Existence of minimizers for a class of quasi-convex functionals with non-standard growth

Barbara Bianconi

Dipartimento di Matematica "U. Dini" - Universitá di Firenze
Viale Morgagni 67/A - 50123 FIRENZE (Italy)
bianconi@math.unifi.it

Matteo Focardi

Scuola Normale Superiore
P.zza dei Cavalieri 7 - 56126 PISA (Italy)
focardi@cibs.sns.it

Elvira Mascolo

Dipartimento di Matematica "U. Dini" - Universitá di Firenze
Viale Morgagni 67/A - 50123 FIRENZE (Italy)
mascolo@math.unifi.it

Abstract

We study the lower semicontinuity properties of non-autonomous variational integrals whose energy densities satisfy general growth conditions. We apply these results to solve Dirichlet's boundary value problems for such functionals.

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1 Introduction

In this paper we consider the variational approach to prove the existence of equilibrium solution in non-linear elasticity. We take into account only elastic materials possessing stored energy functions, for such materials the problem consists in finding a vector field $u : \Omega \to \mathbb{R}^N$, where Ω is a bounded open subset of $\mathbb{R}^n$, which is a solution of

$$m = \inf \{F(u, \Omega) : u = u_0 \text{ on } \partial\Omega\}, \tag{1.1}$$

where

$$F(u, \Omega) = \int_\Omega f(x, u(x), Du(x))\, dx, \tag{1.2}$$

with $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ a Carathéodory's integrand. As usual, we study this problem by using the Direct Methods of the Calculus of Variations, therefore the main questions is to determine conditions on f ensuring coercivity and sequential lower semicontinuity for $F(\cdot, \Omega)$ with respect to the same topology.

A suitable condition on stored energy function f, termed *quasi-convexity*, was introduced by Morrey in a fundamental paper in 1952: f is quasi-convex in z in Morrey' sense if for every $(x_0, s_0, z_0) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ and $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$ there holds

$$f(x_0, s_0, z_0)\mathcal{L}^n(\Omega) \leq \int_\Omega f(x_0, s_0, z_0 + D\varphi(y))dy,$$

denoting with $\mathcal{L}^n(\Omega)$ the n dimensional Lebesgue's measure of Ω. Morrey showed under strong regularity assumptions on f, that $F(\cdot, \Omega)$ is sequentially lower semicontinuous in the weak$*$ topology of $W^{1,\infty}(\Omega, \mathbb{R}^N)$ if and only if f is quasi-convex.

In the last years a great interest has raised around quasi-convex integrals of type (1.2), satisfying the so called (p, q) *growth* conditions, i.e.,

$$c_0(|z|^p - 1) \leq f(x, s, z) \leq c_1(1 + |z|^q) \tag{1.3}$$

with $1 \leq p \leq q$. Indeed, in non-linear elasticity, conditions $N = n = q$ and $q > p$ play a fundamental role in the study of cavitation since they allow discontinuous deformations of the elastic body.

When $p = q$, the case of *natural growth*, Acerbi-Fusco [2] and Marcellini [32] proved the sequential weak lower semicontinuity of $F(\cdot, \Omega)$ in the weak topology of $W^{1,p}(\Omega, \mathbb{R}^N)$.

Many authors have studied the lower semicontinuity and relaxation properties for functionals satisfying (1.3) in the Sobolev space setting obtaining sharp conditions on the mutual dependence of p and q. When $f = f(x, z) \geq 0$, and imposing further structure conditions on f, the lower semicontinuity inequality

$$\liminf_r F(u_r, \Omega) \geq F(u, \Omega) \tag{1.4}$$

has been established by Marcellini [33] along sequences (u_r) in $W^{1,q}(\Omega, \mathbb{R}^N)$ converging in the weak topology of $W^{1,p}(\Omega, \mathbb{R}^N)$ for $p > \frac{n}{n+1}q$. In the autonomous case $f = f(z)$ the lower semicontinuity inequality (1.4) was proven to hold true for $p > q - 1$ by Fonseca-Marcellini [18], and for $p > \frac{n-1}{n}q$ by Fonseca-Malý [17]. See also Malý [30] for related counterexamples and others for refinements (see [1],[5],[11],[22],[23],[27],[29],[31]).

However, this approach cannot be directly applied to establish existence results fo Dirichlet' s boundary value problems since the different topologies with respect to whom the functionals are coercive and lower semicontinuous.

Our aim is to study a particular class of integrands with (p, q) growth, those for which the stored energy function is controlled in terms of suitable convex functions. More precisely, we assume that f is a quasi-convex function satisfying the *non-standard growth* conditions

$$-c_1\Phi_1(|z|) - c_2\Phi_2(|s|) - c_3(x) \leq f(x, s, z) \leq g(x, s)\{1 + \Phi(|z|)\}, \tag{1.5}$$

where c_1,c_2 are positive constants; $c_3 \in L^1(\Omega)$; Φ,Φ_1 and Φ_2 are N-functions suitably related; and $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a positive Carathéodory's function.

With this general growth conditions Orlicz-Sobolev spaces provides the natural setting where to study the lower semicontinuity properties of functionals in (1.2). Indeed we prove that $F(\cdot, \Omega)$ is sequentially lower semicontinuous in the weak$*$ topology of the Orlicz-Sobolev space $W^1 L^\Phi\left(\Omega, \mathbb{R}^N\right)$, assuming that in (1.5) Φ satisfying a sub-homogeneity property at infinity called Δ_2 property, and Φ_1, Φ_2 satisfying some asymptotic conditions with respect to Φ.

Moreover, we estabilish an existence result for such class of integrands. Thus, we are able to study energy densities, depending on the full set of the variables with (p, q) growth and oscillating behaviour. Indeed, the coercivity and lower semicontinuity now holds in the Orlicz-Sobolev spaces setting.

Ball [4] was the first to set some variational problems in the framework of Orlicz-Sobolev spaces considering the poly-convex case. Recently, Focardi in [15] has proved the lower semicontinuity properties for functionals in (1.2) in Orlicz-Sobolev spaces for the integrands $f = f(z)$ satisfying the non-standard growth conditions (1.5) with $\Phi \in \Delta_2$. This result will be an ingredient to prove semicontinuity Theorem 3.1 below.

The case of N-functions Φ not sharing the Δ_2 property, corresponding roughly to exponential growth, has been considered by Focardi-Mascolo in [16] where a suitable semicontinuity property has been proved.

The plan of the paper is the following:

Section 2 is devoted to all the preliminary results about Orlicz-Sobolev spaces and the quasi-convex envelope of a function; moreover we recall the statement of a recent result of higher integrability for local minimizers of integral functionals with general growth conditions proved by Cianchi-Fusco [7]. In Section 3 we prove a semicontinuity result, obtained by suitable modifications of the arguments used in the natural growth case by Marcellini [32]. The proof is based on an approximation procedure of the stored energy function by a non-decreasing sequence of quasi-convex functions. Eventually, Section 4 is devoted to the proof of an existence theorem and to some application to non trivial examples.

2 Notations and Preliminaries

We denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^n$ and with $|\cdot|$ the usual Euclidean norm. Throughout all the paper Ω denotes an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. We denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$ and the notation $a.e.$ stands for almost everywhere with respect to Lebesgue measure. We use standard notations for spaces of classically differentiable functions, Lebesgue and Sobolev spaces. Given any function $u \in L^1(\Omega)$ the symbol $⨍_\Omega u \, dx$ stands for the average of u over Ω, i.e., $\frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega u \, dx$.

2.1 N-functions and Orlicz Spaces

For ease of reference we recall some definitions and known properties of N-functions and Orlicz spaces (see [26],[35]).

A convex function $\Phi : [0,+\infty[\to [0,+\infty[$ is called *N-function* if it satisfies the following conditions: $\Phi(0) = 0$, $\Phi(t) > 0$ for $t > 0$, and

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0, \quad \lim_{t\to +\infty} \frac{\Phi(t)}{t} = +\infty.$$

Such a function Φ has an integral representation of the form

$$\Phi(t) = \int_0^t p(s)ds$$

for every $t \geq 0$, where $p : [0,+\infty[\to [0,+\infty[$ is non-decreasing, right continuous and it satisfies the conditions: $p(0) = 0$, $p(s) > 0$ for $s > 0$, and

$$\lim_{s\to +\infty} p(s) = +\infty.$$

The function p is called the *right derivative* of Φ.
The notion of N-function can be relaxed, in the sense that only the behavior at infinity is important. Indeed, given any convex function $Q : [0,+\infty[\to [0,+\infty[$ satisfying

$$\lim_{t\to +\infty} \frac{Q(t)}{t} = +\infty,$$

there exists a N-function Φ and $t_0 \geq 0$ such that $Q(t) = \Phi(t)$ for every $t \geq t_0$. Such a function Q is called *principal part* of the N-function Φ, since this we will not distinguish the two concepts anylonger.

The set of N-functions can be endowed with a partial ordering, we say that Φ_1 *dominates* Φ_2, and we write $\Phi_2 \prec \Phi_1$, if there exist two constants $k, t_0 > 0$ such that for every $t \geq t_0$ it holds

$$\Phi_2(t) \leq \Phi_1(kt).$$

If, moreover, $\Phi_2 \prec \Phi_1$ and $\Phi_1 \prec \Phi_2$ we say that Φ_1 and Φ_2 are *equivalent*, while if Φ_1 dominates Φ_2 but Φ_1,Φ_2 are not equivalent we say that Φ_1 *dominates strictly* Φ_2, and we write $\Phi_2 \prec\prec \Phi_1$. We remark that if $\Phi_2 \prec\prec \Phi_1$ there exists a N-function Γ such that $\Gamma \circ \Phi_2 \prec \Phi_1$. For instance, Γ can be defined as the primitive of

$$q(s) = \begin{cases} \inf\left\{\frac{\Phi_1(\Phi_2^{-1}(t))}{t} \,:\, t > s\right\} & s \geq 1 \\ \\ q(1)s & 0 \leq s < 1. \end{cases} \tag{2.6}$$

Let Φ be a N-function, define the function

$$\widetilde{\Phi}(t) = \max_{s>0}\{st - \Phi(s)\},$$

$\widetilde{\Phi}$ is a N-function called the *complementary N-function* of Φ. By the very definition of $\widetilde{\Phi}$, the pair Φ, $\widetilde{\Phi}$ satisfies *Young's inequality*, i.e.,

$$st \leq \Phi(s) + \widetilde{\Phi}(t),$$

for every $s, t \geq 0$, with equality holding if $t = p(s)$ or $s = \widetilde{p}(t)$, where $\widetilde{p}$ is the right derivative of $\widetilde{\Phi}$.

In the sequel we will consider a special class of N-functions.

Definition 2.1 *We say that a N-function Φ satisfies the Δ_2 condition, and we write $\Phi \in \Delta_2$, if there exist two constants $k > 1$ and $t_0 \geq 0$ such that for every $t \geq t_0$ there holds*

$$\Phi(2t) \leq k\Phi(t).$$

By taking into account Proposition 2.1 of [10] we infer the following result.

Proposition 2.2 *Let Φ be a N-function, the following conditions are equivalent*

(i) $\Phi \in \Delta_2$;

(ii) there exists $r > 1$ and $t_0 \geq 0$ such that for every $t \geq t_0$ there holds

$$tp(t) \leq r\Phi(t);$$

(iii) there exists $r > 1$ and $t_0 \geq 0$ such that for every $t \geq t_0$ and $\lambda > 1$ there holds

$$\Phi(\lambda t) \leq \lambda^r \Phi(t).$$

Conditions $(ii), (iii)$ above hold true with the same $r > 1$, hence we write $\Phi \in \Delta_2^r$. It is easy to check that $\Phi(t) = t^r$ belongs to $\Delta_2^r, r > 1$, and that $\Phi(t) = t^r \log^\alpha(1+t)$, for $r \geq 1$ and $\alpha > 0$, is a N-function of class $\Delta_2^{r+\varepsilon}$ for every $\varepsilon > 0$. Moreover, the functions $\Phi(t) = \frac{t^r}{log(1+t)}$, with $r \geq \frac{3}{2}$, and $\Phi(t) = t^{a+b\sin(\sin(\log(t)))}$, with $a > 1 + b\sqrt{2}$, are N-functions of class Δ_2. The function $\Phi(t) = e^t - t - 1$ is a N-function which is not in class Δ_2 (for further properties of N-functions of class Δ_2 see [3],[26],[28],[35]).

Let Ω be an open bounded set of $\mathbb{R}^n$, the *Orlicz class* $K^\Phi(\Omega, \mathbb{R}^N)$ is the set of all (equivalence classes modulo equality a.e. in Ω of) measurable functions $u : \Omega \to \mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(|u|)dx < +\infty.$$

The *Orlicz space* $L^\Phi\left(\Omega, \mathbb{R}^N\right)$ is defined to be the linear hull of $K^\Phi(\Omega, \mathbb{R}^N)$. The functional $\|\cdot\|_{\Phi,\Omega} : L^\Phi(\Omega, \mathbb{R}^N) \to \mathbb{R}$, defined by

$$\|u\|_{\Phi,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1\right\},$$

is a norm, called *Luxemburg norm*, and $L^{\Phi}(\Omega,\mathbb{R}^N)$ is a Banach space if endowed with it. In the sequel we will denote by $s - L^{\Phi}\left(\Omega,\mathbb{R}^N\right)$ the norm convergence in $L^{\Phi}\left(\Omega,\mathbb{R}^N\right)$.

The closure of $C_0^{\infty}(\Omega,\mathbb{R}^N)$ in the norm topology of $L^{\Phi}\left(\Omega,\mathbb{R}^N\right)$ is denoted by $E^{\Phi}(\Omega,\mathbb{R}^N)$, the inclusions $E^{\Phi}(\Omega,\mathbb{R}^N) \subseteq K^{\Phi}(\Omega,\mathbb{R}^N) \subseteq L^{\Phi}\left(\Omega,\mathbb{R}^N\right)$ are trivial with equalities holding if and only if $\Phi \in \Delta_2$.

The following result on the integral convergence in Orlicz spaces has been proved in [16].

Proposition 2.3 *Let $u \in E^{\Phi}\left(\Omega,\mathbb{R}^N\right)$, then for every $(u_r) \to u$ in $s - L^{\Phi}\left(\Omega,\mathbb{R}^N\right)$ and for every $\lambda > 0$ there holds*

$$\lim_r \int_{\Omega} \Phi(\lambda |u_r|)dx = \int_{\Omega} \Phi(\lambda |u|)dx.$$

The partial ordering introduced in the set of N-functions induces topological embeddings among Orlicz spaces.

Proposition 2.4 *Let Φ_1,Φ_2 be two N-functions such that $\Phi_2 \prec \Phi_1$, then the embedding*

$$L^{\Phi_1}\left(\Omega,\mathbb{R}^N\right) \hookrightarrow L^{\Phi_2}\left(\Omega,\mathbb{R}^N\right)$$

is continuous. Moreover, if $\Phi_2 \prec\prec \Phi_1$ then

$$L^{\Phi_1}\left(\Omega,\mathbb{R}^N\right) \hookrightarrow E^{\Phi_2}\left(\Omega,\mathbb{R}^N\right).$$

The *Orlicz-Sobolev space* $W^1L^{\Phi}\left(\Omega,\mathbb{R}^N\right)$ is defined to be the set of all functions in $L^{\Phi}\left(\Omega,\mathbb{R}^N\right)$ whose first order distributional derivatives are in $L^{\Phi}\left(\Omega,\mathbb{R}^N\right)$. It is a Banach space if endowed with the norm

$$\|u\|_{1,\Phi,\Omega} = \|u\|_{\Phi,\Omega} + \|\nabla u\|_{\Phi,\Omega}.$$

As in the case of ordinary Sobolev spaces $W_0^1E^{\Phi}\left(\Omega,\mathbb{R}^N\right)$ is taken to be the closure of $C_0^{\infty}(\Omega,\mathbb{R}^N)$ in the norm of $W^1L^{\Phi}\left(\Omega,\mathbb{R}^N\right)$.

Let Φ be a given N-function, we may suppose that

$$\int_0^1 \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{n}}}ds < +\infty,$$

replacing, if necessary Φ by an equivalent N-function. Assume, moreover, that

$$\int_1^{+\infty} \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{n}}}ds = +\infty, \tag{2.7}$$

then we define the *Sobolev's conjugate function* Φ_* of Φ by

$$(\Phi_*)^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{n}}}ds,$$

for every $t \geq 0$. The following compact embedding theorem generalizes to Orlicz-Sobolev spaces Rellich-Kondrakov's one (see ch. VII of [3],[6] and sect. 7.4 of [28]).

Theorem 2.5 *Let Φ be a N-function.*

(i) If (2.7) holds, the embedding

$$W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)\hookrightarrow L^{\Phi_*}(\Omega,\mathbb{R}^N)$$

is continuous. Moreover, the embedding

$$W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)\hookrightarrow L^{\Phi_1}(\Omega,\mathbb{R}^N)$$

is compact for every N-function $\Phi_1 \prec\prec \Phi_*$.

(ii) If (2.7) does not hold, the embedding

$$W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)\hookrightarrow C^0(\overline{\Omega},\mathbb{R}^N)$$

is compact.

We now introduce the weak$*$ convergence in $L^\Phi\left(\Omega,\mathbb{R}^N\right)$, denoted by $*w-L^\Phi\left(\Omega,\mathbb{R}^N\right)$. Since the Orlicz space $L^\Phi\left(\Omega,\mathbb{R}^N\right)$ is isometrically isomorphic to the dual space of $E^{\widetilde{\Phi}}\left(\Omega,\mathbb{R}^N\right)$, a sequence $u_r \to u\ \ *w-L^\Phi\left(\Omega,\mathbb{R}^N\right)$ if and only if for every $v\in E^{\widetilde{\Phi}}\left(\Omega,\mathbb{R}^N\right)$ there holds

$$\lim_r \int_\Omega \langle u_r, v\rangle dx = \int_\Omega \langle u, v\rangle dx.$$

Thus, by means of the Hahn-Banach theorem, we are able to characterize the weak$*$ convergence in $W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$, denoted by $*w-W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$, that is: $u_r\to u\ \ *w-W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$ if and only if (u_r) and (D_iu_r), $1\le i\le n$, converge to u and $D_iu\ \ *w-L^\Phi\left(\Omega,\mathbb{R}^N\right)$, respectively.

Following the notations of [13],[25],[36] $W_0^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$ denotes the weak$*$ closure of $W_0^1E^\Phi\left(\Omega,\mathbb{R}^N\right)$ in $W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$, hence the inclusion $W_0^1E^\Phi\left(\Omega,\mathbb{R}^N\right)\subseteq W_0^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$ is trivial. By taking into account Corollary 1.10 of [25], the intersection of $W_0^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$ with $\prod_{i=1}^{n+1}E^\Phi(\Omega,\mathbb{R}^N)$ is exactly $W_0^1E^\Phi\left(\Omega,\mathbb{R}^N\right)$, therefore we can infer, when $\Phi\in\Delta_2$, the equality $W_0^1E^\Phi\left(\Omega,\mathbb{R}^N\right)=W_0^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$.

2.2 Quasi-convex Envelope

In this section we state the main properties of the quasi-convex envelope of a given measurable function satisfying non-standard growth conditions. The case of natural growth has been established by Dacorogna [8] (see also [24]). The generalization to the non-standard case is not difficult, we report only the statements of the results without the arguments of their proofs.

Let $h:\mathbb{R}^{Nn}\to\mathbb{R}^+$ be a measurable function, define

$$\gamma_\Omega(z)=\inf\left\{ \fint_\Omega h(z+D\varphi(y))dy:\quad \varphi\in C_0^\infty(\Omega,\mathbb{R}^N)\right\},$$

by taking into account Proposition 5.3 of [24] we have that if Ω_1 and Ω_2 are bounded open sets of $\mathbb{R}^n$ then $\gamma_{\Omega_1} \equiv \gamma_{\Omega_2} \equiv \gamma_\Omega$, so that we can drop the dependence on Ω in the definition of γ_Ω and denote it just by γ.

Moreover, assume that

(i) there exists a N-function $\Phi \in \Delta_2$ such that for every $z \in \mathbb{R}^{Nn}$

$$c_0\Phi(|z|) \leq h(z) \leq c_1(1+\Phi(|z|)); \tag{2.8}$$

(ii) there exists $w \in C^0(\mathbb{R}^+, \mathbb{R}^+)$, with $w(0)=0$, such that

$$|h(z)-h(w)| \leq c(1+\Phi(|z|+|w|+1))w(|z-w|). \tag{2.9}$$

By taking into account the growth condition (2.8) and the continuity assumption (2.9) it is easy to check that

$$\gamma(z) = \inf\left\{ -\!\!\!\!\!\!\int_\Omega h(z+D\varphi(y))dy : \quad \varphi \in W_0^1E^\Phi\left(\Omega, \mathbb{R}^N\right)\right\}.$$

Moreover, the same assumptions imply that γ is a continuous function.

Define Q_ch, the *quasi-convex envelope* of h, to be

$$Q_ch = \sup\{\phi : \ \phi \leq h \text{ quasi-convex}\},$$

then, arguing as in the case of natural growth we can prove the following characterization of Q_ch.

Theorem 2.6 *Let $h : \mathbb{R}^{Nn} \to \mathbb{R}^+$ be satisfying (2.8) and (2.9), then $\gamma \equiv Q_ch$.*

2.3 A Regularity Result

In the proof of Theorem 3.1 we will need a regularity result for local minimizers of functionals with non-standard growth. In particular, we will make use of higher integrability properties recently proved by Cianchi-Fusco [7].

Let $h : \mathbb{R}^{Nn} \to \mathbb{R}$ be a continuous function such that there exists a N-function $\Phi \in \Delta_2$ for which

$$c_0\Phi(|z|) \leq h(z) \leq c_1\left(1+\Phi(|z|)\right)$$

for every $z \in \mathbb{R}^{Nn}$.

We say that u is a *quasi-minimum*, or equivalently *Q-minimum*, for the functional $H : W^1L^\Phi\left(\Omega, \mathbb{R}^N\right) \to \mathbb{R}$ defined by

$$H(v,\Omega) = \int_\Omega h(Dv(x))dx,$$

if there exists a constant Q such that for every open set $\Omega_0 \subset\subset \Omega$ there hold

$$\int_{\Omega_0} \Phi(|Du|)dx < +\infty,$$

and

$$H(u,\Omega_0) \le QH(u+\psi,\Omega_0),$$

for every weakly differentiable function $\psi : \Omega_0 \to \mathbb{R}^N$ with compact support and such that $\int_{\Omega_0} \Phi(|D\psi|)dx < +\infty$.

The following result holds true (see Theorem 1.1 of [7]).

Theorem 2.7 *Let u be a Q-minimum of H with h as above, then for every open subset $\Omega_0 \subset\subset \Omega$ there exists $\delta > 0$ depending on n, Φ, Ω_0, $dist(\Omega_0, \partial\Omega)$ and $\int_{\Omega_0} \Phi(|Du|)dx$ such that*

$$\int_{\Omega_0} \Phi(|Du|)\left(\frac{\Phi(|Du|)}{|Du|}\right)^{\delta} dx < +\infty. \tag{2.10}$$

We remark that by using the same arguments of [7], it is possible to prove that given a sequence (u_k) of Q-minima for $H(\cdot,\Omega)$ such that

$$\sup_k \int_{\Omega_0} \Phi(|Du_k|)dx < +\infty,$$

with fixed $\Omega_0 \subset\subset \Omega$, there exist a positive constant δ independent from k such that

$$\sup_k \int_{\Omega_0} \Phi(|Du_k|)\left(\frac{\Phi(|Du_k|)}{|Du_k|}\right)^{\delta} dx < +\infty.$$

3 A Semicontinuity Result

Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ be a *Carathéodory's* function, i.e., f is measurable with respect to x for every $(s,z) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$ and continuous with respect to (s,z) a.e. in Ω, satisfying the growth condition

$$-c_1\Phi_1(|z|) - c_2\Phi_2(|s|) - c_3(x) \le f(x,s,z) \le g(x,s)\left(1+\Phi(|z|)\right), \tag{3.11}$$

where c_1, c_2 are positive constants, $c_3 \in L^1(\Omega)$, Φ is a N-function of class Δ_2^r, Φ_1 and Φ_2 are N-functions such that $\Phi_1 \prec\prec \Phi$ and either $\Phi_2 \prec\prec \Phi_*$ if (2.7) holds or Φ_2 is arbitrarily chosen otherwise and $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a positive Carathéodory's function.

Assume that f is *quasi-convex* with respect to z, i.e., for every $(x_0,s_0,z_0) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ and $\varphi \in C_0^\infty(\Omega,\mathbb{R}^N)$ there holds

$$f(x_0,s_0,z_0) \le \fint_\Omega f(x_0,s_0,z_0 + D\varphi(y))dy. \tag{3.12}$$

The following semicontinuity theorem holds true.

Theorem 3.1 *Let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ *be as above, then the functional* $F : W^1L^\Phi\left(\Omega, \mathbb{R}^N\right) \to \mathbb{R}$ *defined by*

$$F(u, \Omega) = \int_\Omega f(x, u(x), Du(x))dx$$

is sequentially lower semicontinuous with respect to $*w - W^1L^\Phi\left(\Omega, \mathbb{R}^N\right)$.

The proof of the Theorem 3.1 is based on the following approximation result.

Theorem 3.2 *Let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ *be as above, assume that (3.11) is substituted by*

$$c_0\Phi(|z|) \le f(x, s, z) \le g(x, s)(1 + \Phi(|z|)) \tag{3.13}$$

where c_0 *is a positive constant.*

Then there exists a sequence (f_k) *of Carathéodory's functions quasi-convex with respect to* z *such that* $f_k : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ *satisfies*

$$c_0\Phi(|z|) \le f_k(x, s, z) \le k(1 + \Phi(|z|)) \tag{3.14}$$

$$f_k(x, s, z) = c_0\Phi(|z|) \quad |s| \ge k, \quad |z| \ge k \tag{3.15}$$

$$f_k \le f_{k+1} \quad \sup_k f_k = f \tag{3.16}$$

Proof. In the first part of the proof the arguments are similar to those of Theorem 1.2 of [32]. However, for the sake of completness we outline the main ideas.

First we perform a truncation with respect to variables (x, s). For $i \in \mathbb{N}$, let $\phi_i : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\phi_i(t) = 1 \quad 0 \le t \le i - 1, \quad \phi_i(t) = 0 \quad t \ge i,$$

set

$$\eta_i(x, s) = \begin{cases} \phi_i(|s|) & g(x, s) \le i \\ \frac{i\phi_i(|s|)}{g(x,s)} & g(x, s) > i \end{cases}$$

and define

$$g_i(x, s, z) = \eta_i(x, s)f(x, s, z) + (1 - \eta_i(x, s))c_0\Phi(|z|).$$

The functions g_i are Carathéodory's functions quasi-convex with respect to z such that

$$\begin{gathered} c_0\Phi(|z|) \le g_i(x, s, z) \le (i + c_0)(1 + \Phi(|z|)), \\ g_i(x, s, z) = c_0\Phi(|z|) \qquad |s| \ge i, \\ g_i(x, s, z) = f(x, s, z) \qquad i > g(x, s) + |s| + 1, \\ \lim_i g_i(x, u, z) = \sup_i g_i(x, s, z) = f(x, s, z). \end{gathered} \tag{3.17}$$

Now, we perform a truncation with respect to z. Define the Carathéodory's function

$$g_{im}(x,s,z) = \phi_m(|z|)g_i(x,s,z) + (1-\phi_m(|z|))c_0\Phi(|z|). \tag{3.18}$$

The functions g_{im} are not quasi-convex with respect to z, therefore we consider their quasi-convex envelopes G_{im}. Hence, the quasi-convexity of G_{im} and condition $(3.17)_1$ imply

$$c_0\Phi(|z|) \leq G_{im}(x,s,z) \leq (i+c_0)(1+\Phi(|z|)), \tag{3.19}$$

moreover, $(3.17)_2$ yields

$$G_{im}(x,s,z) = c_0\Phi(|z|) \tag{3.20}$$

for $|z| \geq m$, $|s| \geq i$. By taking into account (3.19) and (3.20), we may apply Theorem 2.6 and find the following integral representation formula for G_{im}

$$G_{im}(x,s,z) = \inf\left\{ \fint_\Omega g_{im}(x,s,z+D\varphi(y))dy : \ \varphi \in W_0^1E^\Phi\left(\Omega,\mathbb{R}^N\right)\right\}. \tag{3.21}$$

The more significant and technically difficult part of the proof is worked out in Lemma 3.3 below in which we prove that $(G_{im})_{m\in\mathbb{N}}$ converges to g_i pointwise on Ω. Assuming Lemma 3.3 to hold true, we can conclude the proof of Theorem 3.2. Indeed, for $k \geq 2 + c_0$ define

$$f_k(x,s,z) = \max\{G_{im}(x,s,z) : i+m \leq k\},$$

then f_k satisfies (3.14), (3.15) and (3.16). $\square$

Lemma 3.3 *For every $i \in \mathbb{N}$ the sequence $(G_{im})_{m\in\mathbb{N}}$ converges to g_i pointwise on Ω.*

Proof. With fixed $i \in \mathbb{N}$ let $(x_0,s_0,z_0) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$. First notice that $(G_{im}(x_0,s_0,z_0))_{m\in\mathbb{N}}$ is a non decreasing sequence and that for every $m \in \mathbb{N}$ inequality $G_{im}(x_0,s_0,z_0) \leq g_i(x_0,s_0,z_0)$ is trivial by the very definition of G_{im} (see (3.18) and (3.21)).

By the representation formula (3.21), for fixed $m \in \mathbb{N}$ there exists $w_m \in W_0^1E^\Phi\left(\Omega,\mathbb{R}^N\right)$ such that

$$\fint_\Omega g_{im}(x_0,s_0,z_0+Dw_m)dy \leq G_{im}(x_0,s_0,z_0) + \frac{1}{m}.$$

Consider the functional

$$v \in W_0^{1,1}(\Omega,\mathbb{R}^N) \to \fint_\Omega g_{im}(x_0,s_0,z_0+Dv)dy,$$

by taking into account Ekeland's Variational Principle (see [14],[24]) there exists $u_m \in W_0^{1,1}(\Omega,\mathbb{R}^N)$ such that

$$\fint_\Omega g_{im}(x_0,s_0,z_0+Du_m)dy \leq G_{im}(x_0,s_0,z_0) + \frac{1}{m}, \tag{3.22}$$

and such that for every $\varphi \in W_0^{1,1}(\Omega, \mathbb{R}^N)$ there holds

$$\begin{aligned}&\fint_\Omega g_{im}(x_0, s_0, z_0 + Du_m) dy \\ &\qquad \leq \fint_\Omega g_{im}(x_0, s_0, z_0 + D\varphi) dy + \frac{1}{m} \int_\Omega |Du_m - D\varphi| dy. \qquad (3.23)\end{aligned}$$

Let us prove that (u_m) is a sequence of Q-minima, with Q independent on m, of the functional

$$v \in W_0^1 E^\Phi \left(\Omega, \mathbb{R}^N\right) \to \int_\Omega (1 + \Phi(|Dv|)) dy,$$

i.e., there exists Q such that for every $\varphi \in W_0^1 E^\Phi \left(\Omega, \mathbb{R}^N\right)$, denoted $\Sigma = supp(\varphi - u_m)$, there holds

$$\int_\Sigma (1 + \Phi(|Du_m|)) \, dy \leq Q \int_\Sigma (1 + \Phi(|D\varphi|)) \, dy. \qquad (3.24)$$

First notice that formulas $(3.17)_1$,(3.19) and (3.23) yield for m sufficiently large

$$\begin{aligned}&c_0 \int_\Sigma \Phi(|z_0 + Du_m|) dy \\ &\qquad \leq (i + c_0) \int_\Sigma (1 + \Phi(|z_0 + D\varphi|)) \, dy + \int_\Sigma |Du_m - D\varphi| dy. \qquad (3.25)\end{aligned}$$

Without loss of generality we may assume $c_0 \leq 1$, hence Young's inequality yields

$$|Du_m - D\varphi| \leq \frac{c_0}{2^r} \Phi(|Du_m|) + \frac{c_0}{2^r} \Phi(|D\varphi|) + \widetilde{\Phi}\left(\frac{2^r}{c_0}\right), \qquad (3.26)$$

then combining (3.25),(3.26) and assumption $\Phi \in \Delta_2^r$ we get

$$\begin{aligned}&\int_\Sigma \Phi(|Du_m|) dy \\ &\qquad \leq \left(\frac{i}{c_0} + 2\right) \int_\Sigma \Phi(|D\varphi|) dy + 4^r \left(\Phi(|z_0|) + \widetilde{\Phi}\left(\frac{2^r}{c_0}\right) + \frac{i}{c_o} + 1\right) \mathcal{L}^n(\Omega),\end{aligned}$$

which implies (3.24) with $Q = Q(c_0, r, i, \Phi(|z_0|))$.

Let $\Omega_0 \subset\subset \Omega$ be fixed, formulas (3.19) and (3.22) give

$$c_0 \fint_\Omega \Phi(|z_0 + Du_m|) dy \leq (i + c_0) \left(1 + \Phi(|z_0|)\right),$$

hence, setting $\Omega_m = \{y \in \Omega_0 : |z_0 + Du_m(y)| > m - 1\}$, we get

$$\Phi(m-1) \mathcal{L}^n(\Omega_m) \leq \left(\frac{i}{c_0} + 1\right) \left(1 + \Phi(|z_0|)\right) \mathcal{L}^n(\Omega),$$

and then $\mathcal{L}^n(\Omega_m) \to 0$ for $m \to +\infty$.

Moreover, by (3.19) and (3.22) it follows

$$\sup_m \int_\Omega \Phi(|Du_m|)dy < +\infty, \tag{3.27}$$

hence by taking into account the regularity result of Theorem 2.7, we have that there exists $\delta > 0$, independent on m, such that

$$\sup_m \int_{\Omega_0} \Phi(|Du_m|)\left(\frac{\Phi(|Du_m|)}{|Du_m|}\right)^\delta dy < +\infty. \tag{3.28}$$

Define $\Phi_\delta(t) = \Phi(t)\left[\frac{\Phi(t)}{t}\right]^\delta$ and let $\Gamma_\delta(s) = \Phi_\delta(\Phi^{-1}(s))$, the function

$$\Psi_\delta(t) = \int_0^t \frac{\Gamma_\delta(s)}{s} ds$$

is a N-function of class Δ_2 such that $\Psi_\delta(t) \le \Gamma_\delta(t)$ for every $t \in [0, +\infty)$. Hence, by taking into account Young's inequality we get

$$\int_{\Omega_m} \Phi(|Du_m|)dy \le \|\Phi(|Du_m|)\|_{L^{\Psi_\delta}(\Omega_m)} \|1_{\Omega_m}\|_{L^{\tilde{\Psi}_\delta}(\Omega_m)}.$$

Notice that

$$\|1_{\Omega_m}\|_{L^{\tilde{\Psi}_\delta}(\Omega_m)} = \mathcal{L}^n(\Omega_m)\Psi_\delta{}^{-1}\left(\frac{1}{\mathcal{L}^n(\Omega_m)}\right),$$

and thus we infer $\|1_{\Omega_m}\|_{L^{\tilde{\Psi}_\delta}(\Omega_m)} \to 0$ for $m \to +\infty$. Moreover, the very definition of the Orlicz norm yields

$$\|\Phi(|Du_m|)\|_{L^{\Psi_\delta}(\Omega_m)} \le 1 + \int_{\Omega_m} \Gamma_\delta(\Phi(|Du_m|))dy,$$

and therefore by (3.28) we can conclude

$$\lim_m \int_{\Omega_m} \Phi(|Du_m|)dy = 0. \tag{3.29}$$

By the generalization of Poincaré inequality to N-functions (see Lemma 5.7 of [24]) and (3.27) the sequence $(u_m) \subset W_0^1 E^\Phi\left(\Omega, \mathbb{R}^N\right)$ has equibounded norms in $W^1 L^\Phi\left(\Omega, \mathbb{R}^N\right)$. Therefore, there exists a subsequence, still denoted by (u_m), converging to a function $u \in W_0^1 E^\Phi\left(\Omega, \mathbb{R}^N\right)$ in $*w - W^1 L^\Phi\left(\Omega, \mathbb{R}^N\right)$.

Consider inequality (3.22), then the choice of Ω_m and formulas $(3.17)_1$, (3.18) yield

$$\begin{aligned} G_{im}(x_0, s_0, z_0) + \frac{1}{m} &\ge \frac{1}{\mathcal{L}^n(\Omega)} \int_{\Omega_0 \setminus \Omega_m} g_i(x_0, s_0, z_0 + Du_m)dy \\ &\ge \frac{1}{\mathcal{L}^n(\Omega)} \int_{\Omega_0} g_i(x_0, s_0, z_0 + Du_m)dy - \frac{i + c_0}{\mathcal{L}^n(\Omega)} \int_{\Omega_m} (1 + \Phi(|z_0 + Du_m|))\, dy. \end{aligned} \tag{3.30}$$

Hence, by taking into account the lower semicontinuity result in the autonomous case $f = f(z)$ (see Theorem 3.1 of [15]) and formula (3.29), passing to the limit in (3.30) we have

$$\begin{aligned}
&\lim_m G_{im}(x_0, s_0, z_0) \\
&\qquad \geq \fint_\Omega g_i(x_0, s_0, z_0 + Du)dy - \frac{1}{\mathcal{L}^n(\Omega)} \int_{\Omega \setminus \Omega_0} g_i(x_0, s_0, z_0 + Du)dy \\
&\qquad \geq g_i(x_0, s_0, z_0) - \frac{1}{\mathcal{L}^n(\Omega)} \int_{\Omega \setminus \Omega_0} g_i(x_0, s_0, z_0 + Du)dy, \qquad (3.31)
\end{aligned}$$

where the last inequality follows by the quasi-convexity of g_i. Indeed, condition $(3.17)_1$ assures the continuity of the functional

$$v \in W^1 L^\Phi \left(\Omega, \mathbb{R}^N\right) \to \int_\Omega g_i(x, v, Dv)dx$$

in the strong topology of $W^1 L^\Phi \left(\Omega, \mathbb{R}^N\right)$, thus quasi-convexity inequality for g_i can be extended also to test functions in $W_0^1 E^\Phi \left(\Omega, \mathbb{R}^N\right)$.

Eventually, by letting $\mathcal{L}^n(\Omega \setminus \Omega_0) \to 0$ in (3.31) we get

$$\lim_m G_{im}(x_0, s_0, z_0) \geq g_i(x_0, s_0, z_0),$$

which concludes the proof of the lemma. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1 With fixed $\varepsilon > 0$, define

$$f_\varepsilon(x, s, z) = f(x, s, z) + c_2 \Phi_2(|s|) + c_3(x) + \varepsilon \Phi(|z|) + c_\varepsilon,$$

where $c_\varepsilon > 0$ is chosen such that

$$f_\varepsilon(x, s, z) \geq \frac{\varepsilon}{2} \Phi(|z|),$$

for every $(x, s, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$. The existence of such c_ε follows by the growth conditions (3.11) of f and the assumption $\Phi_1 \prec\prec \Phi$.

Let $(f_{\varepsilon,k})_k$ be the sequence of quasi-convex functions provided by Theorem 3.2, then arguing like in Lemma 4.3 of [32] the functionals $F_\varepsilon^k : W^1 L^\Phi \left(\Omega, \mathbb{R}^N\right) \to \mathbb{R}$ defined by

$$F_\varepsilon^k(u, \Omega) = \int_\Omega f_{\varepsilon,k}\left(x, u, Du\right) dx$$

are sequentially lower semicontinuous in $*w - W^1 L^\Phi \left(\Omega, \mathbb{R}^N\right)$. Moreover, consider the functional $F_\varepsilon : W^1 L^\Phi \left(\Omega, \mathbb{R}^N\right) \to \mathbb{R}$ defined by

$$F_\varepsilon(u, \Omega) = \int_\Omega f_\varepsilon\left(x, u, Du\right) dx,$$

since $F_\varepsilon(\cdot,\Omega) = \sup_k F_\varepsilon^k(\cdot,\Omega)$ we infer the lower semicontinuity of $F_\varepsilon(\cdot,\Omega)$ in $*w-W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$.

Let (u_r) be a sequence weakly$*$ converging to u in $W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$, since the choice of Φ_2, Proposition 2.4 and the embedding Theorem 2.5 yield the convergence of (u_r) to u in the norm topology of $L^{\Phi_2}(\Omega,\mathbb{R}^N)$ and $u \in E^{\Phi_2}(\Omega,\mathbb{R}^N)$. Hence, by applying Proposition 2.3 we infer

$$\lim_r \int_\Omega \Phi_2(|u_r|)dx = \int_\Omega \Phi_2(|u|)dx,$$

and since the $*w-W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$ convergence of (u_r) to u yields

$$\sup_r \int_\Omega \Phi(|Du_r|)dx \le M,$$

we get

$$\begin{aligned}
&\liminf_r \int_\Omega f(x,u_r,Du_r)dx \\
&\quad\ge \liminf_r \int_\Omega f_\varepsilon(x,u_r,Du_r)dx - \int_\Omega \left(c_2\Phi_2(|u|) + c_3(x) + c_\varepsilon\right)dx - \varepsilon M \\
&\quad\ge \int_\Omega f_\varepsilon(x,u,Du)dx - \int_\Omega \left(c_2\Phi_2(|u|) + c_3(x) + c_\varepsilon\right)dx - \varepsilon M \\
&\quad\ge \int_\Omega f(x,u,Du)dx - \varepsilon M,
\end{aligned}$$

which concludes the proof as $\varepsilon \to 0$. $\square$

4 Existence and Applications

Let us first recall few facts about trace operator in Orlicz-Sobolev spaces. Let Ω be such that $\partial\Omega$ is Lipschitz regular, in this case one can define a trace operator from $W^1E^\Phi\left(\Omega,\mathbb{R}^N\right)$ to $E^\Phi\left(\partial\Omega,\mathbb{R}^N\right)$ whose kernel is exactly $W_0^1E^\Phi\left(\Omega,\mathbb{R}^N\right)$. Note that in case $\Phi \in \Delta_2$ we have $W^1L^\Phi\left(\Omega,\mathbb{R}^N\right) \equiv W^1E^\Phi\left(\Omega,\mathbb{R}^N\right)$, $L^\Phi\left(\partial\Omega,\mathbb{R}^N\right) \equiv E^\Phi\left(\partial\Omega,\mathbb{R}^N\right)$ and $W_0^1E^\Phi\left(\Omega,\mathbb{R}^N\right)$ is $*w-W^1L^\Phi\left(\Omega,\mathbb{R}^N\right)$ closed (see [20],[28]).

The last statement enable us to consider Dirichlet's boundary values problems in Orlicz-Sobolev spaces in case $\Phi \in \Delta_2$.

In the vectorial setting, as pointed out in [24], the most natural growth conditions to impose on the stored energy densities f are the ones given below, i.e.,

$$-c_1\Phi_1(|z|) - b(x)\Phi_2(|s|) - c_3(x) \le f(x,s,z) \le c_2\Phi(|z|) + b(x)\Phi_2(|s|) + c_3(x), \tag{4.32}$$

$$f(x,s,z) \ge \widetilde{f}(z) - b(x)\Phi_2(|s|) - c_3(x) \tag{4.33}$$

where in (4.32) $c_i > 0$ for $i = 1, 2$, $c_3 \in L^1(\Omega)$, Φ, Φ_1 and Φ_2 are N-functions such that $\Phi \in \Delta_2$ and $\Phi_i \prec\prec \Phi$ for $i = 1, 2$. Moreover in (4.33) $b \in E^{\widetilde{\Gamma}}(\Omega)$ with Γ defined by (2.6) is such that $\Gamma \circ \Phi_2 \prec \Phi$ and $\widetilde{f} : \mathbb{R}^{Nn} \to \mathbb{R}$ is a *strictly quasi-convex* function in $z = 0$, i.e., $\widetilde{f}$ is a continuous function such that

$$c_4 \int_\Omega \Phi(|D\varphi|)dx + \widetilde{f}(0)\mathcal{L}^n(\Omega) \leq \int_\Omega \widetilde{f}(D\varphi)dx \tag{4.34}$$

for every $\varphi \in W_0^1 E^\Phi \left(\Omega, \mathbb{R}^N\right)$ with $c_4 > 0$. Without loss of generality we may also assume $\widetilde{f}(0) = 0$.

Let us state and prove the following existence result (for related results in the poly-convex case see [4]).

Theorem 4.1 *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to [0, +\infty)$ be a Carathéodory's function, quasi-convex with respect to z satisfying (4.32),(4.33).*

Let $u_0 \in W^1 L^\Phi \left(\Omega, \mathbb{R}^N\right)$, consider the Dirichlet's class

$$V_0 = u_0 + W_0^1 E^\Phi \left(\Omega, \mathbb{R}^N\right),$$

then the variational problem

$$\inf\{F(u, \Omega) : u \in V_0\}$$

has solution.

Proof. Let $u \in V_0$ and set $\varphi = (u - u_0)$, then assumption $\Phi \in \Delta_2$ and formulas (4.33),(4.34) yield

$$\begin{aligned}\int_\Omega \Phi(|Du|)dx &\leq c \int_\Omega \Phi(|D\varphi|)dx + c \int_\Omega \Phi(|Du_0|)dx \\ &\leq c \int_\Omega \widetilde{f}(D\varphi)dx + c \int_\Omega \Phi(|Du_0|)dx \\ &\leq c \int_\Omega f(x, u, D\varphi)dx + c \int_\Omega \left(b(x)\Phi_2(|u|) + c_3(x) + \Phi(|Du_0|)\right) dx,\end{aligned} \tag{4.35}$$

denoting with c a generic constant which may varies from line to line. By taking into account Proposition 3.2 of [15] we have

$$\begin{aligned}&\int_\Omega f(x, u, D\varphi)dx \\ &\quad = \int_\Omega \left(f(x, u, D\varphi) - f(x, u, Du)\right) dx + \int_\Omega f(x, u, Du)dx \\ &\quad \leq c \int_\Omega \big(p(|\theta|) + p(|Du|) + p(|Du_0|)\big)|Du_0|dx + \int_\Omega f(x, u, Du)dx,\end{aligned} \tag{4.36}$$

where $\theta(x) = \Phi^{-1}\big(c_3(x) + b(x)\Phi_2(|u(x)|)\big)$ and recall that p is the right derivative of Φ. Notice that since $b \in E^{\widetilde{\Gamma}}(\Omega)$ then by Young's inequality

$$\int_\Omega b(x)\Phi_2(|u|)dx \leq \int_\Omega \widetilde{\Gamma}(\frac{1}{\varepsilon}|b(x)|)dx + \varepsilon \int_\Omega \Phi(|u|)dx, \tag{4.37}$$

Which implies that $\theta \in L^\Phi\left(\Omega, \mathbb{R}^N\right)$. Moreover, by taking into account assumption $\Phi \in \Delta_2$, (4.37), Young's inequality and Poincaré's inequality for N-functions it follows

$$\begin{aligned}
&\int_\Omega \big(p(|\theta|) + p(|Du|) + p(|Du_0|)\big)|Du_0|dx \qquad (4.38)\\
&\quad\leq c\int_\Omega \big(c_3(x) + \Phi(|Du_0|) + \widetilde{\Gamma}(c_\varepsilon|b(x)|)\big)dx + \varepsilon\int_\Omega \left(\Phi(|u|) + \Phi(|Du|)\right)dx\\
&\quad\leq c\int_\Omega \big(c_3(x) + \Phi(|u_0|) + \Phi(|Du_0|) + \widetilde{\Gamma}(c_\varepsilon|b(x)|)\big)dx + 2\varepsilon\int_\Omega \Phi(|Du|)dx.
\end{aligned}$$

Hence, collecting (4.35),(4.36) and (4.38) we get

$$\begin{aligned}
\int_\Omega \Phi(|Du|)dx \leq c\int_\Omega f(x,u,Du)dx&\\
+c\int_\Omega \big(c_3(x) + \Phi(|u_0|) + \Phi(|Du_0|) + \widetilde{\Gamma}(c_\varepsilon|b(x)|)\big)dx&,
\end{aligned}$$

which yields the coercivity of $F(\cdot,\Omega)$ on V_0.

Eventually, by applying Theorem 3.1, the Direct Methods yields the existence of a minimizer for $F(\cdot,\Omega)$ on V_0. □

We now give some applications of our result.

Zhang in [37] developed a method to construct non trivial, i.e., non convex, quasi-convex functions $g_p = g_p(z)$ with polynomial growth $p \geq 1$ at infinity. Under additional assumptions the resulting functions g_p are not even poly-convex.

In [15],[16] a suitable modification of Zhang's method, i.e., using N-functions instead of powers, enabled the construction of quasi-convex functions $g_\Phi = g_\Phi(z)$ satisfying the non-standard growth conditions

$$c_0(\Phi(|z|) - 1) \leq g_\Phi(z) \leq c_1(\Phi(|z|) + 1)$$

with $c_0, c_1 > 0$. Therefore, given a function $a \in L^\infty(\Omega \times \mathbb{R}^N)$ such that $a(x,s) \geq c_2 > 0$ a.e., the function $f_\Phi(x,s,z) = a(x,s)g_\Phi(z)$ satisfies conditions (4.32),(4.33) of the existence Theorem 4.1. Hence, we may apply the result above to solve Dirichlet's boundary values problems for the integral functionals

$$F_\Phi(u,\Omega) = \int_\Omega f_\Phi(x,u,Du)dx. \tag{4.39}$$

We remark that in the case of variational integrals whose integrands f have (p,q) growth $(p<q)$ and depend on the full set of variables, weak lower semicontinuity results in $W^{1,p}(\Omega,\mathbb{R}^N)$ are available only under additional continuity assumptions on the dependence of f on (x,s) (see Remark 4.3 of [17]). Moreover, there is a restriction on the mutual dependence of p and q. Our approach bypasses these limitations for functionals whose energy densities are controlled in terms of N-functions of class Δ_2.

Let, for instance, $\Phi(t)=\frac{t^2}{\log(1+t)}$, then the corresponding f_Φ has (p,q) growth with $p=2-\varepsilon$ and $q=2$ for every $\varepsilon>0$. The known results in ordinary Sobolev spaces implies the sequential lower semicontinuity of $F_\Phi(\cdot,\Omega)$ in the weak topology of $W^{1,2}(\Omega,\mathbb{R}^N)$, but the functional is coercive only on $W^{1,2-\varepsilon}(\Omega,\mathbb{R}^N)$. Thus, the Sobolev spaces setting does not allow the use of the Direct Methods to solve Dirichlet's boundary value problems unless one is able to exhibit a minimizing sequence possessing higher integrability properties.

Eventually, let $\Phi(t)=t^{a+b\sin(\sin(\log(t)))}$ with $a>1+b\sqrt{2}$, and let $f_\Phi=f_\Phi(x,z)$. Notice that f_Φ has (p,q) growth with $p=a-b$ and $q=a+b$, thus the results of Marcellini [33] ensures the weak lower semicontinuity of $F_\Phi(\cdot,\Omega)$ in $W^{1,a-b}(\Omega,\mathbb{R}^N)$ provided $a>(2n+1)b$ (see the Introduction), that is when p,q are relatively close to each other, i.e., $q-p<\frac{a-b}{n}$. On the other hand, Theorem 3.1 does not impose any further restriction on a and b, and thus Theorem 4.1 can be applied to solve Dirichlet's boundary value problems for any a,b chosen as above.

References

1. E. Acerbi and G. Dal Maso *New lower semicontinuity results for polyconvex integrals case,* Calc. Var., **2** (1994), 329-372.
2. E. Acerbi and N. Fusco *Semicontinuity problems in the calculus of variations,* Arch. Rat. Mech. Anal., **86** (1984), 125-145.
3. R. A. Adams *Sobolev Spaces,* Academic Press, New York, 1975.
4. J. M. Ball *Convexity conditions and existence theorems in nonlinear elasticity,* Arch. Rat. Mech. Anal., **63** (1977), 337-403.
5. P. Celada and G. Dal Maso *Further remarks on the lower semicontinuity of polyconvex integrals,* Ann. Inst. H. Poincaré (Anal. non Linéarie), **11** (1995), 661-691.
6. A. Cianchi *A sharp embedding theorem for Orlicz-Sobolev spaces,* Indiana Univ. Math. J. , **45** (1996), 39-65.
7. A. Cianchi and N. Fusco *Gradient regularity for minimizers under general growth conditions,* J. Reine Angew. Math., **507** (1999), 15-36.
8. B. Dacorogna *Direct methods in the calculus of variations,* Appl. Math Sci., **78** Springer Verlag, Heidelberg and New York, 1989.
9. B. Dacorogna and P. Marcellini *Semicontinuité pour des intégrandes polyconvexes sans continuité des déterminants,* C.R.Acad. Sci. Paris, **311** Série **I** (1990), 393-396.
10. A. Dall'Aglio, E. Mascolo and G. Papi *Regularity for local minima of functionals with non-standard growth conditions,* Rend. Mat., **18** Serie **VII** Roma (1998), 305-326.
11. G. Dal Maso and C. Sbordone *Weak lower semicontinuity of polyconvex integrals: a borderline case,* Math. Z., **218** (1995), 603-609.

12. E. De Giorgi *Teoremi di semicontinuità nel Calcolo delle Variazioni,* INdAM, Roma, 1968-69.
13. T. K. Donaldson and N. S. Trudinger *Orlicz-Sobolev spaces and embedding theorems,* J. Funct. Anal., **8** (1971), 42-75.
14. I. Ekeland *On the Variational Principle,* J. Math. Anal. Appl., **47** (1974), 324-353.
15. M. Focardi *Semicontinuity of vectorial functionals in Orlicz-Sobolev spaces,* Rend. Ist. Mat. Univ. Trieste, vol. **XXIX** (1997), 141-161.
16. M. Focardi and E. Mascolo *Lower semicontinuity of quasi-convex functionals with non standard growth,* to appear on Journal of Convex Analysis.
17. I. Fonseca and J. Malý *Relaxation of multiple integrals in Sobolev spaces below the growth exponent for the energy density,* Ann. Inst. H. Poincaré (Analyse non Linéaire), **14** (1997), 309-338.
18. I. Fonseca and P. Marcellini *Relaxation of multiple integrals in subcritical Sobolev spaces,* J. Geom. Anal., **7** (1997), 57-81.
19. I. Fonseca and S. Müller *Quasi-convex integrands and lower semicontinuity in L^1,* SIAM J. Math. Anal., **23** (1992), 1081-98.
20. A. Fougères *Théorèmes de trace et de prolongement dans les espaces de Sobolev et de Sobolev-Orlicz,* C.R. Acad.Sci. Paris, **274** Série **A** (1972), 181-184.
21. N. Fusco *Quasi-convessità e semicontinuità per integrali multipli di ordine superiore,* Ricerche di Mat., **29** (1980), 307-323.
22. N.Fusco and J.E.Hutchinson *A direct proof of lower semicontinuity of polyconvex functionals,* Manuscripta Mat., **87** (1995), 35-50.
23. W. Gangbo *On the weak lower semicontinuity of energies with polyconvex integrands,* J. Math Pures et Appl., Vol. **73**, 5, 1994.
24. E. Giusti *Metodi diretti nel Calcolo delle Variazioni,* U.M.I., Bologna 1994.
25. J. P. Gossez *Nonlinear elliptic problems for equations with rapidly (or slowly) increasing coefficients,* Trans. Amer. Math Soc., **55** (1974), 163-205.
26. M. A. Krasnosel'skij and Ya. B. Rutickii *Convex functions and Orlicz spaces,* Noordhoff, Groningen, 1961.
27. J. Kristensen *Lower semicontinuity in Sobolev spaces below the growth exponent of the integrand,* Proc. Roy. Soc. Edimburgh Sect A, **127** (1997), 797-817.
28. A. Kufner, O. John and S. Fučik *Function Spaces,* Academia, Praha, 1977.
29. J. Malý *Weak lower semicontinuity of polyconvex integrals,* Proc. Roy. Soc. Edimburgh Sect A, **123** (1993), 681-691.
30. J. Malý *Weak lower semicontinuity of polyconvex and quasi-convex integrals,* Vortagsreihe 1993, Bonn.
31. J. Malý *Lower semicontinuity of quasi-convex integrals,* Manus. Math., **85** (1994), 419-428.
32. P. Marcellini *Approximation of quasi-convex functions and lower semicontinuity of multiple integrals,* Manus.Math., **51** (1985), 1-28.
33. P. Marcellini *On the definition and the lower semicontinuity of certain quasi-convex integrals,* Ann. Inst. H. Poincaré (Analyse non Linéaire), **3** (1986), 391-409.
34. C. B. Jr Morrey *Quasi-convexity and the lower semicontinuity of multiple integrals,* Pacific J. Math., **2** (1952), 25-53.
35. M. M. Rao and Z. D. Ren *Theory of Orlicz spaces,* Pure and Appl. Math., Marcel Dekker, New York, 1981.
36. N. S. Trudinger *On imbeddings into Orlicz spaces and some applications,* J. Math.

Mech., **17** (1967), 473-483.
37. K. Zhang *A construction of quasi-convex functions with linear growth at infinity,* Ann. Scuola Norm. Sup. Pisa, **19** (1992), 313-326.
38. W. Ziemer *Weakly Differentiable Functions,* GTM **120**, Springer Verlag, 1989.