mathbb{R}^N) \end{cases},$$

and denote by $\overline{F}(u, A)$ the relaxation of $F(u, A)$ in the strong $L^1(\Omega; \mathbb{R}^N)$ topology.

The following two theorems are due to Fonseca and Leoni (Theorems 1.8 and 1.9 [24]), and will be used to identify the Lebesgue and the Cantor part of the Γ-limit in the proof of Theorem 3.2.

Theorem 2.14 *Assume that*

(i) $f(x, u, \cdot)$ is quasiconvex for every $(x, u) \in \Omega \times \mathbb{R}^N$ and there exists $c > 0$ such that

$$0 \leq f(x, u, z) \leq c(|z| + 1)$$

for every $(x, u, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$;

(ii) for all $(x_o, u_o) \in \Omega \times \mathbb{R}^N$ either $f(x_o, u_o, z) \equiv 0$ for all $z \in \mathbb{R}^{N\times n}$, or for every $\eta > 0$ there exist $c_0, c_1, \delta > 0$ such that

$$f(x_o, u_o, z) - f(x, u, z) \leq \eta \left(1 + f(x, u, z)\right), \tag{2.10}$$

$$f(x, u, z) \geq c_1 |z| - c_0$$

for all $(x, u) \in \Omega \times \mathbb{R}^N$ with $|x - x_o| + |u - u_o| \leq \delta$ and for all $z \in \mathbb{R}^{N\times n}$.

Then for $u \in BV\left(\Omega; \mathbb{R}^N\right)$ we get

$$\overline{F}(u, A) \geq \int_A f\left(x, u, \nabla u)\right) dx + \int_A f^\infty \left(x, \tilde{u}, dD^c u\right).$$

Theorem 2.15 *Let f satisfies condition (i) of Theorem 2.14.*

Let $u \in BV\left(\Omega; \mathbb{R}^N\right)$, then $\overline{F}(u, \cdot)$ is the trace of a finite Radon measure on $\mathcal{A}(\Omega)$, and

(1) if f is Carathéodory or $f(\cdot, \cdot, z)$ is upper semicontinuous then

$$\overline{F}\left(u, A \setminus (J_u \cup C_u)\right) \leq \int_A f\left(x, u, \nabla u)\right) dx;$$

(2) if $f^\infty(\cdot, \cdot, z)$ is upper semicontinuous then

$$\overline{F}(u, A \cap C_u) \leq \int_A f^\infty \left(x, \tilde{u}, dD^c u\right).$$

Eventually, let us recall part of the integral representation result of Theorem 3.7 [12], in a form which is useful for our purposes.

Theorem 2.16 *Let $\mathcal{F} : BV\left(\Omega; \mathbb{R}^N\right) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be such that $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure, for every $A \in \mathcal{A}(\Omega)$ $\mathcal{F}(\cdot, A)$ is $L^1\left(A; \mathbb{R}^N\right)$ lower semicontinuous and*

$$0 \leq \mathcal{F}(u; A) \leq c \left(\mathcal{L}^n(A) + \|Du\|(A)\right)$$

for every $u \in BV\left(\Omega; \mathbb{R}^N\right)$.

Then, for every $u \in BV\left(\Omega; \mathbb{R}^N\right)$ and $A \in \mathcal{A}(\Omega)$

$$\mathcal{F}(u; A \cap J_u) = \int_{J_u \cap A} f_J\left(x, u^+, u^-, \nu_u\right) d\mathcal{H}^{n-1},$$

where $f_J : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1} \to [0, +\infty)$ is defined as

$$\begin{aligned} f_J\left(x_o, a, b, \nu\right) = \limsup_{\delta \to 0^+} \frac{1}{\delta^{n-1}} \inf \Big\{ & \mathcal{F}\left(w, x_o + \delta Q_\nu\right) : \\ & w \in BV\left(\Omega; \mathbb{R}^N\right), w = u_{a,b,\nu} \text{ on } x_o + \delta \partial Q_\nu \Big\}, \end{aligned} \tag{2.11}$$

with

$$u_{a,b,\nu}(x) := \begin{cases} b & \textit{if } \langle x,\nu\rangle \geq 0 \\ a & \textit{if } \langle x,\nu\rangle < 0 \end{cases}. \tag{2.12}$$

3 Γ-Convergence Result

In this section we prove a variational approximation for functionals defined on $(GBV(\Omega))^N$ as

$$\mathcal{F}(u) = \int_\Omega f(x,u,\nabla u)\,dx + \int_\Omega f^\infty(x,\tilde{u},dD^c u) + \int_{J_u} K\big(x,u^+,u^-,\nu_u\big)\,d\mathcal{H}^{n-1}, \tag{3.1}$$

where the assumptions on all the quantities appearing above are specified below.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to [0,+\infty)$ be a Borel integrand satisfying

(f1) there exist three constants $c_0 \geq 0$, c_1 and $c_2 > 0$ such that

$$c_1\,|z| - c_0 \leq f(x,u,z) \leq c_2\big(|z|+1\big) \tag{3.2}$$

for every $(x,u,z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$;

(f2) $f(x,u,\cdot)$ is quasiconvex in z for every $(x,u) \in \Omega \times \mathbb{R}^N$, and either f is Carathéodory or $f(\cdot,\cdot,z)$ is upper semicontinuous for every $z \in \mathbb{R}^{N\times n}$;

(f3) for every $(x_o,u_o) \in \Omega \times \mathbb{R}^N$ and $\eta > 0$ there exists δ, depending on (x_o,u_o) and η, such that

$$f(x_o,u_o,z) - f(x,u,z) \leq \eta\,(1 + f(x,u,z)) \tag{3.3}$$

for every $(x,u) \in \Omega\times\mathbb{R}^N$ with $|x-x_o|+|u-u_o| \leq \delta$ and for every $z \in \mathbb{R}^{N\times n}$;

(f4) for every $x_o \in \Omega$ and $\eta > 0$ there exists $\delta, L > 0$ (all these quantities depend on x_o and η) such that

$$\left| f^\infty(x,u,z) - \frac{f(x,u,tz)}{t} \right| \leq \eta \left(1 + \frac{f(x,u,tz)}{t}\right), \tag{3.4}$$

for every $t > L$ and $x \in \Omega$ with $|x-x_o| \leq \delta$ and for every $(u,z) \in \mathbb{R}^N\times\mathbb{R}^{N\times n}$;

(f5) for every $x_o \in \Omega$ and $\eta > 0$ there exists δ (depending on x_o and η) such that

$$f^\infty(x_o,u,z) - f^\infty(x,u,z) \leq \eta f^\infty(x,u,z), \tag{3.5}$$

for every $x \in \Omega$ with $|x-x_o| \leq \delta$ and for every $(u,z) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$;

(f6) $f^\infty(\cdot,\cdot,z)$ is upper semicontinuous for every $(x,u)\in\Omega\times\mathbb{R}^N$.

Remark 3.1 *It is well known that $f^\infty(x,u,\cdot)$ inherits from $f(x,u,\cdot)$ the quasiconvexity property in z. Moreover, by the growth condition (3.2) for every $(x,u,z)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ there holds*

$$c_1|z|\le f^\infty(x,u,z)\le c_2|z|. \tag{3.6}$$

To perform the approximation we introduce an extra variable v and define the functional $F:L^1\left(\Omega;\mathbb{R}^{N+1}\right)\to[0,+\infty]$ by

$$F(u,v):=\begin{cases}\mathcal{F}(u) & \text{if } u\in(GBV(\Omega))^N\ v=1 \text{ a.e. in } \Omega\\ \\ +\infty & \text{otherwise}\end{cases}, \tag{3.7}$$

which is equivalent to $\mathcal{F}$ as far as minimum problems are concerned. The approximating functionals $F_\varepsilon:L^1\left(\Omega;\mathbb{R}^{N+1}\right)\to[0,+\infty]$ have the form

$$F_\varepsilon(u,v):=\begin{cases}\displaystyle\int_\Omega f_\varepsilon(x,(u,v),\nabla(u,v))\,dx & \text{if } (u,v)\in W^{1,1}\left(\Omega;\mathbb{R}^{N+1}\right),\\ & 0\le v\le 1 \text{ a.e.}\\ \\ +\infty & \text{otherwise}\end{cases}, \tag{3.8}$$

where $f_\varepsilon:\Omega\times\mathbb{R}^{N+1}\times\mathbb{R}^{(N+1)\times n}\to[0,+\infty)$ is defined by

$$f_\varepsilon(x,(u,v),(z,\zeta)):=\psi(v)f(x,u,z)+\frac{1}{\varepsilon}W(v)+\varepsilon|\zeta|^2,$$

with $\psi:[0,1]\to[0,1]$ any lower semicontinuous increasing function such that $\psi(0)=0$, $\psi(1)=1$ and $\psi(t)>0$ if $t>0$; and $W:[0,1]\to[0,+\infty)$ is any continuous function such that $W(1)=0$ and $W(t)>0$ if $t\in[0,1)$.

Let us state and prove the main result of the paper.

Theorem 3.2 *Let $(F_\varepsilon)_{\varepsilon>0}$ be as above, then*

$$\Gamma\left(L^1\left(\Omega;\mathbb{R}^{N+1}\right)\right)\text{-}\lim_{\varepsilon\to0^+}F_\varepsilon(u,v)=F(u,v).$$

where F is given by (3.7) and the function $K:\Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1}\to[0,+\infty)$ is defined by

$$\begin{aligned}&K(x_o,a,b,\nu):=\\ &\quad\inf\left\{\int_{Q_\nu}\left(\psi(v)f^\infty(x_o,u,\nabla u)+L\,W(v)+\frac{1}{L}|\nabla v|^2\right)dx:\right.\\ &\quad\left.(u,v)\in\mathcal{A}(a,b,\nu),L>0\right\},\end{aligned} \tag{3.9}$$

with

$$\mathcal{A}(a,b,\nu) := \left\{(u,v) \in W^{1,1}\left(Q_\nu;\mathbb{R}^{N+1}\right) : (u,v) = (u_{a,b,\nu},1) \text{ on } \partial Q_\nu\right\}, \tag{3.10}$$

where $u_{a,b,\nu}$ is defined in (2.12).

In the rest of the paper we will denote $\Gamma\left(L^1\left(\Omega;\mathbb{R}^{N+1}\right)\right)$ by $\Gamma\left(L^1\right)$ for simplicity of notation.

Remark 3.3 *We will prove Theorem 3.2 in case (3.2) of (f1) is substituted by*

$$c_1\,|z| \le f(x,u,z) \le c_2\big(|z|+1\big), \tag{3.11}$$

for every $(x,u,z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$. This is not restrictive, by considering the approximating functionals obtained by substituting f with $f_1 = f + c_o$, which now satisfies (3.11) above, and by noting that, calling F_1 their Γ-limit, $F_1 = F + c_o\mathcal{L}^n(\Omega)$.

Remark 3.4 *The result of Theorem 3.2 generalizes that of Theorem 5.1 in [1], in which they consider the particular case $N = 1$ and $f(x,u,z) = \tilde{f}(|z|)$, where $\tilde{f} : [0,+\infty) \to [0,+\infty)$ is convex, increasing and $\lim_{t\to+\infty}\frac{\tilde{f}(t)}{t} = 1$, that is $f^\infty(x,u,z) = |z|$. In Section 3.1, under these assumptions on f and for all $N \ge 1$, we will show that $K(x_o,a,b,\nu) = g(|b-a|)$, where $g : [0,+\infty) \to [0,+\infty)$ is the concave function defined in [1] by*

$$g(t) := \inf_{r\in[0,1]}\left\{\psi(r)t + 4\int_r^1 \sqrt{W(s)}\,ds\right\}. \tag{3.12}$$

So we recover the results of [1] also in the vector-valued case.

Remark 3.5 *Let us notice that by a comparison argument and by the Γ-convergence result of [1], we immediately derive a bound for the lower and upper Γ-limits of the family $(F_\varepsilon)_{\varepsilon>0}$. Indeed, consider the scalar functional*

$$I(u,v) := \begin{cases} \|Du\|(\Omega\setminus S_u) + \displaystyle\int_{J_u} g\left(|u^+ - u^-|\right)\,d\mathcal{H}^{n-1} & \text{if } u \in GBV(\Omega),\\ & v = 1 \text{ a.e.}\\ +\infty & \text{otherwise,}\end{cases}$$

g being given by (3.12). Then, by the growth condition (3.2) and by virtue of Theorem 5.1 [1], there exists three positive constants c_0, c_1 and c_2 such that

$$\begin{aligned} c_1\sum_{i=1}^N I(u_i,v) - c_0 &\le \Gamma\text{-}\liminf_{\varepsilon\to0^+} F_\varepsilon(u,v)\\ &\le \Gamma\text{-}\limsup_{\varepsilon\to0^+} F_\varepsilon(u,v) \le c_2\sum_{i=1}^N I(u_i,v) + c_2. \end{aligned}$$

In particular, we deduce that the domains of the lower and upper Γ-limits of the family $(F_\varepsilon)_{\varepsilon>0}$ coincide and are contained in $(GBV(\Omega))^N \times \{1\}$.

Remark 3.6 *We provide an equivalent characterization of the jump energy density K defined in (3.9) which will be useful in the sequel (see Section 3.1).*

Let $\tilde{K}$ be the function obtained by substituting in the minimization formula (3.9) defining K the class $\mathcal{A}(a,b,\nu)$ with

$$\tilde{\mathcal{A}}(a,b,\nu) := \Big\{(u,v) \in W^{1,1}_{loc}\left(S_\nu;\mathbb{R}^{N+1}\right) : (u,v) \text{ 1-periodic in } \nu_i, 1 \le i \le n-1,$$
$$(u,v) = (a,1) \text{ on } \langle x,\nu\rangle = -\frac{1}{2},\ (u,v) = (b,1) \text{ on } \langle x,\nu\rangle = \frac{1}{2}\Big\}$$

where $S_\nu := \left\{x \in \mathbb{R}^N : |\langle x,\nu\rangle| < \frac{1}{2}\right\}$ (see [10]).

Then, since $\mathcal{A}(a,b,\nu) \subseteq \tilde{\mathcal{A}}(a,b,\nu)$ we have $\tilde{K} \le K$. The opposite inequality can be proved by exploiting the same arguments we will use in Lemma 4.2. However, we will obtain it as a consequence of Proposition 4.1 and inequality (5.6) in the proof of Proposition 5.3.

First notice that assumption (f5) implies that with fixed $(x_o,a,b,\nu) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1}$ and $\eta > 0$ there exists $\delta > 0$ such that

$$K(x_o,a,b,\nu) - K(x,a,b,\nu) \le \eta K(x,a,b,\nu) \tag{3.13}$$

for every $x \in \Omega$ with $|x - x_o| \le \delta$.

Let $u_{a,b,\nu}$ be the function defined in (2.12), then by (3.13), Proposition 4.1 and inequality (5.6), we get

$$\frac{1}{1+\eta}K(x_o,a,b,\nu) \le \liminf_{\delta\to0^+}\frac{1}{\delta^{n-1}}\int_{(x_o+\Pi_\nu)\cap(x_o+\delta Q_\nu)} K(x,a,b,\nu)\,d\mathcal{H}^{n-1}$$
$$\le \limsup_{\delta\to0^+}\frac{1}{\delta^{n-1}}\left(\Gamma\text{-}\lim_{\varepsilon\to0^+} F_\varepsilon\left(u_{a,b,\nu}(\cdot - x_o),1;x_o+\delta Q_\nu\right)\right) \le \tilde{K}(x_o,a,b,\nu).$$

The conclusion then follows by letting $\eta \to 0^+$.

Let us introduce the localized versions of the approximating and limiting functionals. For every $A \in \mathcal{A}(\Omega)$ set

$$F(u,v;A) := \begin{cases} \mathcal{F}(u;A) & \text{if } u \in (GBV(A))^N,\ v = 1 \text{ a.e. in } A \\ +\infty & \text{otherwise in } L^1\left(A;\mathbb{R}^{N+1}\right), \end{cases}$$

where $\mathcal{F}(\cdot,A)$ is defined as $\mathcal{F}(\cdot)$ in (3.1) by taking A as domain of integration in place of Ω. Moreover, let

$$F_\varepsilon(u,v;A) := \begin{cases} \displaystyle\int_A f_\varepsilon(x,(u,v),\nabla(u,v))\,dx & \text{if } (u,v) \in W^{1,1}\left(\Omega;\mathbb{R}^{N+1}\right), \\ & 0 \le v \le 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise in } L^1\left(\Omega;\mathbb{R}^{N+1}\right), \end{cases}$$

and

$$G_\varepsilon(v;A) := \begin{cases} \displaystyle\int_A \left(\frac{1}{\varepsilon}W(v) + \varepsilon\left|\nabla v\right|^2\right) dx & \text{if } v \in W^{1,1}(\Omega), \\ & 0 \le v \le 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Eventually, with fixed $x_o \in \Omega$, denote by $F_\varepsilon(x_o;\cdot,\cdot;A)$, $F_\varepsilon^\infty(x_o;\cdot,\cdot;A)$ the functionals defined analogously to $F_\varepsilon(\cdot,\cdot;A)$ and obtained by substituting in the definition of f_ε the function f with $f(x_o,\cdot,\cdot)$, $f^\infty(x_o,\cdot,\cdot)$, respectively. With this notation we get

$$K(x_o,a,b,\nu) = \inf\left\{F^\infty_{\frac{1}{L}}(x_o;u,v;Q_\nu) : (u,v) \in \mathcal{A}(a,b,\nu), L > 0\right\}. \tag{3.14}$$

3.1 Properties of the surface density function

Before proving Theorem 3.2 we state some properties of the surface energy density K and we show a more explicit characterization of it in some particular cases. The proofs are in the spirit of the papers [10], [11], [26], [27].

We remark that Lemma 3.7 below will be exploited only in the proof of Lemma 6.1 in order to extend the Γ-convergence result from BV on the whole GBV.

Lemma 3.7 *Let $K : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1} \to [0,+\infty)$ be defined as in (3.9), then*

(a) for every (x_o,a,b,ν), $(x_o,a',b',\nu) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1}$ there holds

$$|K(x_o,a,b,\nu) - K(x_o,a',b',\nu)| \le c\left(|a-a'| + |b-b'|\right);$$

(b) for every $(x_o,a,b,\nu) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1}$ there holds

$$c_1 g(|b-a|) \le K(x_o,a,b,\nu) \le c_2 g(|b-a|).$$

where c_1, c_2 are positive constants, and g is given by (3.12).

Proof. (a) We use the different characterization of K discussed in Remark 3.6. Let $(u,v) \in \tilde{\mathcal{A}}(a,b,\nu)$, let $\varphi \in C^\infty(\mathbb{R})$ be a function such that $0 \le \varphi \le 1$, $\varphi = 1$ for $t \le \frac{1}{4}$, $\varphi = 0$ for $t \ge \frac{1}{2}$, then define

$$(\tilde{u},\tilde{v})(x) := \begin{cases} \varphi(-x\cdot\nu)(a,1) + (1-\varphi(-x\cdot\nu))(a',1) & \text{if } -\frac{1}{2} \le x\cdot\nu < -\frac{1}{4} \\ (u(2x),v(2x)) & \text{if } |x\cdot\nu| \le \frac{1}{4} \\ \varphi(x\cdot\nu)(b,1) + (1-\varphi(x\cdot\nu))(b',1) & \text{if } \frac{1}{4} < x\cdot\nu \le \frac{1}{2} \end{cases}.$$

Then $(\tilde{u},\tilde{v}) \in \tilde{\mathcal{A}}(a',b',\nu)$ and, for $L>0$, we get

$$
\begin{aligned}
&K(x_o,a',b',\nu) \le F^{\infty}_{\frac{1}{L}}(x_o;\tilde{u},\tilde{v};Q_\nu)\\
&= \int_{Q_\nu\cap\{|x\cdot\nu|<\frac{1}{4}\}} \psi(v(2x)) f^{\infty}(x_o,u(2x),2\nabla u(2x))\,dx\\
&+ \int_{Q_\nu\cap\{|x\cdot\nu|<\frac{1}{4}\}} \left(L\,W(v(2x)) + \frac{4}{L}|\nabla v(2x)|^2\right) dx\\
&+ \int_{Q_\nu\cap\{\frac{1}{4}<x\cdot\nu<\frac{1}{2}\}} f^{\infty}\big(x_o,\varphi(x\cdot\nu)\,b + (1-\varphi(x\cdot\nu))\,b',(b-b')\otimes\varphi'(x\cdot\nu)\,\nu\big)\,dx\\
&+ \int_{Q_\nu\cap\{\frac{1}{4}<x\cdot\nu<\frac{1}{2}\}} f^{\infty}\big(x_o,\varphi(x\cdot\nu)\,a + (1-\varphi(x\cdot\nu))\,a',(a'-a)\otimes\varphi'(x\cdot\nu)\,\nu\big)\,dx.
\end{aligned}
$$

Since the periodicity of (u,v) and by the growth assumption (3.6), there follows

$$K(x_o,a',b',\nu) \le F^{\infty}_{\frac{2}{L}}(x_o;u,v;Q_\nu) + c\,(|a-a'|+|b-b'|)$$

and so by taking the infimum on $(u,v)\in\tilde{\mathcal{A}}(a,b,\nu)$ and $L>0$ we conclude that

$$K(x_o,a',b',\nu) \le K(x_o,a,b,\nu) + c\,(|a-a'|+|b-b'|)\,.$$

Analogously, we can prove the opposite inequality.
(b) Use the growth condition (3.6) and consider the characterization of K given by Lemma 3.8 (b) and Remark 3.9 when $f^{\infty}(x,u,z)=|z|$. □

In the following we characterize the function K under isotropy assumptions on f^{∞}. In such a case we show that K can be calculated by restricting the infimum to functions (u,v) with one-dimensional profile.

Lemma 3.8 *Let $K:\Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1}\to[0,+\infty)$ and $g:[0,+\infty)\to[0,+\infty)$ be defined by (3.9) and (3.12), respectively. Then*

(a) for every $(x_o,a,b,\nu)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1}$ there holds

$$K(x_o,a,b,\nu) \le g\left(K_f(x_o,a,b,\nu)\right),$$

where $K_f:\Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1}\to[0,+\infty)$ is defined by

$$
\begin{aligned}
K_f(x_o,a,b,\nu) := \inf\bigg\{ \int_{Q_\nu} f^{\infty}(x_o,u,\nabla u)\,dy : u\in W^{1,1}\left(Q_\nu;\mathbb{R}^N\right),\\
u = u_{a,b,\nu} \text{ on } \partial Q_\nu \bigg\};
\end{aligned}
\tag{3.15}
$$

(b) if f^{∞} is isotropic, i.e., for every $(x_o,u,z)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{N+1}$ and $\nu\in\mathbb{S}^{n-1}$ there holds

$$f^{\infty}(x_o,u,z\nu\otimes\nu) \le f^{\infty}(x_o,u,z)\,,$$

then $K=g(K_f)$.

Proof. (a) With fixed $r \in [0,1)$ and $\eta > 0$, let $T_\eta > 0$, $v_\eta \in W^{1,1}(0,T_\eta)$ be such that $v_\eta(0) = r$, $v_\eta(T_\eta) = 1$ and

$$\int_0^{T_\eta} \left(W(v_\eta) + |v_\eta'|^2\right) dt \le 2\int_r^1 \sqrt{W(s)}\, ds + \eta,$$

(see Remark 3.11 [13]). Then define

$$v_{\eta,L}(y) := \begin{cases} v_\eta\left(\frac{-y\cdot\nu-\alpha_L}{\beta_L}\right) & -\frac{1}{2} \le y\cdot\nu < -\alpha_L \\ r & |y\cdot\nu| \le \alpha_L \\ v_\eta\left(\frac{y\cdot\nu-\alpha_L}{\beta_L}\right) & \alpha_L < y\cdot\nu \le \frac{1}{2} \end{cases},$$

where α_L is any positive infinitesimal as $L \to +\infty$, and $\beta_L := \frac{\frac{1}{2}-\alpha_L}{T_\eta}$. If $r = 1$ simply take $v_{\eta,L} \equiv 1$.

Let u be admissible for K_f and extend it by periodicity to $\mathbb{R}^n$, then set

$$u_L(y) := \begin{cases} a & -\frac{1}{2} \le y\cdot\nu < -\alpha_L \\ u\left(\frac{y}{2\alpha_L}\right) & |y\cdot\nu| \le \alpha_L \\ b & \alpha_L < y\cdot\nu \le \frac{1}{2} \end{cases}.$$

Notice that $(u_L, v_{\eta,L}) \in \tilde{\mathcal{A}}(a,b,\nu)$. Let us compute $F^\infty_{\frac{1}{\beta_L}}(u_L, v_L; Q_\nu)$. Let R^ν be a rotation such that $R^\nu Q = Q_\nu$. Then, since $f^\infty(x,u,\cdot)$ is positively one homogeneous, we get by simple changes of variables and by Fubini's Theorem

$$\begin{aligned}
&\int_{Q_\nu} \psi(v_L) f^\infty(x_o, u_L, \nabla u_L)\, dy \\
&\quad = \psi(r)\int_{Q_\nu\cap\{|y\cdot\nu|<\alpha_L\}} f^\infty\left(x_o, u\left(\frac{y}{2\alpha_L}\right), \frac{1}{2\alpha_L}\nabla u\left(\frac{y}{2\alpha_L}\right)\right) dy \\
&\quad = \psi(r)\int_{-\frac{1}{2}}^{\frac{1}{2}} dt \int_{Q'} f^\infty\left(x_o, u\left(R^\nu\left(\frac{y'}{2\alpha_L}, t\right)\right), \nabla u\left(R^\nu\left(\frac{y'}{2\alpha_L}, t\right)\right)\right) dy' \\
&\quad = \psi(r)\int_Q f^\infty(x_o, u(R^\nu y), \nabla u(R^\nu y))\, dy + o(1) \\
&\quad = \psi(r)\int_{Q_\nu} f^\infty(x_o, u(y), \nabla u(y))\, dy + o(1),
\end{aligned}$$

where the last equality follows by Riemann-Lebesgue's Lemma. Moreover, there holds

$$G_{\frac{1}{\beta_L}}(v_{\eta,L}; Q_\nu) = \int_{-\frac{1}{2}}^{-\alpha_L} \frac{1}{\beta_L}\left(W\left(v_\eta\left(\frac{-t-\alpha_L}{\beta_L}\right)\right) + \left|v_\eta'\left(\frac{-t-\alpha_L}{\beta_L}\right)\right|^2\right) dt$$

$$+2\frac{\alpha_L}{\beta_L}W(r)+\int_{\alpha_L}^{\frac{1}{2}}\frac{1}{\beta_L}\left(W\left(v_\eta\left(\frac{t-\alpha_L}{\beta_L}\right)+\left|v_\eta'\left(\frac{t-\alpha_L}{\beta_L}\right)\right)\right|^2\right)dt$$
$$=2\int_0^{T_\eta}\left(W(v_\eta)+|v_\eta'|^2\right)dt+o(1)\leq 4\int_r^1\sqrt{W(s)}\,ds+c\eta+o(1).$$

Hence, there follows

$$K\left(x_o,a,b,\nu\right)\leq F^\infty_{\frac{1}{\beta_L}}\left(u_L,v_{\eta,L};Q_\nu\right)$$
$$\leq\psi(r)\int_{Q_\nu}f^\infty\left(x_o,u(y),\nabla u(y)\right)\,dy+4\int_r^1\sqrt{W(s)}\,ds+c\eta+o(1),$$

and so by letting $L\to+\infty$, $\eta\to 0^+$ and by passing to the infimum on u we get for every $r\in[0,1]$

$$K\left(x_o,a,b,\nu\right)\leq\psi(r)K_f\left(x_o,a,b,\nu\right)+4\int_r^1\sqrt{W(s)}\,ds.$$

Eventually, the desired inequality follows by the very definition of g.

(b) Following [6] define

$$D_f\left(x_o,a,b,\nu\right):=\inf\left\{\int_{Q_\nu}f^\infty\left(x_o,u,\nabla u\right)\,dy: u\in W^{1,1}\left(Q_\nu;\mathbb{R}^N\right),\right.$$
$$\left.u(y)=\xi(y\cdot\nu),\xi\left(-\frac{1}{2}\right)=a,\,\xi\left(\frac{1}{2}\right)=b\right\}$$
$$=\inf\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}}f^\infty\left(x_o,\xi,\dot{\xi}\otimes\nu\right)dt:\xi\in W^{1,1}\left(\left(-\frac{1}{2},\frac{1}{2}\right);\mathbb{R}^N\right),\right.$$
$$\left.\xi\left(-\frac{1}{2}\right)=a,\,\xi\left(\frac{1}{2}\right)=b\right\};$$

then it is obvious that $K_f\leq D_f$. Moreover, in case f^∞ is isotropic, $D_f=K_f$ (see Proposition 2.6 [27]). Hence, by (a), we have to prove only that $K\geq g\left(D_f\right)$.

The isotropy condition on f^∞ implies that for every $(u,v)\in\mathcal{A}(a,b,\nu)$, $L>0$ there holds

$$F^\infty_{\frac{1}{L}}\left(u,v;Q_\nu\right)\geq I(u,v;Q_\nu)$$
$$:=\int_{Q_\nu}\left(\psi(v)f^\infty\left(x_o,u,\nabla u\,\nu\otimes\nu\right)+2\sqrt{W(v)}\left|\nabla v\,\nu\otimes\nu\right|\right)\,dy.$$

For every $y'\in Q_\nu'$ and $t\in[-\frac{1}{2},\frac{1}{2}]$, let $\left(u^{\nu,y'}(t),v^{\nu,y'}(t)\right):=(u(y'+t\nu),v(y'+t\nu))$, then by Fubini's Theorem there holds

$$I(u,v;Q_\nu)$$
$$=\int_{Q_\nu'}dy'\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\psi\left(v^{\nu,y'}\right)f^\infty\left(x_o,u^{\nu,y'},\dot{u}^{\nu,y'}\otimes\nu\right)+2\sqrt{W\left(v^{\nu,y'}\right)}\left|\dot{v}^{\nu,y'}\right|\right)\,dt,$$

and thus

$$K\left(x_o,a,b,\nu\right)\geq\inf\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\psi\left(\xi_2\right)f^{\infty}\left(x_o,\xi_1,\dot{\xi}_1\otimes\nu\right)+2\sqrt{W\left(\xi_2\right)}\left|\dot{\xi}_2\right|\right)\,dt:\right.$$
$$\left(\xi_1,\xi_2\right)\in W^{1,1}\left(\left(-\frac{1}{2},\frac{1}{2}\right);\mathbb{R}^{N+1}\right),$$
$$\left.\left(\xi_1,\xi_2\right)\left(-\frac{1}{2}\right)=(a,1),\left(\xi_1,\xi_2\right)\left(\frac{1}{2}\right)=(b,1)\right\}.$$

In order to conclude, with fixed (ξ_1,ξ_2) as above, let $m=\inf_{[-\frac{1}{2},\frac{1}{2}]}\xi_2$, then

$$\begin{aligned}&\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\psi\left(\xi_2\right)f^{\infty}\left(x_o,\xi_1,\dot{\xi}_1\otimes\nu\right)+2\sqrt{W\left(\xi_2\right)}\left|\dot{\xi}_2\right|\right)\,dt\\&\quad\geq\psi(m)\int_{-\frac{1}{2}}^{\frac{1}{2}}f^{\infty}\left(x_o,\xi_1,\dot{\xi}_1\otimes\nu\right)dt+4\int_m^1\sqrt{W(s)}\,ds\\&\quad\geq\psi(m)D_f\left(x_o,a,b,\nu\right)+4\int_m^1\sqrt{W(s)}\,ds\geq g\left(D_f\left(x_o,a,b,\nu\right)\right).\end{aligned}$$

□

Remark 3.9 *The characterization of K in the isotropic case, given in Lemma 3.8 (b), is relevant when an explicit expression of K_f is given, for instance in the autonomous and scalar case.*

Indeed, if $f=f(x,z)$ then $K_f\left(x_o,a,b,\nu\right)=f^{\infty}\left(x_o,(b-a)\otimes\nu\right)$ (see Remark 2.17 [26]).

In the scalar setting $N=1$, since f satisfies conditions (f1), (f4)-(f6), Corollary 1.4 and Theorem 1.10 [24] (see also [17]) yield the equality

$$K_f\left(x_o,a,b,\nu\right)=\begin{cases}\displaystyle\int_b^a f^{\infty}\left(x_o,u,\nu\right)\,du & \text{if } a>b\\ \displaystyle\int_a^b f^{\infty}\left(x_o,u,-\nu\right)\,du & \text{if } a<b\end{cases}.$$

In particular, if $f(z)=|z|$ we recover the surface energy density of [1], that is $K\left(x_o,a,b,\nu\right)=g(|b-a|)$.

4 Γ-liminf inequality

In this section we establish the lower bound inequality when restricting the target functional to $BV\left(\Omega;\mathbb{R}^N\right)\times L^1(\Omega)$. We treat separately the diffuse and jump part. Indeed, we recover straightforward the estimate on the diffuse part by using the semicontinuity result Theorem 2.14, while we apply the blow-up argument of Fonseca-Müller to estimate the surface energy density.

Proposition 4.1 *For every* $(u,v) \in BV\left(\Omega;\mathbb{R}^N\right) \times L^1(\Omega)$*,* $A \in \mathcal{A}(\Omega)$ *we have*

$$\Gamma\left(L^1\right)\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon\left(u,v;A\right) \geq F\left(u,v;A\right).$$

Proof. For the sake of simplicity we only prove the case $A = \Omega$.

Let $\varepsilon_j \to 0^+$ and $(u_j,v_j) \to (u,v)$ in $L^1\left(\Omega;\mathbb{R}^{N+1}\right)$. Without loss of generality we may assume the inferior limit $\liminf_j F_{\varepsilon_j}\left(u_j,v_j\right)$ to be finite and to be a limit. Then, we get

$$\liminf_j \int_\Omega W(v_j)\,dx \leq \liminf_j \left(\varepsilon_j G_{\varepsilon_j}\left(v_j;\Omega\right)\right) = 0,$$

so that by Fatou's lemma there follows

$$W(v) \leq \liminf_j W(v_j) = 0$$

for a.e. $x \in \Omega$, and then $v = 1$ for a.e. $x \in \Omega$.

Since $f_{\varepsilon_j} \geq 0$, up to passing to a subsequence, we may assume that there exists a non-negative finite Radon measure μ on Ω such that

$$f_{\varepsilon_j}(\cdot,(u_j(\cdot),v_j(\cdot)),\nabla(u_j(\cdot),v_j(\cdot))\mathcal{L}^n \llcorner \Omega \to \mu$$

weakly* in the sense of measures. Using the Radon-Nykodim's Theorem we decompose μ in the sum of four mutually orthogonal measures

$$\mu = \mu_a\mathcal{L}^n + \mu_c|D^c u| + \mu_J|u^+ - u^-|\mathcal{H}^{n-1} \llcorner J_u + \mu_s,$$

we claim that

$$\mu_a(x_o) \geq f\left(x_o,u(x_o),\nabla u(x_o)\right) \tag{4.1}$$

for a.e. $x_o \in \Omega$;

$$\mu_c(x_o) \geq f^\infty\left(x_o,\tilde{u}(x_o),\frac{dD^c u}{d\|D^c u\|}(x_o)\right) \tag{4.2}$$

for $\|D^c u\|$ a.e. $x_o \in \Omega$;

$$\mu_J(x_o) \geq \frac{1}{|u^+(x_o) - u^-(x_o)|} K\left(x_o,u^+(x_o),u^-(x_o),\nu_u(x_o)\right) \tag{4.3}$$

for $|u^+ - u^-|\mathcal{H}^{n-1} \llcorner J_u$ a.e. $x_o \in \Omega$.

Assuming the previous inequalities shown, to conclude consider an increasing sequence of smooth cut-off functions $(\varphi_i) \subset C_0^\infty(\Omega)$ such that $0 \leq \varphi_i \leq 1$ and $\sup_i \varphi_i(x) = 1$ on Ω, then for every $i \in \mathbb{N}$ we have

$$\begin{aligned}
\lim_j F_{\varepsilon_j}\left(u_j,v_j;\Omega\right) &\geq \liminf_j \int_\Omega f_{\varepsilon_j}\left(x,(u_j,v_j),\nabla(u_j,v_j)\right)\varphi_i\,dx \\
&= \int_\Omega \varphi_i d\mu \geq \int_\Omega f\left(x,u,\nabla u\right)\varphi_i\,dx + \int_\Omega f^\infty\left(x,\tilde{u},\frac{dD^c u}{d\|D^c u\|}\right)\varphi_i d\|D^c u\| \\
&+ \int_{J_u} K\left(x,u^+,u^-,\nu_u\right)\varphi_i\,d\mathcal{H}^{n-1}.
\end{aligned}$$

Eventually, let $i \to +\infty$ and apply the Monotone Convergence Theorem. $\square$

In the following subsections we prove (4.1), (4.2), (4.3).

4.1 The density of the diffuse part

Consider the auxiliary function $\Phi : [0,1] \to [0,+\infty)$ defined by

$$\Phi(t) = 2\int_0^t \sqrt{W(s)}\, ds, \tag{4.4}$$

then notice that Φ is increasing, $\Phi(t) = 0$ if and only if $t = 0$ and $\Phi \in W^{1,\infty}([0,1])$.

Define the function $\tilde{f} : \Omega \times \mathbb{R}^{N+1} \times \mathbb{R}^{(N+1)\times n} \to [0,+\infty)$ by

$$\tilde{f}(x,(u,v),(z,\zeta)) := \psi\left(\Phi^{-1}\left(v \vee 0 \wedge \Phi(1)\right)\right)\left(f(x,u,z) + |\zeta|\right),$$

then notice that for every $(u,v) \in W^{1,1}\left(\Omega;\mathbb{R}^{N+1}\right)$, $\varepsilon > 0$ and $A \in \mathcal{A}(\Omega)$ Young's inequality yields

$$G_\varepsilon(v;A) \geq 2\int_A \sqrt{W(v)}\,|\nabla v|\, dx = \int_A |\nabla \Phi(v)|\, dx,$$

from which we infer that

$$F_\varepsilon(u,v;A) \geq \int_A \tilde{f}\left(x,(u,\Phi(v)),\nabla(u,\Phi(v))\right) dx.$$

It can be easily seen, by the hypotheses on f and ψ, that $\tilde{f}$ satisfies all the assumptions of Theorem 2.14.

Moreover, if $v_j \to 1$ in $L^1(\Omega;[0,1])$, then $\Phi(v_j) \to \Phi(1)$ in $L^1(\Omega;[0,\Phi(1)])$. Hence, given (u_j,v_j) as in the proof of Proposition 4.1, for every $A \in \mathcal{A}(\Omega)$ there holds

$$\begin{aligned}
\liminf_j F_{\varepsilon_j}(u_j,v_j;A) &\geq \liminf_j \int_A \tilde{f}\left(x,(u_j,\Phi(v_j)),\nabla(u_j,\Phi(v_j))\right) dx \\
&\geq \int_A \tilde{f}\left(x,(u,\Phi(1)),\nabla(u,\Phi(1))\right) dx + \int_A \tilde{f}^\infty\left(x,(\tilde{u},\Phi(1)),dD^c(u,\Phi(1))\right) \\
&= \int_A f(x,u,\nabla u)\, dx + \int_A f^\infty\left(x,\tilde{u},dD^c u\right).
\end{aligned}$$

From this, it is easy to infer (4.1) and (4.2).

4.2 The density of the jump part

To prove (4.3) recall that Lemma 2.7, Theorem 3.77 [5] and Radon-Nykodym's Theorem yield for $\mathcal{H}^{n-1}$ a.e. $x_o \in J_u$

$$\lim_{t\to 0^+} \frac{1}{t^{n-1}} \int_{J_u \cap \left(x_o + tQ_{\nu_u(x_o)}\right)} \left|u^+(x) - u^-(x)\right|\, d\mathcal{H}^{n-1} = \left|u^+(x_o) - u^-(x_o)\right|, \tag{4.5}$$

$$\lim_{t\to 0^+} \frac{1}{t^n} \int_{x_o+tQ^{\pm}_{\nu_u(x_o)}} \left|u(x) - u^{\pm}(x_o)\right| \, dx = 0, \tag{4.6}$$

$$\mu_J(x_o) = \lim_{t\to 0^+} \frac{\mu\left(x_o + tQ_{\nu_u(x_o)}\right)}{|u^+ - u^-| \, \mathcal{H}^{n-1}\left(J_u \cap \left(x_o + tQ_{\nu_u(x_o)}\right)\right)}, \tag{4.7}$$

exists and is finite.

By (4.5) and (4.7), and since the function $\mathcal{X}_{x_o+tQ_{\nu_u(x_o)}}$ is upper semicontinuous and with compact support in Ω if t is sufficiently small, we get

$$\begin{aligned}
&\left|u^+(x_o) - u^-(x_o)\right| \mu_J(x_o) = \lim_{t\to 0^+} \frac{1}{t^{n-1}} \int_{x_o+tQ_{\nu_u(x_o)}} d\mu(x) \\
&\geq \limsup_{t\to 0^+} \limsup_{j} \frac{1}{t^{n-1}} \int_{x_o+tQ_{\nu_u(x_o)}} f_{\varepsilon_j}\left(x, (u_j, v_j), \nabla(u_j, v_j)\right) dx \\
&= \limsup_{t\to 0^+} \limsup_{j} \int_{Q_{\nu_u(x_o)}} t f_{\varepsilon_j}\left(x_o + ty, (u_j, v_j)(x_o + ty), \nabla(u_j, v_j)(x_o + ty)\right) dy \\
&= \limsup_{t\to 0^+} \limsup_{j} \int_{Q_{\nu_u(x_o)}} \left(t\psi\left(v_j^t(y)\right) f\left(x_o + ty, u_j^t(y), \frac{1}{t}\nabla u_j^t(y)\right) \right. \\
&\qquad\qquad \left. + \frac{t}{\varepsilon_j} W\left(v_j^t(y)\right) + \frac{\varepsilon_j}{t}\left|\nabla v_j^t(y)\right|^2 \right) dy,
\end{aligned} \tag{4.8}$$

where $(u_j^t(y), v_j^t(y)) := (u_j(x_o + ty), v_j(x_o + ty))$. Notice that $(u_j^t(y), v_j^t(y)) \to (u(x_o+ty), 1)$ in $L^1\left(Q_{\nu_u(x_o)}; \mathbb{R}^{N+1}\right)$ as $j \to +\infty$, and by (4.6) there follows $(u(x_o+ty), 1) \to (u_o(x), 1)$ in $L^1\left(Q_{\nu_u(x_o)}; \mathbb{R}^{N+1}\right)$ as $t \to 0^+$, where

$$u_o(x) := \begin{cases} u^+(x_o) & \langle x - x_o, \nu_u(x_o)\rangle \geq 0 \\ u^-(x_o) & \langle x - x_o, \nu_u(x_o)\rangle < 0 \end{cases}.$$

With fixed $\eta > 0$, let δ, $L > 0$ be given by (f4) and (f5). Then, by (3.4) of (f4), if $t < \frac{1}{L} \wedge \frac{2}{\sqrt{n}}\delta$ we get

$$\begin{aligned}
&\int_{Q_{\nu_u(x_o)}} \psi\left(v_j^t(y)\right) t f\left(x_o + ty, u_j^t(y), \frac{1}{t}\nabla u_j^t(y)\right) dx \\
&\qquad \geq \frac{1}{1+\eta} \int_{Q_{\nu_u(x_o)}} \psi\left(v_j^t(y)\right) f^\infty\left(x_o + ty, u_j^t(y), \nabla u_j^t(y)\right) dx - \frac{c\eta}{1+\eta}.
\end{aligned}$$

On the other hand, by (3.5) of (f5) there follows

$$\begin{aligned}
&\int_{Q_{\nu_u(x_o)}} \psi\left(v_j^t(y)\right) f^\infty\left(x_o + ty, u_j^t(y), \nabla u_j^t(y)\right) dy \\
&\qquad \geq \frac{1}{1+\eta} \int_{Q_{\nu_u(x_o)}} \psi\left(v_j^t(y)\right) f^\infty\left(x_o, u_j^t(y), \nabla u_j^t(y)\right) dy.
\end{aligned}$$

Therefore, letting $\eta \to 0$, from (4.8) we obtain

$$
\begin{aligned}
&\left|u^+(x_o) - u^-(x_o)\right| \mu_J(x_o) \\
&\quad \geq \limsup_{t\to 0^+} \limsup_{j} \int_{Q_{\nu_u(x_o)}} \Bigg(\psi\left(v_j^t(y)\right) f^\infty\left(x_o, u_j^t(y), \nabla u_j^t(y)\right) \\
&\qquad\qquad\qquad + \frac{t}{\varepsilon_j} W\left(v_j^t(y)\right) + \frac{\varepsilon_j}{t}\left|\nabla v_j^t(y)\right|^2 \Bigg)\, dy \\
&\quad = \limsup_{t\to 0^+} \limsup_{j} F^\infty_{\frac{\varepsilon_j}{t}}\left(x_o; u_j^t, v_j^t; Q_{\nu_u(x_o)}\right). \qquad (4.9)
\end{aligned}
$$

By using a diagonal argument for every $h \in \mathbb{N}$ there exists indexes $j_h \in \mathbb{N}$ and $t_h \in (0, +\infty)$ such that $\gamma_h := \frac{\varepsilon_{j_h}}{t_h} \leq \frac{1}{h}$, the sequence $\left(u_{j_h}^{t_h}, v_{j_h}^{t_h}\right) \to (u_o, 1)$ in $L^1\left(Q_{\nu_u(x_o)}; \mathbb{R}^{N+1}\right)$, and

$$\left|u^+(x_o) - u^-(x_o)\right| \mu_J(x_o) \geq \lim_{h} F^\infty_{\gamma_h}\left(x_o; u_{j_h}^{t_h}, v_{j_h}^{t_h}; Q_{\nu_u(x_o)}\right). \qquad (4.10)$$

In order to establish (4.3) and taking into account the definition of K, we need to modify $\left(u_{j_h}^{t_h}, v_{j_h}^{t_h}\right)$ near $\partial Q_{\nu_u(x_o)}$ without increasing the energy in the limit and in such a way that the new sequence belongs to $\mathcal{A}\left(u^+(x_o), u^-(x_o), \nu_u(x_o)\right)$. Assuming Lemma 4.2 below proved, we are done. □

Let us prove the following De Giorgi's type averaging-slicing lemma.

Lemma 4.2 *For every $x_o \in \Omega$, $(a, b, \nu) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1}$, $\gamma_j \to 0^+$, $(u_j, v_j) \to (u_{a,b,\nu}, 1)$ in $L^1\left(Q_\nu; \mathbb{R}^{N+1}\right)$ there exists $(\hat{u}_j, \hat{v}_j) \in \mathcal{A}(a, b, \nu)$ such that $(\hat{u}_j, \hat{v}_j) \to (u_{a,b,\nu}, 1)$ in $L^1\left(Q_\nu; \mathbb{R}^{N+1}\right)$ and*

$$\limsup_{j} F^\infty_{\gamma_j}\left(x_o; \hat{u}_j, \hat{v}_j; Q_\nu\right) \leq \liminf_{j} F^\infty_{\gamma_j}\left(x_o; u_j, v_j; Q_\nu\right) \qquad (4.11)$$

Proof. Without loss of generality we may assume the inferior limit in (4.11) to be finite and to be a limit. Moreover, we denote by c a generic positive constant which may vary from line to line.

Let $(w_j) \subset W^{1,1}\left(Q_\nu; \mathbb{R}^N\right)$ be such that $w_j \to u_{a,b,\nu}$ in $L^1\left(Q_\nu; \mathbb{R}^N\right)$, $w_j = u_{a,b,\nu}$ on ∂Q_ν and $\|Dw_j\|(Q_\nu) \to \|Du_{a,b,\nu}\|(Q_\nu)$ (see Lemma 2.5 [12]).
Let $a_j \to 0^+$, $b_j \in \mathbb{N}$ to be chosen suitably and such that $s_j := \frac{a_j}{b_j} \to 0$, then set $Q_\nu^{j,i} := (1 - a_j + i s_j) Q_\nu$, $0 \leq i \leq b_j$. Let $(\varphi_{j,i}) \subset C_0^\infty\left(Q_\nu^{j,i}\right)$, $1 \leq i \leq b_j$, be a family of cut-off functions such that $0 \leq \varphi_{j,i} \leq 1$, $\varphi_{j,i} = 1$ on $Q_\nu^{j,i-1}$, $\|\nabla \varphi_{j,i}\|_\infty = O(s_j^{-1})$.
Define

$$u_j^i := \varphi_{j,i-1} u_j + (1 - \varphi_{j,i-1}) w_j;\ v_j^i := \varphi_{j,i} v_j + (1 - \varphi_{j,i}),$$

then $\left(u_j^i, v_j^i\right) \in \mathcal{A}(a, b, \nu)$ and tends to $(u_{a,b,\nu}, 1)$ in $L^1\left(\Omega; \mathbb{R}^{N+1}\right)$ as $j \to +\infty$ for every $i \in \mathbb{N}$. Moreover

$$F^\infty_{\gamma_j}\left(x_o; u_j^i, v_j^i; Q_\nu\right) \leq F^\infty_{\gamma_j}\left(x_o; u_j, v_j; Q_\nu^{j,i-2}\right)$$

$$
+\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} \psi(v_j) f^\infty(x_o,u_j^i,\nabla u_j^i)\,dx + G_{\gamma_j}(v_j;Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2})
$$

$$
+\int_{Q_\nu\setminus Q_\nu^{j,i-1}} \psi\left(v_j^i\right) f^\infty(x_o;w_j,\nabla w_j)\,dx + G_{\gamma_j}\left(v_j^i;Q_\nu^{j,i}\setminus Q_\nu^{j,i-1}\right). \quad (4.12)
$$

We estimate separately the terms appearing above. To begin with, we have that

$$
F^\infty_{\gamma_j}\left(x_o;u_j,v_j;Q_\nu^{j,i-2}\right) + G_{\gamma_j}(v_j;Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}) \le F^\infty_{\gamma_j}\left(x_o,u_j,v_j;Q_\nu\right). \quad (4.13)
$$

Moreover, since $\nabla u_j^i = \varphi_{j,i-1}\nabla u_j + (1-\varphi_{j,i-1})\nabla w_j + \nabla\varphi_{j,i-1}\otimes(u_j-w_j)$, by the growth condition (3.2) we have

$$
\begin{aligned}
&\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} \psi(v_j) f^\infty(x_o,u_j^i,\nabla u_j^i)\,dx \\
&\quad\le c\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} \psi(v_j)\left(|\nabla u_j| + |\nabla w_j| + |\nabla\varphi_{j,i-1}|\;|u_j-w_j|\right)\,dx \\
&\quad\le c\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} \psi(v_j) f^\infty(x_o,u_j,\nabla u_j)\,dx + c\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} |\nabla w_j|\,dx \\
&\quad+\frac{c}{s_j}\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} |u_j-w_j|. \qquad (4.14)
\end{aligned}
$$

Analogously, there follows

$$
\int_{Q_\nu\setminus Q_\nu^{j,i-1}} \psi\left(v_j^i\right) f^\infty(x_o,w_j,\nabla w_j)\,dx \le c\int_{Q_\nu\setminus Q_\nu^{j,i-1}} |\nabla w_j|\,dx. \quad (4.15)
$$

Eventually, since $\nabla v_j^i = \varphi_{j,i}\nabla v_j + (v_j-1)\nabla\varphi_{j,i}$, we get

$$
\begin{aligned}
&G_{\gamma_j}\left(v_j^i;Q_\nu^{j,i}\setminus Q_\nu^{j,i-1}\right) \\
&\quad\le c\int_{Q_\nu^{j,i}\setminus Q_\nu^{j,i-1}} \left(\frac{1}{\gamma_j} + \gamma_j\,|\nabla v_j|^2 + \frac{\gamma_j}{s_j^2}|v_j-1|^2\right)\,dx \\
&\quad\le \frac{c}{\gamma_j}\mathcal{L}^n\left(Q_\nu^{j,i}\setminus Q_\nu^{j,i-1}\right) + c\;G_{\gamma_j}\left(v_j;Q_\nu^{j,i}\setminus Q_\nu^{j,i-1}\right) \\
&\quad+c\frac{\gamma_j}{s_j^2}\int_{Q_\nu^{j,i}\setminus Q_\nu^{j,i-1}} |v_j-1|^2\,dx. \qquad (4.16)
\end{aligned}
$$

By collecting (4.13), (4.14), (4.15) and (4.16) in (4.12) above, by adding up on i and averaging, we have that there exists an index $i_j\in\mathbb{N}$, $2\le i_j\le b_j$, such that

$$
\begin{aligned}
F^\infty_{\gamma_j}\left(x_o;u_j^{i_j},v_j^{i_j};Q_\nu\right) &\le \frac{1}{b_j-1}\sum_{i=2}^{b_j} F^\infty_{\gamma_j}\left(x_o;u_j^i,v_j^i;Q_\nu\right) \\
&\le F^\infty_{\gamma_j}\left(x_o;u_j,v_j;Q_\nu\right) + \frac{c}{b_j}F^\infty_{\gamma_j}\left(x_o;u_j,v_j;Q_\nu\right)
\end{aligned}
$$

$$
\begin{aligned}
&+c\int_{Q_\nu\setminus Q_\nu^{j,0}}|\nabla w_j|\,dx+\frac{c}{a_j}\int_{Q_\nu\setminus Q_\nu^{j,0}}|u_j-w_j|\,dx\\
&+\frac{c}{\gamma_j b_j}\mathcal{L}^n\left(Q_\nu\setminus Q_\nu^{j,0}\right)+c\frac{\gamma_j b_j}{a_j^2}\int_{Q_\nu\setminus Q_\nu^{j,0}}|v_j-1|^2\,dx. \qquad (4.17)
\end{aligned}
$$

Eventually, choose $a_j=\|u_j-w_j\|^{\frac{1}{2}}_{L^1(Q_\nu;\mathbb{R}^N)}+\|v_j-1\|^{\frac{1}{2}}_{L^2(Q_\nu)}$, $b_j=\left[\gamma_j^{-1}\right]$ and set $(\hat{u}_j,\hat{v}_j)=\left(u_j^{i_j},v_j^{i_j}\right)$. The conclusion then follows by passing to the limit on $j\to+\infty$ in (4.17), and noticing that $\mathcal{L}^n\left(Q_\nu\setminus Q_\nu^{j,0}\right)=O(a_j)$ and $\|Dw_j\|\left(Q_\nu\setminus Q_\nu^{j,0}\right)\to 0$. The last assertion follows since $(\|Dw_j\|)$ weakly* converges to $\|Du_{a,b,\nu}\|$ in the sense of measures, $\|Dw_j\|\left(Q_\nu\right)\to\|Du_{a,b,\nu}\|\left(Q_\nu\right)$ and $\|Du_{a,b,\nu}\|\left(\partial Q_\nu^{j,0}\right)=0$ for every $j\in\mathbb{N}$. □

5 Γ-limsup inequality

In order to prove Theorem 3.2 on $BV\left(\Omega;\mathbb{R}^N\right)$, we follow an abstract approach (see [4], [16]). Indeed, first we prove that the $\overline{\Gamma}\left(L^1\right)$-limit of any subsequence of $(F_\varepsilon)_{\varepsilon>0}$, as a set function, is a Borel measure and, by Proposition 2.5, coincides with its Γ-limit. Then, by using Theorems 2.15 and 2.16, in Proposition 5.3 in the sequel we provide an upper estimate of the limiting functional, which, combined with the lower estimate of Proposition 4.1, allows us to conclude that the $\Gamma\left(L^1\right)$-limit does not depend on the chosen subsequence and it is equal to F. Hence, by Urysohn's property the whole family $(F_\varepsilon)_{\varepsilon>0}$ $\Gamma\left(L^1\right)$-converges to F.

As a first step we prove the following crucial lemma, in which we establish the so called weak subadditivity for $F''(u,1,\cdot)$ (see [16], [18]).

The argument used is a careful modification of well known techniques in this kind of problems, and it is strictly related to the ones exploited in Lemma 4.2.

Lemma 5.1 *Let $u\in BV\left(\Omega;\mathbb{R}^N\right)$, let $A',A,B\in\mathcal{A}(\Omega)$ with $A'\subset\subset A$, then*

$$
F''\left(u,1;A'\cup B\right)\le F''\left(u,1;A\right)+F''\left(u,1;B\right).
$$

Proof. Let $(w_j)\subset C^\infty\left(\Omega;\mathbb{R}^N\right)$ be strictly converging to u, i.e., such that $w_j\to u$ in $L^1\left(\Omega;\mathbb{R}^N\right)$ and $\|Dw_j\|(\Omega)\to\|Du\|(\Omega)$, and let $\left(u_j^A,v_j^A\right)$, $\left(u_j^B,v_j^B\right)$ be converging to $(u,1)$ in $L^1\left(\Omega;\mathbb{R}^{N+1}\right)$ and such that

$$
\begin{aligned}
&\limsup_j F_{\varepsilon_j}\left(u_j^A,v_j^A;A\right)=F''(u,1;A),\\
&\limsup_j F_{\varepsilon_j}\left(u_j^B,v_j^B;B\right)=F''(u,1;B),
\end{aligned}
$$

respectively. Set $\delta:=d\left(A',\partial A\right)$, let $M\in\mathbb{N}$ and define

$$
\begin{cases}
A_i^M:=\left\{x\in A:d\left(x,A'\right)\le\frac{\delta}{M}i\right\} & 1\le i\le M\\
A_0^M:=A'
\end{cases}.
$$

Let $(\varphi_i) \subset C_0^\infty(A_i^M)$, $1 \le i \le M$, be a family of cut-off functions such that $0 \le \varphi_i \le 1$, $\varphi_i = 1$ on A_{i-1}^M, $\|\nabla\varphi_i\|_\infty \le \frac{2M}{\delta}$. Define

$$u_j^i := \begin{cases} \varphi_{i-1} u_j^A + (1-\varphi_{i-1}) w_j & A_{i-1}^M \\ \\ w_j & A_i^M \setminus A_{i-1}^M \\ \\ (1-\varphi_{i+1}) u_j^B + \varphi_{i+1} w_j & \Omega \setminus A_i^M \end{cases},$$

and

$$v_j^i := \varphi_i v_j^A + (1-\varphi_i) v_j^B,$$

then $\left(u_j^i, v_j^i\right) \to (u,1)$ in $L^1\left(\Omega;\mathbb{R}^{N+1}\right)$ for every $1 \le i \le M$. Moreover, there follows

$$\begin{aligned} F_{\varepsilon_j}\left(u_j^i, v_j^i; A' \cup B\right) &\le F_{\varepsilon_j}\left(u_j^A, v_j^A; A_{i-2}^M\right) \\ &+ \int_{\left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right)\cap B} \psi\left(v_j^A\right) f\left(x, u_j^i, \nabla u_j^i\right) dx + G_{\varepsilon_j}\left(v_j^A; \left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right) \cap B\right) \\ &+ c\int_{\left(A_i^M \setminus \overline{A_{i-1}^M}\right)\cap B} \left(1 + |\nabla w_j|\right) dx + G_{\varepsilon_j}\left(v_j^i; \left(A_i^M \setminus \overline{A_{i-1}^M}\right) \cap B\right) \\ &+ \int_{\left(A_{i+1}^M \setminus \overline{A_i^M}\right)\cap B} \psi\left(v_j^B\right) f\left(x, u_j^i, \nabla u_j^i\right) dx + G_{\varepsilon_j}\left(v_j^B; \left(A_{i+1}^M \setminus \overline{A_i^M}\right) \cap B\right) \\ &+ F_{\varepsilon_j}\left(u_j^B, v_j^B; B \setminus \overline{A_{i+1}^M}\right). \end{aligned}$$

Let us estimate only the terms above depending on the superscript A, analogous computations holds for the one with B. First, it is easy to check that

$$F_{\varepsilon_j}\left(u_j^A, v_j^A; A_{i-2}^M\right) + G_{\varepsilon_j}\left(v_j^A; \left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right) \cap B\right) \le F_{\varepsilon_j}\left(u_j^A, v_j^A; A\right), \tag{5.1}$$

and

$$\begin{aligned} \int_{\left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right)\cap B} \psi\left(v_j^A\right) f\left(x, u_j^i, \nabla u_j^i\right) dx &\le c\, F_{\varepsilon_j}\left(u_j^A, v_j^A; \left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right) \cap B\right) \\ + c\int_{\left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right)\cap B} \left(1 + |\nabla w_j|\right) dx &+ \int_{\left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right)\cap B} |\nabla\varphi_{i-1}| \left|u_j^A - w_j\right| dx. \end{aligned} \tag{5.2}$$

Moreover, there holds

$$\begin{aligned} &G_{\varepsilon_j}\left(v_j^i; \left(A_i^M \setminus \overline{A_{i-1}^M}\right) \cap B\right) \\ &\le c\, G_{\varepsilon_j}\left(v_j^A; \left(A_i^M \setminus \overline{A_{i-1}^M}\right) \cap B\right) + c\, G_{\varepsilon_j}\left(v_j^B; \left(A_i^M \setminus \overline{A_{i-1}^M}\right) \cap B\right) \\ &+ c\,\varepsilon_j \int_{\left(A_i^M \setminus \overline{A_{i-1}^M}\right)\cap B} |\nabla\varphi_i|^2 \left|v_j^A - v_j^B\right|^2 dx + \frac{c}{\varepsilon_j}\mathcal{L}^n\left(\left(A_i^M \setminus \overline{A_{i-1}^M}\right) \cap B\right). \end{aligned} \tag{5.3}$$

Then, from (5.1), (5.2), (5.3), by adding up on i and averaging, there exists an index $2 \le i_j \le M-1$ such that

$$
\begin{aligned}
F_{\varepsilon_j}\left(u_j^{i_j}, v_j^{i_j}; A' \cup B\right) &\le \frac{1}{M-2} \sum_{i=2}^{M-1} F_{\varepsilon_j}\left(u_j^i, v_j^i; A' \cup B\right) \\
&\le \left(1 + \frac{c}{M-2}\right)\left(F_{\varepsilon_j}\left(u_j^A, v_j^A; A\right) + F_{\varepsilon_j}\left(u_j^B, v_j^B; B\right)\right) \\
&+ \frac{cM}{(M-2)\delta}\left(\int_{(A\setminus A')\cap B} \left|u_j^A - w_j\right| dx + \int_{(A\setminus A')\cap B} \left|u_j^B - w_j\right| dx\right) \\
&+ \frac{c\,\varepsilon_j M^2}{(M-2)\delta^2} \int_{(A\setminus A')\cap B} \left|v_j^A - v_j^B\right|^2 dx + \frac{c}{M-2} \int_{(A\setminus A')\cap B} |\nabla w_j|\, dx \\
&+ \frac{c}{\varepsilon_j (M-2)} \mathcal{L}^n\left((A \setminus A') \cap B\right).
\end{aligned}
$$

Now choose $M_j = \left[\varepsilon_j^{-1} \left\|v_j^A - v_j^B\right\|_{L^2(\Omega)}^{-1}\right]$, then by passing to the superior limit on $j \to +\infty$ and by the definition of F'' we get the conclusion. $\square$

By virtue of Lemma 5.1 we get the following.

Corollary 5.2 *Assume that* $\left(F_{\varepsilon_j}\right)_{j\in\mathbb{N}}$ $\overline{\Gamma}\left(L^1\right)$ *-converges to* $\hat{F}$*, then for every* $u \in BV\left(\Omega; \mathbb{R}^N\right)$ *the set function* $\hat{F}(u,1;\cdot)$ *is a Borel measure.*

Moreover, for every $A \in \mathcal{A}(\Omega)$

$$
\hat{F}(u,1;A) \le c\left(\mathcal{L}^n(A) + \|Du\|(A)\right),
$$

and

$$
\hat{F}(u,1;A) = \Gamma\left(L^1\right)\text{-}\lim_j F_{\varepsilon_j}(u,1;A).
$$

Proof. It suffices to take into account that the growth assumptions (3.2) on f and to apply Propositions 2.4 and 2.5. $\square$

We now are able to prove Theorem 3.2 in the BV case.

Proposition 5.3 *For every* $u \in BV\left(\Omega; \mathbb{R}^N\right)$ *we have*

$$
\Gamma\left(L^1\right)\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u,1) = F(u,1).
$$

Proof. Let $\varepsilon_j \to 0^+$ be such that for every $u \in BV\left(\Omega; \mathbb{R}^N\right)$ and $A \in \mathcal{A}(\Omega)$ there exists $\hat{F}(u,1;A) := \Gamma\left(L^1\right)\text{-}\lim_j F_{\varepsilon_j}(u,v;A)$.

Then, by Proposition 4.1, we are done if we show that

$$
\hat{F}(u,1;\Omega) \le F(u,1).
$$

Since by Corollary 5.2 $\hat{F}(u,1;\cdot)$ is a Borel measure, it suffices to prove that

$$\hat{F}(u,1;\Omega\setminus J_u) \le \int_\Omega f\left(x,u,\nabla u\right)\,dx + \int_\Omega f^\infty\left(x,\tilde{u},dD^c u\right), \tag{5.4}$$

and

$$\hat{F}(u,1;J_u) \le \int_{J_u} \tilde{K}\left(x,u^+,u^-,\nu_u\right)\,d\mathcal{H}^{n-1}, \tag{5.5}$$

where $\tilde{K}$ is the function defined in Remark 3.6 (recall that $\tilde{K}\le K$).

To prove (5.4), note that for every $j\in\mathbb{N}$

$$F_{\varepsilon_j}(u,1;A)\equiv F_0(u;A),$$

with

$$F_0(u;A) = \begin{cases} \displaystyle\int_A f\left(x,u,\nabla u\right)\,dx & \text{if } u\in W^{1,1}\left(\Omega;\mathbb{R}^N\right) \\ +\infty & \text{if } u\in L^1(\Omega;\mathbb{R}^N)\setminus W^{1,1}\left(\Omega;\mathbb{R}^N\right) \end{cases}.$$

Hence, for every $B\in\mathcal{B}(\Omega)$

$$\hat{F}(u,1;B)\le \overline{F}_0(u;B).$$

By Theorems 2.14 and 2.15, we get that for every $u\in BV\left(\Omega;\mathbb{R}^N\right)$

$$\overline{F}_0\left(u;\Omega\setminus J_u\right) = \int_\Omega f\left(x,u,\nabla u\right)\,dx + \int_\Omega f^\infty\left(x,\tilde{u},dD^c u\right),$$

from which (5.4) is easily deduced.

By (2.11) of Theorem 2.16, to prove (5.5), it suffices to show that for every $(x_o,a,b,\nu)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1}$

$$\limsup_{\delta\to0^+}\frac{\hat{F}\left(u_{a,b,\nu}(\cdot-x_o),1;x_o+\delta Q_\nu\right)}{\delta^{n-1}}\le\tilde{K}(x_o,a,b,\nu). \tag{5.6}$$

Without loss of generality we prove (5.6) assuming $x_o=0$ and $\nu=e_n$ (recall that Q_{e_n} is denoted by Q).

With the same notations of formula (3.14) for $\tilde{K}$, given $\gamma>0$, let $(u,v)\in\tilde{\mathcal{A}}(a,b,e_n)$ and $L>0$ be such that

$$F^\infty_{\frac{1}{L}}(0,u,v;Q)\le\tilde{K}(0,a,b,e_n)+\gamma.$$

Define $(u_j,v_j)\in W^{1,1}\left(S_{e_n};\mathbb{R}^{N+1}\right)$ by

$$(u_j,v_j)(x) = \begin{cases} (b,1) & \text{if } x_n>\frac{\varepsilon_j L}{2} \\ (u,v)\left(\frac{x}{\varepsilon_j L}\right) & \text{if } |x_n|\le\frac{\varepsilon_j L}{2} \\ (a,1) & \text{if } x_n<-\frac{\varepsilon_j L}{2}, \end{cases}$$

hence, $(u_j, v_j) \to (u_{a,b,e_n}, 1)$ in $L^1\left(Q; \mathbb{R}^{N+1}\right)$, and thus

$$\hat{F}(u_{a,b,e_n}, 1; \delta Q) \leq \limsup_j F_{\varepsilon_j}\left(u_j, v_j; \delta Q\right). \tag{5.7}$$

Set $Q_\delta^j := \delta Q \cap \left\{|x_n| \leq \frac{\varepsilon_j L}{2}\right\}$ and $Q'_\delta = \delta Q \cap \{x_n = 0\}$, then we have

$$\begin{aligned} F_{\varepsilon_j}\left(u_j, v_j; \delta Q\right) = & \int_{\delta Q \cap \left\{x_n < -\frac{\varepsilon_j L}{2}\right\}} f(x, a, 0)\, dx \\ & + \int_{\delta Q \cap \left\{x_n > \frac{\varepsilon_j L}{2}\right\}} f(x, b, 0)\, dx + F_{\varepsilon_j}\left(u_j, v_j; Q_\delta^j\right). \end{aligned} \tag{5.8}$$

The change of variables $t = \frac{x_n}{\varepsilon_j L}$ yields for j large

$$\begin{aligned} & F_{\varepsilon_j}\left(u_j, v_j; Q_\delta^j\right) = \\ & \int_{-1/2}^{1/2} \varepsilon_j L\, dt \int_{Q'_\delta} \psi\left(v\left(\frac{x'}{\varepsilon_j L}, t\right)\right) f\left(\left(x', \varepsilon_j L t\right), u\left(\frac{x'}{\varepsilon_j L}, t\right), \frac{1}{\varepsilon_j L} \nabla u\left(\frac{x'}{\varepsilon_j L}, t\right)\right) dx' \\ & + \int_{-1/2}^{1/2} dt \int_{Q'_\delta} \left(L\, W\left(v\left(\frac{x'}{\varepsilon_j L}, t\right)\right) + \frac{1}{L}\left|\nabla\left(v\left(\frac{x'}{\varepsilon_j L}, t\right)\right)\right|^2\right) dx' \\ & =: I_{j,\delta}^1 + I_{j,\delta}^2. \end{aligned} \tag{5.9}$$

With fixed $\eta > 0$, by (3.4), we can choose δ small enough such that for j large we have

$$\begin{aligned} I_{j,\delta}^1 \leq & \frac{1}{1-\eta}\Bigg(\eta \delta^{n-1} \\ & + \int_{-1/2}^{1/2} dt \int_{Q'_\delta} \psi\left(v\left(\frac{x'}{\varepsilon_j L}, t\right)\right) f^\infty\left(\left(x', \varepsilon_j L t\right), u\left(\frac{x'}{\varepsilon_j L}, t\right), \nabla u\left(\frac{x'}{\varepsilon_j L}, t\right)\right) dx'\Bigg). \end{aligned} \tag{5.10}$$

Now consider the Yosida's Transform of f^∞ defined, for $\lambda > 0$, as

$$f_\lambda^\infty(x, u, z) := \sup_{y \in \mathbb{R}^n} \{f^\infty(y, u, z) - \lambda |y - x|\}.$$

Recall that

$$f^\infty(x, u, z) \leq f_{\lambda_1}^\infty(x, u, z) \leq f_{\lambda_2}^\infty(x, u, z) \tag{5.11}$$

if $0 < \lambda_2 \leq \lambda_1$ and, since $f^\infty(\cdot, u, z)$ is upper semicontinuous, $f_\lambda^\infty(\cdot, u, z) \to f^\infty(\cdot, u, z)$ pointwise as $\lambda \to +\infty$. Moreover, f_λ^∞ is λ-lipschitzian, i.e.,

$$|f_\lambda^\infty(x_1, u, z) - f_\lambda^\infty(x_2, u, z)| \leq \lambda |x_1 - x_2| \tag{5.12}$$

and, by (3.2), for every $z \in \mathbb{R}^N$ there holds

$$0 < f_\lambda^\infty(x, u, z) \leq c(1 + |z|).$$

Thus, given $\lambda > 0$, by (5.10), (5.11) and (5.12), we get

$$\begin{aligned} I_{j,\delta}^1 \leq \frac{1}{1-\eta} \Bigg(& \eta \delta^{n-1} + 2\lambda \delta^n \\ & + \int_{-1/2}^{1/2} dt \int_{Q'_\delta} \psi \left(v \left(\frac{x'}{\varepsilon_j L}, t \right) \right) f_\lambda^\infty \left(0, u \left(\frac{x'}{\varepsilon_j L}, t \right), \nabla u \left(\frac{x'}{\varepsilon_j L}, t \right) \right) dx' \Bigg). \end{aligned} \tag{5.13}$$

Let now $j \to +\infty$ in (5.8) and take into account the inequalities (5.9) and (5.13); then by virtue of the Riemann-Lebesgue's Lemma we have

$$\begin{aligned} \limsup_j F_{\varepsilon_j}(u_j, v_j; \delta Q) \leq & \frac{1}{1-\eta} \delta^{n-1} \int_Q \psi(v) f_\lambda^\infty \left(0, u, \nabla u\right) \, dx \\ & + \delta^{n-1} \int_Q \left(LW(v) + \frac{1}{L} |\nabla v|^2 \right) \, dx + \frac{\eta}{1-\eta} \delta^{n-1} + \left(\frac{2\lambda}{1-\eta} + c \right) \delta^n. \end{aligned}$$

Thus, by (5.7), we get

$$\begin{aligned} \limsup_{\delta \to 0^+} \frac{\hat{F}\left(u_{a,b,e_n}, 1; \delta Q\right)}{\delta^{n-1}} \leq & \frac{1}{1-\eta} \int_Q \psi(v) f_\lambda^\infty \left(0, u, \nabla u\right) \, dx \\ & + \int_Q \left(LW(v) + \frac{1}{L} |\nabla v|^2 \right) \, dx + \frac{\eta}{1-\eta}. \end{aligned}$$

Eventually, by letting $\eta \to 0^+$ and $\lambda \to +\infty$, by Lebesgue's Theorem we get

$$\begin{aligned} & \limsup_{\delta \to 0^+} \frac{\hat{F}\left(u_{a,b,e_n}, 1; \delta Q\right)}{\delta^{n-1}} \\ & \quad \leq \int_Q \left(\psi(v) f^\infty \left(0, u, \nabla u\right) + LW(v) + \frac{1}{L} |\nabla v|^2 \right) dx \\ & \quad = F_{\frac{1}{L}}^\infty(0; u, v; Q) \leq \tilde{K}(0, a, b, e_n) + \gamma, \end{aligned}$$

and by the arbitrariness of $\gamma > 0$ we obtain (5.6). □

6 The GBV case

In this section we prove the full result stated in Theorem 3.2. We recall that we have already shown the Γ-convergence result if the target function $u \in BV\left(\Omega; \mathbb{R}^N\right)$, here we extend the proof to all functions $u \in L^1\left(\Omega;\mathbb{R}^{N+1}\right)$, and we identify the domain of the limit functional in a subset of $(GBV(\Omega))^N \times \{1\}$.

We first state and prove a preliminary lemma on the continuity of $F(\cdot,1)$ with respect to truncations.

Lemma 6.1 *Let $u \in (GBV(\Omega))^N$ with $F(u,1;\Omega) < +\infty$ and let $u^i := \Psi_i(u)$, $i \in \mathbb{N}$, where Ψ_i are defined in (2.4). Then*

$$\lim_i F\left(u^i,1\right) = F\left(u,1\right).$$

Proof. We prove separately the convergence of the different terms of F.

Since $\nabla u(x) = \nabla u^i(x)$ for a.e. $x \in \Omega_i := \{x \in \Omega : |\tilde{u}(x)| < a_i\}$, we have

$$\int_\Omega f(x,u^i,\nabla u^i)\,dx = \int_{\Omega_i} f(x,u,\nabla u)\,dx + \int_{\Omega\setminus\Omega_i} f(x,u^i,\nabla u^i)\,dx.$$

By the growth assumption (3.2), we get

$$\left|\int_{\Omega\setminus\Omega_i} f\left(x,u^i,\nabla u^i\right)\,dx\right| \le c\int_{\Omega\setminus\Omega_i}\left(1+|\nabla u|\right)\,dx,$$

and so, being the term on the right hand side above infinitesimal, we deduce that

$$\lim_i \int_\Omega f(x,u^i,\nabla u^i)\,dx = \int_\Omega f\left(x,u,\nabla u\right)\,dx.$$

Let us prove the convergence of the Cantor part of the energy. Since the measures $D^c u^i$ are absolutely continuous with respect to $\|D^c u\|$ and $D^c u^i \llcorner \Omega_i \equiv D^c u \llcorner \Omega_i$, we have

$$\begin{aligned}\int_\Omega f^\infty\left(x,\tilde{u}^i,dD^cu^i\right) &= \int_\Omega f^\infty\left(x,\tilde{u}^i,\frac{dD^cu^i}{d\|D^cu\|}\right)\,d\|D^cu\| \\ &= \int_{\Omega_i} f^\infty\left(x,\tilde{u},dD^cu\right) + \int_{\Omega\setminus\Omega_i} f^\infty\left(x,\tilde{u}^i,\frac{dD^cu^i}{d\|D^cu\|}\right)\,d\|D^cu\|. \end{aligned} \tag{6.1}$$

Moreover, by (3.6), we have

$$\left|\int_{\Omega\setminus\Omega_i} f^\infty\left(x,\tilde{u}^i,\frac{dD^cu^i}{d\|D^cu\|}\right)\,d\|D^cu\|\right| \le c\,\|D^cu\|\left(\Omega\setminus\Omega_i\right),$$

and thus, since $\|D^cu\|\left(\Omega\setminus\Omega_i\right) \to 0$ as $i \to +\infty$, from (6.1) we conclude that

$$\lim_i \int_\Omega f^\infty\left(x,\tilde{u}^i,dD^cu^i\right) = \int_\Omega f^\infty\left(x,\tilde{u},dD^cu\right).$$

Eventually, for what the surface energy is concerned, note that $\mathcal{H}^{n-1}\left(J_u^\infty\right) = 0$ (see Theorem 2.12, Remark 2.13 and Remark 3.5) and $J_{u^i} \subseteq J_u$ for every $i \in \mathbb{N}$ with $\nu_{u^i} = \nu_u$ for $\mathcal{H}^{n-1}$ a.e. $x \in J_{u^i}$. Then, $\left(u^i\right)^\pm \to u^\pm$, $\mathcal{X}_{J_{u^i}} \to \mathcal{X}_{J_u}$ for $\mathcal{H}^{n-1}$ a.e. $x \in J_u$ as $i \to +\infty$. Hence, there follows

$$\begin{aligned}
\lim_i \int_{J_{u^i}} & K\left(x, \left(u^i\right)^+, \left(u^i\right)^-, \nu_{u^i}\right) \, d\mathcal{H}^{n-1} \\
&= \lim_i \int_{J_u} K\left(x, \left(u^i\right)^+, \left(u^i\right)^-, \nu_u\right) \mathcal{X}_{J_{u^i}} \, d\mathcal{H}^{n-1} \\
&= \int_{J_u} K\left(x, u^+, u^-, \nu_u\right) \, d\mathcal{H}^{n-1},
\end{aligned}$$

by Lebesgue's Theorem and taking into account properties (a) and (b) of Lemma 3.7 . □

The idea of the proof of the Γ-liminf inequality in the next proposition is based again on De Giorgi's averaging-slicing method but now the truncation is performed on the range rather than on the domain (see Lemma 3.7 [11], Lemma 3.5 [15]).

Proposition 6.2 *For every* $(u,v) \in L^1\left(\Omega; \mathbb{R}^{N+1}\right)$ *we have*

$$\Gamma\left(L^1\right)\text{-}\lim_{\varepsilon \to 0^+} F_\varepsilon(u,v) = F(u,v).$$

Proof. We divide the proof in two steps, dealing with the Γ-liminf and the Γ-limsup inequality separately.

Step 1 (liminf inequality): for every $(u,v) \in L^1\left(\Omega; \mathbb{R}^{N+1}\right)$ there holds

$$\Gamma\left(L^1\right)\text{-}\liminf_{\varepsilon \to 0^+} F_\varepsilon(u,v) \geq F(u,v). \tag{6.2}$$

Let $(u_j, v_j) \to (u,v)$ in $L^1\left(\Omega; \mathbb{R}^{N+1}\right)$ be such that

$$\lim_j F_{\varepsilon_j}(u_j, v_j) = \Gamma\left(L^1\right)\text{-}\liminf_j F_{\varepsilon_j}(u,v). \tag{6.3}$$

We may also assume such a limit to be finite; hence, as already shown in Proposition 4.1, we have that $v_j \to 1$ in $L^1(\Omega)$, and, as observed in Remark 3.5, $u \in (GBV(\Omega))^N$.

Define $u_j^i := \Psi_i(u_j)$, $u^i := \Psi_i(u)$, where Ψ_i are the auxiliary functions in (2.4), then $u_j^i \in W^{1,1}\left(\Omega; \mathbb{R}^N\right)$, $u^i \in BV\left(\Omega; \mathbb{R}^N\right)$ and $u_j^i \to u^i$ in $L^1\left(\Omega; \mathbb{R}^N\right)$ for every $i \in \mathbb{N}$. Moreover, notice that

$$F_{\varepsilon_j}\left(u_j^i, v_j\right) = \int_\Omega \psi(v_j) f\left(x, u_j^i, \nabla u_j^i\right) dx + G_{\varepsilon_j}(v_j; \Omega). \tag{6.4}$$

Fix $j \in \mathbb{N}$, then we have

$$\int_\Omega \psi(v_j) f\left(x, u_j^i, \nabla u_j^i\right) dx = \int_{\{|u_j|<a_i\}} \psi(v_j) f(x, u_j, \nabla u_j) dx$$
$$+\int_{\{a_i\le|u_j|\le a_{i+1}\}} \psi(v_j) f\left(x, u_j^i, \nabla u_j^i\right) dx + \int_{\{|u_j|>a_{i+1}\}} \psi(v_j) f(x,0,0) dx$$
$$\le \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j) dx + c\int_{\{a_i\le|u_j|\le a_{i+1}\}} \psi(v_j)(1+|\nabla u_j|) dx$$
$$+c\mathcal{L}^n(\{|u_j|>a_{i+1}\}).$$

With fixed $\eta>0$ there exists $i_o \in \mathbb{N}$, $i_o \ge \frac{1}{\eta}$, such that $c\mathcal{L}^n(\{|u_j|\ge a_{i_o}\}) \le \eta$. Let $M \in \mathbb{N}$, then for every $j \in \mathbb{N}$ there exists $i_j \in \{i_o, i_o+1, \ldots, i_o+M-1\}$ such that

$$\int_\Omega \psi(v_j) f\left(x, u_j^{i_j}, \nabla u_j^{i_j}\right) dx \le \frac{1}{M}\sum_{i=i_o}^{i_o+M-1}\int_\Omega \psi(v_j) f\left(x, u_j^i, \nabla u_j^i\right) dx$$
$$\le \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j) dx + \frac{c}{M}\int_{\{|u_j|\ge a_{i_o}\}} \psi(v_j)(1+|\nabla u_j|) dx + \eta$$
$$\le \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j) dx + 2\eta, \tag{6.5}$$

by (3.2), (6.3) and by choosing $M \in \mathbb{N}$ suitably. Note that M is independent of j and depends only on η. Moreover, (6.4) and (6.5) yield

$$F_{\varepsilon_j}\left(u_j^{i_j}, v_j\right) \le F_{\varepsilon_j}(u_j, v_j) + 2\eta. \tag{6.6}$$

Since $i_j \in \{i_o, i_o+1, \ldots, i_o+M-1\}$ for every $j \in \mathbb{N}$, up to extracting a subsequence not relabelled for convenience, we may assume $i_j \equiv i_\eta$ to be constant. Hence, $u_j^{i_\eta} \to u^{i_\eta}$ in $L^1(\Omega;\mathbb{R}^N)$ and so by (6.3), (6.6) and Subsection 4.1 there follows

$$F\left(u^{i_\eta}, v\right) \le \lim_j F_{\varepsilon_j}\left(u_j^{i_\eta}, v_j\right)$$
$$\le \Gamma\left(L^1\right)\text{-}\liminf_j F_{\varepsilon_j}(u,v) + 2\eta. \tag{6.7}$$

Eventually, letting $\eta \to 0^+$ in (6.7), by Lemma 6.1 we obtain (6.2).

Step 2 (limsup inequality): for every $(u,v) \in L^1\left(\Omega;\mathbb{R}^{N+1}\right)$ we have

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u,v) \le F(u,v). \tag{6.8}$$

It suffices to prove (6.8) for $u \in (GBV(\Omega))^N$ with $F(u,1)<+\infty$ and $v\equiv 1$. Let u^i be the truncation of u defined before, then, since $u^i \in BV\left(\Omega;\mathbb{R}^N\right)$, Proposition 5.3 yields

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon\left(u^i,1\right) = F\left(u^i,1\right). \tag{6.9}$$

Letting $i \to +\infty$ in (6.9), the conclusion follows by Lemma 6.1 and the lower semicontinuity of $\Gamma\left(L^1\right)$ - $\limsup_{\varepsilon\to 0^+} F_\varepsilon$. $\square$

7 Compactness and Convergence of Minimizers

Let us state an equicoercivity result for the approximating functionals defined in (3.8). The proof follows the one of Lemma 4.1 [23], we outline it here for the reader's convenience.

Lemma 7.1 *Let $(u_j, v_j) \in L^1\left(\Omega; \mathbb{R}^{N+1}\right)$ be such that*

$$\liminf_j \left(F_{\varepsilon_j}\left(u_j, v_j\right) + \int_\Omega |u_j|^q \, dx \right) < +\infty, \tag{7.1}$$

with $q > 1$. Then there exists a subsequence (u_{j_h}, v_{j_h}) and $u \in (GBV(\Omega))^N$ such that $(u_{j_h}, v_{j_h}) \to (u, 1)$ in $L^1\left(\Omega; \mathbb{R}^{N+1}\right)$.

Proof. Up to an increasing approximation argument using the Yosida's transforms, we may assume $\psi \in W^{1,\infty}([0,1])$.

Condition (7.1) and the bound $\|v_j\|_\infty \le 1$ imply that $v_j \to 1$ in $L^1(\Omega)$. Fix $i \in \mathbb{N}$, consider the sequence $\left(\psi\left(\frac{\Phi(v_j)}{\|\Phi'\|_\infty}\right) u_j^i\right) \subset W^{1,1}\left(\Omega; \mathbb{R}^N\right)$, where $u_j^i := \Psi_i\left(u_j\right)$ with Ψ_i the auxiliary functions defined in (2.4) and Φ is the one defined in (4.4). Let us show that $\left(\psi\left(\frac{\Phi(v_j)}{\|\Phi'\|_\infty}\right) u_j^i\right)$ is bounded in $BV\left(\Omega; \mathbb{R}^N\right)$. Indeed, by the lipschitz continuity and the monotonicity of Φ and ψ, Young's inequality yields

$$\begin{aligned}
&\int_\Omega \left|\psi\left(\frac{\Phi\left(v_j\right)}{\|\Phi'\|_\infty}\right) u_j^i\right| dx + \int_\Omega \left|\nabla\left(\psi\left(\frac{\Phi\left(v_j\right)}{\|\Phi'\|_\infty}\right) u_j^i\right)\right| dx \\
&\quad \le c\, i\mathcal{L}^n(\Omega) + \int_\Omega \psi(v_j)|\nabla u_j| \, dx + \frac{\|\psi'\|_\infty}{\|\Phi'\|_\infty} i \int_\Omega |\nabla\Phi(v_j)| \, dx \\
&\quad \le c\, i\big(1 + F_{\varepsilon_j}\left(u_j, v_j\right)\big),
\end{aligned}$$

denoting by c a positive constant independent of i.

By (7.1) and the convergence $v_j \to 1$ in $L^1(\Omega)$, by applying the BV Compactness Theorem and a diagonal argument we may suppose that, up to a subsequence not relabelled for convenience, for every $i \in \mathbb{N}$ there exists $w^i : \Omega \to \mathbb{R}^N$, with $\left\|w^i\right\|_\infty \le i$, such that for a.e. in Ω

$$\lim_j u_j^i(x) = w^i(x). \tag{7.2}$$

Let us prove that for a.e. x in Ω there exists $u : \Omega \to \mathbb{R}^N$ such that

$$\lim_i w^i(x) = u(x). \tag{7.3}$$

Indeed, let $x \in \Omega$ be such that (7.2) holds, then either $|u_j(x)| \to +\infty$ or there exist $w \in \mathbb{R}^N$ and $(u_{j_h}) \subseteq (u_j)$ such that $u_{j_h}(x) \to w$. In the first case $w^i(x) = 0$ for every $i \in \mathbb{N}$, and then (7.3) holds with $u(x) = 0$; while in the second case $u^i_{j_h}(x) \to w$ for every $i > |w|$ as $j \to +\infty$ and thus $u(x) = w$ by (7.2).
Let us prove the convergence of (u_j) to u in measure on Ω. Indeed, condition (7.1) yields

$$\mathcal{L}^n\big(\{x \in \Omega : |u_j(x)| > i\}\big) \leq c\, i^{-q},$$

thus for every $\varepsilon > 0$, since the decomposition

$$\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\} = \big\{x \in \Omega : |u^i_j(x) - u(x)| > \varepsilon\big\} \cup \\ \big(\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\} \cap \{x \in \Omega : |u_j(x)| > i\}\big),$$

we have

$$\mathcal{L}^n\big(\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\}\big) \leq \mathcal{L}^n\big(\big\{x \in \Omega : |u^i_j(x) - u(x)| > \varepsilon\big\}\big) + c\, i^{-q},$$

and the claimed convergence follows by (7.2) and (7.3).
Moreover, since $q > 1$, by (7.1) we have that the sequence (u_j) is equi-integrable and so the conclusion follows by Vitali's Theorem.

By (7.1) and by Remark 3.5 we deduce that $u \in (GBV(\Omega))^N$. □

We are now able to state the following result on the convergence of minimum problems.

Theorem 7.2 *For every $g \in L^q\left(\Omega; \mathbb{R}^N\right)$, $q > 1$, and every $\gamma > 0$, define*

$$m_\varepsilon := \inf\left\{F_\varepsilon\left(u, v\right) + \gamma \int_\Omega |u - g|^q \, dx : (u, v) \in L^1\left(\Omega; \mathbb{R}^{N+1}\right)\right\},$$

and let $(u_\varepsilon, v_\varepsilon)$ be asymptotically minimizing, i.e.,

$$F_\varepsilon\left(u_\varepsilon, v_\varepsilon\right) - m_\varepsilon \to 0.$$

Then every cluster point of (u_ε) is a solution of the minimum problem

$$m := \inf\left\{\mathcal{F}(u) + \gamma \int_\Omega |u - g|^q \, dx : u \in (GBV(\Omega))^N\right\},$$

and $m_\varepsilon \to m$ as $\varepsilon \to 0^+$.

8 Generalizations

In this section we discuss a generalization of Theorem 3.2, by considering spatially and directionally anisotropic singular perturbation terms in the definition of the approximating functionals.

With fixed $p > 1$, let $h : \Omega \times \mathbb{R}^n \to [0, +\infty)$ be a Borel integrand satisfying the following set of assumptions:

(h1) there exists three constants $c_3 \geq 0$ and c_4, $c_5 > 0$ such that

$$c_4 \left|\zeta\right| - c_3 \leq h(x,\zeta) \leq c_5 \left(\left|\zeta\right| + 1\right)$$

for every $(x,\zeta) \in \Omega \times \mathbb{R}^n$;

(h2) $h(x,\cdot)$ is locally Lipschitz for every $x \in \Omega$;

(h3) for every $x_o \in \Omega$ and for every $\eta > 0$ there exists $\delta > 0$, depending on x_o and η, such that

$$\left|\left(h^\infty\right)^p (x_o,\zeta) - \left(h^\infty\right)^p (x,\zeta)\right| \leq \eta \left(h^\infty\right)^p (x,\zeta)$$

for every $x \in \Omega$ with $|x - x_o| \leq \delta$ and for every $\zeta \in \mathbb{R}^n$;

(h4) for every $x_o \in \Omega$ and for every $\eta > 0$ there exists $\delta, L > 0$, depending on x_o and η, such that

$$\left|\left(h^\infty\right)^p (x,\zeta) - \frac{h^p (x,t\zeta)}{t^p}\right| \leq \eta \left(1 + \frac{h^p (x,t\zeta)}{t^p}\right)$$

for every $t > L$ and $x \in \Omega$ with $|x - x_o| \leq \delta$ and for every $\zeta \in \mathbb{R}^n$.

Let

$$h_\varepsilon(x,(u,v),(z,\zeta)) := \psi(v) f\left(x,u,z\right) + \frac{W(v)}{p'\varepsilon} + \frac{\varepsilon^{p-1}}{p} h^p\left(x,\zeta\right),$$

with f, ψ and W as in Section 3, $p' = \frac{p}{p-1}$. Then, consider the family of functionals $H_\varepsilon : L^1\left(\Omega;\mathbb{R}^N\right) \to [0,+\infty]$ defined by

$$H_\varepsilon\left(u,v\right) := \begin{cases} \displaystyle\int_\Omega h_\varepsilon\left(x,(u,v),\nabla(u,v)\right) dx & \text{if } (u,v) \in W^{1,1}\left(\Omega;\mathbb{R}^{N+1}\right) \\ & 0 \leq v \leq 1 \text{ a.e.,} \\ +\infty & \text{otherwise} \end{cases}.$$

The proof of the Γ-convergence for the family $\left(H_\varepsilon\right)_{\varepsilon>0}$ follows by exploiting the same arguments used to prove Theorem 3.2 with some minor changes.

Theorem 8.1 *Let $\left(H_\varepsilon\right)_{\varepsilon>0}$ be as above, then*

$$\Gamma\left(L^1\right)\text{-}\lim_{\varepsilon\to 0^+} H_\varepsilon\left(u,v;\Omega\right) = F(u,v;\Omega),$$

where F is the functional defined in (3.7) with surface energy density $K : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1} \to [0,+\infty)$ given by

$$K\left(x_o,a,b,\nu\right) := \inf\left\{\int_{Q_\nu}\left(\psi(v) f^\infty\left(x_o,u,\nabla u\right) + \frac{L}{p'} W(v) + \frac{1}{pL^{p-1}}\left(h^\infty\right)^p\left(x_o,\nabla v\right)\right) dy : (u,v) \in \mathcal{A}(a,b,\nu), L > 0\right\}.$$

Let us remark that Lemma 3.7 still holds true. Moreover, (a) of Lemma 3.8 is valid provided the function g appearing in the statement is substituted by

$$g_h\left(x_o,\nu,t\right) := \inf_{r\in[0,1]}\left\{\psi(r)t + \left(h^\infty(x_o,\nu)+h^\infty(x_o,-\nu)\right)\int_r^1 (W(s))^{\frac{1}{p'}}\,ds\right\}.$$

Eventually, assume $h^\infty(x,\cdot)$ to be isotropic for every $x\in\Omega$, then K can be characterized as in Lemma 3.8 (b) with the function g substituted by g_h defined above.

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