Γ-CONVERGENCE: A TOOL TO INVESTIGATE PHYSICAL PHENOMENA ACROSS SCALES

MATTEO FOCARDI

Abstract. De Giorgi's Γ-convergence is a variational theory modelled upon the convergence of families of (perturbed) minimum problems and of the corresponding minimizers.

In these notes, after reviewing briefly the basic theory and accounting for some recent new insights, we discuss three examples of static mechanical models which can be analyzed by means of Γ-convergence arguments.

1. Introduction

A recurrent question arising in several fields is the determination of sufficient conditions, hopefully necessary as well, ensuring the convergence of sequences of minimum problems and of the related minimizers. In these notes we shall be interested mainly into static variational models in mechanics for which some examples of the issue raised above are listed in what follows:

(i) *Homogenization of composites*: in this setting two (or more) materials are finely mixed in a way that several physical properties of the microscopically heterogeneous medium behave macroscopically like those of a 'fictitious' homogeneous one when looking at samples of the body much bigger than the singular constituents (see Figure 1). Mathematically, this problem corresponds to analysing PDEs with rapidly oscillating coefficients such as, for instance,

$$-\mathrm{div}\,(B_j\nabla w) = f \quad \mathcal{L}^n \text{ a.e. } (0,1)^3, \text{ with } w \in W_0^{1,2}((0,1)^3),$$

with $f \in L^2(\Omega)$ and $B_j \in L^\infty(\Omega, \mathbb{R}^{n\times n})$, and try to determine the asymptotic behaviour of the corresponding solutions as $j \uparrow +\infty$. If the B_j's are symmetric and equi-coercive, those problems can be equivalently reformulated as determining the limit behaviour of the minimizers of

$$\min_{W_0^{1,2}((0,1)^3)} \int_{(0,1)^3} \left(\langle B_j\nabla u, \nabla u\rangle - f\,u\right) dx.$$

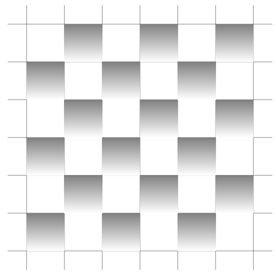

FIGURE 1. A periodic arrangement of two materials.

(ii) *Derivation of linearized elasticity*: the usual argument used to deduce such a theory from finite elasticity hinges upon a Taylor expansion around the identity for Lipschitz deformations in case of (suitably) imposed small external loads. More precisely, for a homogeneous and hyper-elastic body occupying the reference configuration $\Omega \subseteq \mathbb{R}^3$ and with stored energy

$$\int_\Omega W(\nabla v)dx,$$

$v : \Omega \to \mathbb{R}^3$ being the elastic deformation, we assume the body to be at equilibrium when no external loads are applied and that the energy density W is frame indifferent (see Section 4 for the precise assumptions on W). It is then natural to expect small displacements $\varepsilon_j u$ if small external loads $\varepsilon_j \ell$, $\ell \in L^2(\Omega, \mathbb{R}^3)$, are imposed ($\varepsilon_j \downarrow 0^+$). Supposing W to be smooth close to the identity and the deformation $v = x + \varepsilon_j v$ to be Lipschitz continuous, one finds

$$W(\mathrm{Id}_3 + \varepsilon_j \nabla u) = \varepsilon_j^2 D^2 W(\mathrm{Id}_3)[e(u), e(u)] + o(\varepsilon_j^2)$$

with $e(u) = (\nabla u + \nabla^t u)/2$ the linearized strain of u. In turn, the latter formula implies as $\varepsilon_j \downarrow 0^+$

$$\frac{1}{\varepsilon_j^2}\left(\int_\Omega W(\mathrm{Id}_3 + \varepsilon_j \nabla u)dx - \varepsilon_j^2 \int_\Omega \ell \cdot u\, dx\right) = \int_\Omega \left(D^2 W(\mathrm{Id}_3)[e(u), e(u)] - \ell \cdot u\right) dx + o(1). \tag{1.1}$$

This argument is commonly taken as a justification of linearized elasticity. Note that it does not supply any piece of information on the behaviour of the stable states of the energies on the left hand side of (1.1) with respect to those on the right hand side there.

(iii) *Obstacle problems for nonlocal energies*: let us consider a problem of diffusion through semipermeable membranes as described in [37]. Given a cell, whose membrane is modeled by the surface $z = 0$, the outside concentration of molecules of some substance is represented by ψ, the transport of molecules through the membrane is possible only across some given channels (represented, for instance, by the set $T_j \subseteq \{z = 0\}$ in Figure 1, see Section 5 for details), and only from the outside toward the inside, $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, of the cell. At equilibrium, the concentration inside the cell is the unique function $u \in W^{1,2}(\mathbb{R}^3_+)$ satisfying

$$\begin{cases} -\triangle u(x,y,z) = 0 & (x,y,z) \in \mathbb{R}^3_+ \\ u(x,y,0) \geq \psi(x,y) & (x,y) \in T_j \\ \partial_z u(x,y,0) \leq 0 & (x,y) \in \mathbb{R}^3 \\ \partial_z u(x,y,0) = 0 & (x,y) \in \mathbb{R}^3 \setminus T_j,\ x \in T_j \cap \{u(\cdot,0) > \psi\}. \end{cases}$$

Equivalently, in terms of the boundary trace $v(x, y) = u(x, y, 0)$ the previous problem translates into

$$\begin{cases} (-\triangle)^{1/2} v(x,y) \geq 0 & x \in \mathbb{R}^2 \\ (-\triangle)^{1/2} v(x,y) = 0 & x \in \mathbb{R}^2 \setminus T_j, \text{ and } (x,y) \in T_j \cap \{v > \psi\} \\ v(x,y) \geq 0 & x \in T_j \end{cases} \tag{1.2}$$

that is v is the minimizer among $H^{1/2}$-functions of the energy

$$\mathscr{F}_j(w) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|w(x) - w(y)|^2}{|x - y|^3} dxdy$$

under the condition $w \geq \psi$ on T_j. Here, $(-\triangle)^{1/2}$ is the 1/2-fractional Laplace operator (see Section 5 for the definition).

Assuming that the size of each channel vanishes as $j \uparrow +\infty$, we want to determine the limits of the solutions of the minimum problems related to the $\mathscr{F}_j$'s, one reason being that if j is very big the presence of many channel renders the use of numerical analysis tools prohibitive. A limit substitute problem is then looked for, in order to infer qualitative properties of solutions of (1.2) for j big but finite.

In all the previous examples the asymptotic behaviour of minimizers of a family of variational problems depending on a vanishing parameter is under study. In a simplified setting, we can take energies $\mathscr{F}_j$ and $\mathscr{F}$ defined on a Hilbert space X with values into the extended real line $\mathbb{R} \cup \{\pm\infty\}$, and try to define a notion of convergence of $(\mathscr{F}_j)_{j\in\mathbb{N}}$

FIGURE 2. The set T_j is the union of the circles.

to $\mathscr{F}$, indicated in this introduction as $\mathscr{F}_j \twoheadrightarrow \mathscr{F}$, ensuring both the convergence of minimum values

$$\min_X \mathscr{F}_j \to \min_X \mathscr{F}, \tag{1.3}$$

and that of the corresponding minimizers (assuming that they exist)

$$x_j \to x, \text{ with } x_j \in \mathrm{argmin}_X \mathscr{F}_j \text{ and } x \in \mathrm{argmin}_X \mathscr{F}. \tag{1.4}$$

In view of applications, it is also reasonable to require both properties (1.3) and (1.4) to be stable under the addition of continuous perturbations, that is if $\mathscr{F}_j \twoheadrightarrow \mathscr{F}$, then $\mathscr{F}_j + \mathscr{G} \twoheadrightarrow \mathscr{F} + \mathscr{G}$ for all $\mathscr{G} : X \to \mathbb{R}$ continuous, and

$$\lim_j \min_X \{\mathscr{F}_j + \mathscr{G}\} = \min_X \{\mathscr{F} + \mathscr{G}\} \tag{1.5}$$

together with the convergence of minimizers. Clearly, an interplay with the topology of the ambient space X is necessary to guarantee under mild structural assumptions on the functions $\mathscr{F}_j$ the existence of minimizers for the minimum problems in (1.3) and (1.5).

A prototypical setting for which a necessary and sufficient condition for the convergence of minimizers can be devised is that of quadratic forms satisfying a uniform coercivity hypothesis, i.e., there is some $\lambda > 0$ such that for all $j \in \mathbb{N}$ and $x \in X$

$$\mathscr{F}_j(x) = \langle A_j x, x\rangle \geq \lambda \|x\|^2, \text{ and } \mathscr{F}(x) = \langle Ax, x\rangle \geq \lambda \|x\|^2.$$

In such a case we claim that condition (1.5) is equivalent to

$$\liminf_j \mathscr{F}_j(x_j) \geq \mathscr{F}(x), \quad \textit{for all } (x_j)_{j\in\mathbb{N}} \text{ weakly converging to } x, \tag{1.6}$$

and

$$\limsup_j \mathscr{F}_j(y_j) \leq \mathscr{F}(x), \quad \textit{for some } (y_j)_{j\in\mathbb{N}} \text{ weakly converging to } x. \tag{1.7}$$

for all $x \in X$.[1]

The derivation of inequalities (1.6) and (1.7) from (1.5) relies on the simple observation that, with fixed a generic point $x \in X$, the continuous linear perturbation $\mathscr{G}(y) = -2\langle Ax, y\rangle$ renders x the global minimizer of the energy $\mathscr{F} + \mathscr{G}$, so that one can estimate the values $\mathscr{F}_j(x_j)$, $(x_j)_{j\in\mathbb{N}}$ as in (1.6), as follows:

$$\mathscr{F}_j(x_j) = \mathscr{F}_j(x_j) - 2\langle Ax, x_j\rangle + 2\langle Ax, x_j\rangle \geq \min_X\{\mathscr{F}_j(y) - 2\langle Ax, y\rangle\} + 2\langle Ax, x_j\rangle.$$

Since (1.5) holds true, (1.6) is deduced straightforwardly from the weak convergence of $(x_j)_{j\in\mathbb{N}}$ to x, i.e.

$$\begin{aligned} &\liminf_j \mathscr{F}_j(x_j) \geq \lim_j \min_X\{\mathscr{F}_j(y) - 2\langle Ax, y\rangle\} + 2\lim_j\langle Ax, x_j\rangle \\ &= \min_X\{\mathscr{F}(y) - 2\langle Ax, y\rangle\} + 2\langle Ax, x\rangle = \mathscr{F}(x) - 2\langle Ax, x\rangle + 2\langle Ax, x\rangle = \mathscr{F}(x). \end{aligned} \tag{1.8}$$

Actually, the inequality above turns into an equality for the sequence $(y_j)_{j\in\mathbb{N}}$ of minimizers of $\min_X\{\mathscr{F}_j(y) - 2\langle Ax, y\rangle\}$ provided we show that it converges weakly to x. To infer this property, we note that the uniform coercivity condition on the $\mathscr{F}_j$'s implies that $\sup_j \|y_j\| < +\infty$, thus a subsequence $(y_{j_k})_{k\in\mathbb{N}}$ converges weakly to some point z. Therefore, the argument leading to (1.8) yields

$$\begin{aligned} &\mathscr{F}(z) \leq \lim_k \mathscr{F}_{j_k}(y_{j_k}) \\ &= \min_X\{\mathscr{F}(y) - 2\langle Ax, y\rangle\} + 2\langle Ax, z\rangle = \mathscr{F}(x) - 2\langle Ax, x\rangle + 2\langle Ax, z\rangle. \end{aligned}$$

Being x is the unique minimizer of $y \to \mathscr{F}(y) - 2\langle Ax, y\rangle$ by strict convexity of $\mathscr{F}$, the latter inequality implies $z = x$. In turn, from this we deduce that any weakly convergent subsequence has x as limit, so that the whole sequence $(y_j)_{j\in\mathbb{N}}$ converges weakly to x, and (1.7) is established at once.

On the other way round, (1.6) and (1.7) implies the validity of (1.5). Indeed, given any $\mathscr{G}$ linear and continuous, the minimizer x_j of $\mathscr{F}_j + \mathscr{G}$ satisfies $\sup_j \|x_j\| < +\infty$. In particular, a subsequence $(x_{j_k})_{k\in\mathbb{N}}$ converges weakly to some $x \in X$, so that by (1.6)

$$\liminf_k \min_X\{\mathscr{F}_{j_k} + \mathscr{G}\} = \liminf_k \left(\mathscr{F}_{j_k}(x_{j_k}) + \mathscr{G}(x_{j_k})\right) \geq \mathscr{F}(x) + \mathscr{G}(x) \geq \min_X\{\mathscr{F} + \mathscr{G}\}.$$

[1]Given a sequence of real numbers $(\alpha_j)_{j\in\mathbb{N}}$ its *inferior/superior limits* are defined respectively as $\liminf_j \alpha_j := \sup_k \inf_{j\geq k} \alpha_j$, $\limsup_j \alpha_j := \inf_k \sup_{j\geq k} \alpha_j$. We refer to subsection 1.1 for the notation adopted throughout the paper.

By taking into account that for some sequence $(y_j)_{j\in\mathbb{N}}$ weakly converging to x inequality (1.7) holds true, we have

$$\mathscr{F}(x)+\mathscr{G}(x)=\limsup_j\left(\mathscr{F}_j(y_j)+\mathscr{G}(y_j)\right)\geq\limsup_k\min_X\{\mathscr{F}_{j_k}+\mathscr{G}\}.$$

In particular,

$$\mathscr{F}(x)+\mathscr{G}(x)=\lim_k\min_X\{\mathscr{F}_{j_k}+\mathscr{G}\},$$

and actually by repeating the same argument for any weakly convergent subsequence of $(x_j)_{j\in\mathbb{N}}$, we infer that $(x_j)_{j\in\mathbb{N}}$ itself converges to x and then the validity of (1.3).

Note that in the last derivation we have not used the specific structure of the problem, i.e. neither that the relevant energies $\mathscr{F}_j$, $\mathscr{F}$ are quadratic forms nor their regularity properties used to infer that the corresponding (global) minimizers satisfy some Euler-Lagrange equations.

Only inequalities (1.6) and (1.7) have played a role, together with the uniform coercivity condition on the $\mathscr{F}_j$'s in order to guarantee precompactness for the sequence of minimizers and uniqueness of the limit point.

Hence, inequalities (1.6) and (1.7), if satisfied, identify a functional convergence implying (1.5) (provided some additional assumption guarantees the existence of minimizers), this convergence is known in literature as Γ-convergence.

Γ-convergence was introduced systematically in 1975 (in the general setting of topological spaces) by De Giorgi and Franzoni [31] building upon a previous contribution by De Giorgi himself studying the variational limits of families of area-type functionals [28] (see also [29, 30]). Γ-convergence reduces to inequalities (1.6) and (1.7) in case of metric spaces endowed with the distance topology, it is a variational convergence tailor made to deal with the description of the asymptotic behaviour of the *global* minimizers of family of energies depending on parameters of different nature (geometric, constitutive, and so on).

It provides a unified framework summarizing several notions introduced for different purposes in unrelated fields almost contemporarily:

(a) Wijsman's *infimal convergence* characterizing the continuity of Fenchel conjugate in finite dimensional spaces (see [72, 73]),
(b) Spagnolo's *G-convergence* theory investigating the asymptotics of families of solutions to (equi-uniformly) elliptic (and parabolic) PDEs (see [66, 67, 68]),
(c) Mosco's convergence determining the limits of variational inequalities and characterizing the continuity of Fenchel conjugate

in infinite dimensional spaces (see [57, 58] and the contribution by Joly [45]),

(d) Marcellini's *G-convergence* extending the notions by Spagnolo and Mosco quoted above to the (nonlinear) convex setting (see [49, 8]),

(e) Zolezzi's *variational convergence* studying stability issues in mathematical optimization (see [74]),

Despite some of the theories quoted above are more effective in specific problems, Γ-convergence has attracted the attention of many researchers since its introduction and has found interesting application in several fields: homogenization of composites, discrete-to-continuum limits to validate continuum mechanical theories both in static and evolutionary settings, dimension reduction problems in mechanics, approximation of variational models in image segmentation and in fracture mechanics, obstacle problems for local and fractional operators and many others. All these results show the flexibility of the theory and its effectiveness to study families of variational problems and the behaviour of the corresponding global minimizers.

The treatise [25] and the books [7], [11] and [9] are classical references for this subject covering a wide variety of topics and providing very detailed list of references. The aim of the present notes is much more modest: we give to the readers only the essential tools to understand the analysis of the examples mentioned at the beginning of this introduction, so that we survey only on the basic results of the theory. Despite this, we complement the material in the standard references quoted above by including some new theoretical insights, and discussing some recent applications to static mechanical problems.

Let us also point out that we have confined the exposition to static problems, we shall not cover the connections between Γ-convergence of a sequence of functionals and the convergence of the solutions to the corresponding evolution equations, for which we refer to the papers [63], [65] and [22] for what limits of gradient-flows are concerned, to [54] and [53] for limits of rate-independent systems, and to [52] for limits of Hamiltonian systems.

An outline of the paper is as follows. In Section 2 the basics of the abstract theory of Γ-convergence in metric spaces are worked out. We shall not give full proofs of most of the results referring to the books quoted above (giving precise references), but rather complement the classical material including new outcomes on asymptotic Γ-development and the convergence of local minimizers.

In Sections 3-5 we provide three examples of application of Γ-convergence to the analysis of static mechanical models.

Section 3 deals with a classical problem in the theory of composites: the homogenization of equi-uniformly elliptic quadratic forms on Sobolev spaces or equivalently the determination of the asymptotic behaviour of the solutions of the corresponding Euler-Lagrange elliptic PDEs. We shall present only the statements of the abstract theory without providing the proofs referring for them mainly to the books [25, Chapter 13], [11, Chapter 3]. On the other hand, we shall discuss how the analogous problem for non-variational PDEs, related to non-self-adjoint operators, can be recasted into the variational framework of Γ-convergence. Moreover, we shall also highlight an extension to nonlinear elliptic problems and hint to some further generalizations. Links with recent proposals for 'dynamical' versions of Γ-convergence shall also be underlined.

The second example, discussed in Section 4, falls into an intensive field of research: the rigorous mathematical, or better variational, derivation of mechanical theories from nonlinear continuum models. The research in this direction has found a renewed impulse in the last years thanks to the geometric rigidity estimate by Friesecke, James and Müller (see [41, 42] and [48] for an account of many results of this kind and related references). In this presentation we shall only deal with the variational derivation of linearized elasticity from finite elasticity, following the work by Dal Maso, Negri and Percivale [27] and the subsequent developments in [3].

The last example in Section 5 is related to the asymptotics of obstacle problems for quadratic forms. This was one of the first topics in which Γ-convergence theory was successfully applied in the late '70's. Here, we shall consider the case of nonlocal energies, a setting that has recently attracted the attention of many researchers.

We warn the reader that in this notes Γ-convergence analysis is applied mainly to problems involving quadratic forms, this is by no means a limitation of the theory but only a matter of taste of the Author, essentially adopted for presentation purposes. Finally, the bibliography reported at the end of the paper is systematic for the topics exposed here, and has no claim of completeness.

1.1. **Notations.** We recall briefly the main notations recurrently used in the whole paper, quoting this subsection for reminders throughout the text. Instead, specific symbols, of use only in some sections, shall be indicated directly there.

Inferior and superior limits of a real sequence $(\alpha_j)_{j\in\mathbb{N}}$ are defined respectively as

$$\liminf_j \alpha_j := \sup_k \inf_{j\geq k} \alpha_j, \quad \limsup_j \alpha_j := \inf_k \sup_{j\geq k} \alpha_j.$$

Recall that they represent the smallest and biggest cluster point of $(\alpha_j)_{j\in\mathbb{N}}$.

Metric, Banach, Hilbert spaces shall be all denoted generically by the letter X, the precise role of the symbol shall be specified in the context of use. In the former case, the related distance shall be indicated by d, in the other two cases the norm by $\|\cdot\|$, and in the latter $\langle\cdot,\cdot\rangle$ shall be the corresponding scalar product. In addition, we shall use the same symbol for the duality pairing between X and its dual space X^*. More precisely, in the Hilbertian setting we shall often identify X and X^* in view of Riesz's representation theorem.

Given a set A its *characteristic function* $\mathbf{1}_A$ is defined by $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise; instead its *indicator function* χ_A by $\chi_A(x) = 0$ if $x \in A$ and $+\infty$ otherwise.

The letter Ω shall always denote a bounded, smooth and connected open set of an Euclidean space, denoted by $\mathbb{R}^n$ (or $\mathbb{R}^m$), endowed with the corresponding Lebesgue measure $\mathcal{L}^n$ ($\mathcal{L}^m$), and Hausdorff measure $\mathcal{H}^{n-1}$ ($\mathcal{H}^{m-1}$). The real line is simply denoted by $\mathbb{R}$, the extended real line, $\mathbb{R}\cup\{\pm\infty\}$, by $\overline{\mathbb{R}}$.

With fixed $p \in [1,+\infty]$, we use standard notations for Lebesgue spaces $L^p(\Omega,\mathbb{R}^m)$ and Sobolev spaces $W^{1,p}(\Omega,\mathbb{R}^m)$, referring mainly to [14] for the needed (basic) prerequisites. If $m = 1$ we shall only write $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

We shall also consider Sobolev-Slobodeckij spaces $W^{s,2}(\Omega)$, $s \in (0,1)$, for which we refer to the notes [1] and to the book [71] as main sources of references; the recent hitchhicker's guide [36] provides a friendly introduction.

2. Working out the basics

In this section we shall focus on the classical theory of Γ-convergence developing the rudiments. We shall not deal neither with the most general framework, having confined the exposition to metric spaces since in that setting Γ-convergence can be checked sequentially (as it is well-known for several other topological properties), nor with refined tools such as the localization method (termed in literature $\overline{\Gamma}$-convergence theory) nor with the blow-up techniques for integral functionals.

Our choice is motivated both for the sake of conciseness and in view of the applications in the following sections. It is evident that those aspects should not be neglected by an interested reader in order to gain further insight on the theory, and to learn very useful technical tools. We refer to the texts quoted in the Introduction for a comprehensive exposition complemented with several examples.

On the other hand, we shall discuss some recent proposals to include in the theory the convergence of (stable) local minimizers and a discussion of several aspects of the asymptotic Γ-development.

In what follows (X, d) will always denote a metric space.

2.1. **Recalling Tonelli's direct method.** Modern Calculus of Variations is based essentially on the so called Tonelli's direct method to establish the existence of solutions to minimum problem under mild assumptions. Before stating the main result we introduce necessary definitions.

Definition 2.1. *A function $\mathscr{F} : X \to \overline{\mathbb{R}}$ is d-lower semicontinuous provided*

$$\mathscr{F}(x) \leq \liminf_j \mathscr{F}(x_j) \quad \text{for all } x_j \xrightarrow{d} x.$$

$\mathscr{F}$ is called d-coercive on X if for all $t > 0$ the set $\{\mathscr{F} \leq t\}$ is sequentially compact.

$\mathscr{F}$ is called d-mildly coercive on X if there is a sequentially compact subset $K \neq \emptyset$ such that $\inf_X \mathscr{F} = \inf_K \mathscr{F}$.

We are now ready to state the main result on which Tonelli's direct method is based.

Theorem 2.2 (Weierstrass' theorem)**.** *Let $\mathscr{F} : X \to \overline{\mathbb{R}}$ be a d-lower semicontinuous and d-mildly coercive function such that $\inf_X \mathscr{F} > -\infty$, then $\mathscr{F}$ admits a minimizer on X.*

Proof. The role of d-mild coercivity is to extract a converging subsequence from any minimizing sequence. More precisely, if $(x_j)_{j\in\mathbb{N}}$ is a sequence such that $\lim_j \mathscr{F}(x_j) = \inf_X \mathscr{F}$, by assumption there is no loss in generality supposing that it belongs to a sequentially compact set K as in Definition 2.1. Hence, we can extract a subsequence $(x_{j_k})_{k\in\mathbb{N}}$ converging to some $\bar{x} \in K$. Finally, the d-lower semicontinuty of $\mathscr{F}$ assures that

$$\mathscr{F}(\bar{x}) \leq \liminf_k \mathscr{F}(x_{j_k}) = \inf_X \mathscr{F} \Longrightarrow \mathscr{F}(x) = \min_X \mathscr{F}.$$

□

Clearly, when studying a specific problem the main issue to follow the procedure outlined above is to select a suitable metric d ensuring both lower semicontinuity and d-(mild) coercivity of the relevant function $\mathscr{F}$.

On one hand, d-coercivity is more easily fulfilled if the topology induced by d is 'poor' of open sets, on the other hand d-lower semicontinuity calls for 'many' open sets to have 'few' converging sequences to test the required inequality. The balance of these two opposite needs is the essential (and only) criterion to select the appropriate metric d with which the space X has to be endowed.

Dealing with a generic function $\mathscr{F} : X \to \overline{\mathbb{R}}$ it is extremely useful to introduce its *lower semicontinuous envelope* or *relaxation* defined for all $x \in X$ as

$$\overline{\mathscr{F}}(x) = \sup\{\mathscr{G}(x) : \mathscr{G} \ d\text{-lower semicontinuous}, \mathscr{G} \leq \mathscr{F}\}.$$

By the very definition, $\overline{\mathscr{F}}$ is the biggest d-lower semicontinuous function less than $\mathscr{F}$, so that $\mathscr{F} = \overline{\mathscr{F}}$ if and only if $\mathscr{F}$ itself is d-lower semicontinuous.

Since on metric spaces d-lower semicontinuity can be qualified sequentially, the relaxation of $\mathscr{F}$ can be characterized via the equality

$$\overline{\mathscr{F}}(x) = \inf\{\liminf_j \mathscr{F}(x_j) : x_j \xrightarrow{d} x\}. \tag{2.1}$$

A diagonal argument actually shows that the infimum on the right hand side above is a minimum. Hence, formula (2.1) can be equivalently expressed as

$$\overline{\mathscr{F}}(x) \leq \liminf_j \mathscr{F}(x_j) \quad \text{for } \textit{every} \text{ sequence } x_j \xrightarrow{d} x, \tag{2.2}$$

$$\overline{\mathscr{F}}(x) \geq \limsup_j \mathscr{F}(y_j) \quad \text{for } \textit{some} \text{ sequence } y_j \xrightarrow{d} x. \tag{2.3}$$

In particular, noting that $\overline{\mathscr{F}}$ is d-coercive, provided $\mathscr{F}$ is d-coercive (see [25, Proposition 7.7]), from the latter characterization and Theorem 2.2 we infer that

$$\min_X \overline{\mathscr{F}} = \inf_X \mathscr{F}.$$

In conclusion, the relaxation of a function $\mathscr{F}$ is extremely useful to analyze the behaviour of the minimizing sequences of $\mathscr{F}$ itself. More precisely, if $\mathscr{F}$ is d-coercive the cluster points of the minimizing sequences of $\mathscr{F}$ are exactly the minimizers of $\overline{\mathscr{F}}$.

2.2. **Basics of Γ-convergence.** We start off with the definition building upon inequalities (2.2) and (2.3).

Definition 2.3. *A sequence* $\mathscr{F}_j : X \to \overline{\mathbb{R}}$ $\Gamma(d)$-converges *to* $\mathscr{F} : X \to \overline{\mathbb{R}}$*, in short we write* $\mathscr{F} = \Gamma(d)\text{-}\lim_j \mathscr{F}_j$*, if for all* $x \in X$ *it holds*

(LB) Lower Bound *inequality:*

$$\mathscr{F}(x) \le \liminf_j \mathscr{F}_j(x_j) \quad \textit{for every sequence } x_j \xrightarrow{d} x; \tag{2.4}$$

(UB) Upper Bound *inequality:*

$$\mathscr{F}(x) \ge \limsup_j \mathscr{F}_j(y_j) \quad \textit{for some sequence } y_j \xrightarrow{d} x. \tag{2.5}$$

The function $\mathscr{F}$ is uniquely determined by the two conditions (LB) and (UB), and it is called the $\Gamma(d)$-limit *of* $(\mathscr{F}_j)_{j\in\mathbb{N}}$.

In what follows, we shall not highlight the dependence on the metric d of the Γ-limit if no confusion may arise. Clearly, different metrics may give different asymptotic behaviours (see Remark 2.14 for more details).

One of the most considerable issues of Definition 2.3 is that the Γ-limit is selected via an optimization process with no a priori ansatz on its form. Indeed, if on one hand inequality (2.4) requires the search of an asymptotic local lower bound for the family of energies $(\mathscr{F}_j)_{j\in\mathbb{N}}$, on the other hand in (2.5) such a bound is optimized.

In particular, each sequence $(y_j)_{j\in\mathbb{N}}$ for which (2.5) holds is termed a *recovery sequence* (for the relevant point x), in that satisfying both inequalities (2.4) and (2.5) the limit energy computed in x is 'recovered' by the approximating ones computed on the converging points y_j, that is

$$\mathscr{F}(x) = \lim_j \mathscr{F}_j(y_j).$$

We work out next few elementary but nontrivial examples of Γ-convergent sequences. Differences between Γ-limits and pointwise limits will also be highlighted. We consider the most elementary setting with ambient space $X = \mathbb{R}$ endowed with the Euclidean metric. Further examples shall be provided at the end of the section (see Examples 2.21, 2.22 and 2.23).

Example 2.4. *Suppose that $(\mathscr{F}_j)_{j\in\mathbb{N}}$ converges locally uniformly on $\mathbb{R}$ to $\mathscr{F}$, then $\Gamma\text{-}\lim_j \mathscr{F}_j = \overline{\mathscr{F}}$, where $\overline{\mathscr{F}}$ denotes the relaxation of $\mathscr{F}$.*

Let us first check (LB) inequality: given $x \in \mathbb{R}$ and $(x_j)_{j\in\mathbb{R}}$ converging to x we have

$$\mathscr{F}_j(x_j) \ge \mathscr{F}(x_j) - \sup_{y\in[x-1,x+1]} |\mathscr{F}_j(y) - \mathscr{F}(y)| \overset{(2.2)}{\Longrightarrow} \liminf_j \mathscr{F}_j(x_j) \ge \overline{\mathscr{F}}(x).$$

Instead, to enforce (UB) inequality take any sequence $(y_j)_{j\in\mathbb{N}}$ converging to x and such that $(\mathscr{F}(y_j))_{j\in\mathbb{N}}$ converges to $\overline{\mathscr{F}}(x)$, then arguing as *before*

$$\mathscr{F}_j(y_j) \leq \mathscr{F}(y_j) + \sup_{y\in[x-1,x+1]} |\mathscr{F}_j(y) - \mathscr{F}(y)| \stackrel{(2.3)}{\Longrightarrow} \limsup_j \mathscr{F}_j(y_j) \leq \overline{\mathscr{F}}(x).$$

Example 2.5. *Let us now discuss Γ-limits of monotone sequences of real functions $\mathscr{F}_j : \mathbb{R} \to \mathbb{R}$, with pointwise limit denoted by $\mathscr{F} : \mathbb{R} \to \overline{\mathbb{R}}$.*

We start with the monotone increasing case, i.e., $\mathscr{F}_j \leq \mathscr{F}_{j+1} \leq \mathscr{F}$, we claim that $\Gamma\text{-}\lim_j \mathscr{F}_j = \sup_j \overline{\mathscr{F}_j}$. To check (LB) inequality, given any point x and a sequence $(x_j)_{j\in\mathbb{N}}$ converging to it, note that for every $k \leq j$ we have $\mathscr{F}_j(x_j) \geq \mathscr{F}_k(x_j)$, so that

$$\liminf_j \mathscr{F}_j(x_j) \geq \liminf_j \mathscr{F}_k(x_j) \stackrel{(2.2)}{\geq} \overline{\mathscr{F}_k}(x) \Rightarrow \liminf_j \mathscr{F}_j(x_j) \geq \sup_k \overline{\mathscr{F}_k}(x).$$

For what (UB) inequality is concerned, with fixed a point x, for every $j \in \mathbb{N}$ let x_j be satisfying

$$|x_j - x| + |\mathscr{F}_j(x_j) - \overline{\mathscr{F}_j}(x)| \leq \frac{1}{j},$$

then $(x_j)_{j\in\mathbb{N}}$ converges to x, and

$$\limsup_j \mathscr{F}_j(x_j) = \limsup_j \overline{\mathscr{F}_j}(x) \leq \sup_j \overline{\mathscr{F}_j}(x).$$

Instead, in the monotone decreasing case, i.e., $\mathscr{F}_j \geq \mathscr{F}_{j+1} \geq \mathscr{F}$, we claim that $\Gamma\text{-}\lim_j \mathscr{F}_j = \overline{\mathscr{F}}$ (cp. with (2.1)*). Given any point x, for every sequence $(x_j)_{j\in\mathbb{N}}$ converging to it we have*

$$\liminf_j \mathscr{F}_j(x_j) \geq \liminf_j \mathscr{F}(x_j) \stackrel{(2.2)}{\geq} \overline{\mathscr{F}}(x),$$

so that (LB) inequality eventually follows. On the other hand, to check (UB) inequality, fix a point x and let $(x_k)_{k\in\mathbb{N}}$ be satisfying equality (2.3)*. For every $k \in \mathbb{N}$, by pointwise convergence we find an increasing sequence $h_k \uparrow +\infty$ such that*

$$|\mathscr{F}_{h_k}(x_k) - \mathscr{F}(x_k)| \leq \frac{1}{k}.$$

By setting $x_j = x_k$ if $h_k \leq j < h_{k+1}$, the monotonicity of $(\mathscr{F}_j)_{j\in\mathbb{N}}$ yields

$$\mathscr{F}_j(x_j) = \mathscr{F}_j(x_k) \leq \mathscr{F}_{h_k}(x_k) \quad \textit{for } h_k \leq j < h_{k+1},$$

so that

$$\limsup_j \mathscr{F}_j(x_j) = \lim_k \mathscr{F}_{h_k}(x_k) = \lim_k \mathscr{F}(x_k) \stackrel{(2.3)}{=} \overline{\mathscr{F}}(x).$$

Example 2.6. *Consider the sequence* $(\mathscr{F}_j)_{j\in\mathbb{N}}$, *with* $\mathscr{F}_j(x) = \sin(jx)$, *for which it is well known that no pointwise limit exists. Instead, since the period of* $\mathscr{F}_j$ *vanishes as* $j \uparrow +\infty$, *wild oscillations between* -1 *and* 1 *are produced in any neighbourhood of every point, so that the constant* -1 *turns out to be the* Γ*-limit of* $(\mathscr{F}_j)_{j\in\mathbb{N}}$. *More precisely, inequality (LB) inequality is trivial, while to enforce (UB) inequality in a point* x, *it is sufficient to take* x_j *as the minimum point of* $\mathscr{F}_j$ *in the interval* $[x, x + 2\pi/j]$ *for which* $\mathscr{F}_j(x_j) = -1$.

Slightly modifying the sequence above we obtain a sequence neither pointwise convergent nor Γ*-convergent. Let* $\mathscr{G}_j(x) = (-1)^j(1 + \mathscr{F}_j(x))$, *then* $\mathscr{G}_j$ *has no* Γ*-limit since* $\Gamma\text{-}\lim_j \mathscr{G}_{2j} = 0$ *while* $\Gamma\text{-}\lim_j \mathscr{G}_{2j+1} = -2$.

Example 2.7. *Consider the function*[2] $\mathscr{G}(x) = -xe^{-x}\mathbf{1}_{[0,+\infty)}(x)$, *and define* $\mathscr{F}_j(x) = \mathscr{G}(jx)$. *Then,* $(\mathscr{F}_j)_{j\in\mathbb{N}}$ *converges pointwise to the constant* 0. *On the other hand, noting that* $\min_{\mathbb{R}} \mathscr{G} = \mathscr{G}(1)$, *the* Γ*-limit is given by* $\mathscr{F}(x) = \mathscr{G}(1)\mathbf{1}_{\{0\}}(x)$ *since* $(\mathscr{F}_j)_{j\in\mathbb{N}}$ *converges uniformly to* 0 *on* $\mathbb{R} \setminus (-\delta, \delta)$ *for all* $\delta > 0$, *and* $\mathrm{argmin}_{\mathbb{R}} \mathscr{F}_j = \{1/j\}$.

The same construction performed in the preceding example provides a sequence pointwise converging and not Γ*-converging.*

Elementary though non trivial features of the theory follow directly from Definition 2.3, they are summarized in the next theorem (for the proofs see [9, Paragraph 1.5] and [25, Chapter 7]).

Theorem 2.8. *Let* $\mathscr{F}_j$, $\mathscr{F} : X \to \overline{\mathbb{R}}$ *with* $\Gamma\text{-}\lim_j \mathscr{F}_j = \mathscr{F}$, *then*

(i) Lower semicontinuity*:* $\mathscr{F}$ *is d-lower semicontinuous on* X*;*

(ii) Stability under continuous perturbations: *if* $\mathscr{G} : X \to \mathbb{R}$ *is continuous, then*

$$\Gamma\text{-}\lim_j(\mathscr{F}_j + \mathscr{G}) = \mathscr{F} + \mathscr{G};$$

(iii) Stability under relaxation: *if* $\overline{\mathscr{F}_j} : X \to \overline{\mathbb{R}}$ *denotes the d-lower semicontinuous envelope of* $\mathscr{F}_j$, *then*

$$\Gamma\text{-}\lim_j \mathscr{G}_j = \mathscr{F}$$

for all $\mathscr{G}_j : X \to \overline{\mathbb{R}}$ *with* $\overline{\mathscr{F}_j} \leq \mathscr{G}_j \leq \mathscr{F}_j$.

Few remarks are in order. Item (ii) usually simplifies the calculation of Γ-limits, since one can drop a large class of ineffective perturbations.

An interesting corollary of item (iii) is that Γ-convergence is not a convergence induced by an underlying topology in the full generality of Definition 2.3. This is clear if considering the constant sequence $\mathscr{F}_j =$

[2]We recall that, the characteristic function of a set A is defined by $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise.

$\mathscr{F}$ for which the Γ-limit is given by the lower semicontinuos envelope of $\mathscr{F}$ itself (cp. with (2.2), (2.3) in subsection 2.2). Instead, on the set of lower semicontinuous functions Γ-convergence is induced by a topology, and even more by a metric if restricting further to equicoercive lower semicontinuous functions (see [25, Chapter 10]).

The lower semicontinuity of Γ-limits in item (i) is one of the many variational features of the theory, that is tailor made to be operated in the setting of Tonelli's direct method as the result below shows clearly.

Definition 2.9. *A family $\mathscr{F}_j : X \to \overline{\mathbb{R}}$ is d-equi-coercive on X, if for all $t \in \mathbb{R}$ $\{\mathscr{F}_j \leq t\} \subseteq K_t$ for every $j \in \mathbb{N}$, with K_t a sequentially compact set.*

The $\mathscr{F}_j$'s are d-equi-mildly coercive on X if for some non-void sequentially compact set $K \subseteq X$

$$\inf_X \mathscr{F}_j = \inf_K \mathscr{F}_j \quad \textit{for all } j \in \mathbb{N}.$$

Remark 2.10. *It is well-known that $(\mathscr{F}_j)_{j\in\mathbb{N}}$ is d-equi-coercive on X if and only if $\mathscr{F}_j(x) \geq \psi(x)$ for some d-lower semicontinuous coercive function $\psi : X \to \overline{\mathbb{R}}$ (see* [25, Proposition 7.7]*).*

Theorem 2.11 (Convergence of global minimizers)**.** *Suppose that $\mathscr{F}_j : X \to \overline{\mathbb{R}}$ are d-equi-mildly coercive, if $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges to $\mathscr{F}$, every sequence $(x_j)_{j\in\mathbb{N}}$ of asymptotic minimizers, i.e.,*

$$\lim_j \left(\mathscr{F}_j\,(x_j) - \inf_X \mathscr{F}_j\right) = 0,$$

is precompact and each cluster point $\bar{x}$ minimizes $\mathscr{F}$ and

$$\lim_j \left(\inf_X \mathscr{F}_j\right) = \mathscr{F}(\bar{x}) = \min_X \mathscr{F}. \tag{2.6}$$

Proof. Let $(x_j)_{j\in\mathbb{N}} \subseteq K$ be asymptotically minimizing, equi-mild coercivity yields that the sequence converges (up to a subsequence not relabeled) to some $\bar{x} \in K$. Then, by (LB) inequality we find

$$\mathscr{F}(\bar{x}) \leq \liminf_j \mathscr{F}_j(x_j) = \liminf_j \inf_X \mathscr{F}_j.$$

In addition, with fixed any $x \in X$, consider a recovery sequence $(y_j)_{j\in\mathbb{N}}$ for x. Hence, we find

$$\mathscr{F}(\bar{x}) \leq \limsup_j \inf_X \mathscr{F}_j \leq \lim_j \mathscr{F}_j(y_j) = \mathscr{F}(x).$$

By collecting the previous inequalities we get

$$\mathscr{F}(\bar{x}) = \min_X F = \lim_j \inf_X \mathscr{F}_j.$$

□

The variational nature of Γ-convergence is now evident. Theorem 2.11 expresses the convergence of the minimum problems related to the $\mathscr{F}_j$'s to the corresponding one for $\mathscr{F}$, and moreover the existence of solutions to the latter problem is also guaranteed. Thus, Theorem 2.11 can be considered as a 'sequential' version of Theorem 2.2. Actually, this link can be made more precise by appealing to the abstract definition of Γ-convergence (cp. with [30]). Thus, the same comments to select a metric for the ambient space X under which the Γ-convergence of $(\mathscr{F}_j)_{j\in\mathbb{N}}$ has to be investigated are pertinent. Two competing needs have to be balanced in choosing the metric: on one hand a topology with 'few' open sets ensures easily that any sequence of almost minimizers $(x_j)_{j\in\mathbb{N}}$ is d-precompact, on the other hand Γ-convergence is more likely to occur provided the topology is 'rich' of open sets in order to have 'few' converging sequences on which (LB) and (UB) inequalities have to be tested.

Thus, the metric on X is one of the unknown of the problem under study and has to be carefully selected relying upon the considerations above.

There are two main ways in which Theorem 2.11 can be exploited in applications: first when studying variational models in which a family of functionals is given, representing for instance a physical energy depending on some small scale parameter, often a more elementary and tractable model is needed for theoretical or computational reasons. Dealing with variational models finding the stable states of the system is the main object of investigation. Hence, Theorem 2.11 gives a first answer solving the case of global minimizers and providing an 'effective' model to which the original ones are asymptotically close to.

On the other way round, if dealing with a variational model in which the relevant energy is difficult to be studied either directly or numerically, one can try to build up more elementary approximating models in the variational sense provided by Γ-convergence and infer qualitative properties of the minimizers of the problem of interest from those of the approximating ones.

Eventually, we stress that *global* minimizers are only a subset of the broader class of stable states which are central object of interest in mechanical models. Hence, it is of some importance to infer the asymptotic behaviour of sequences of those points from Γ-convergence type arguments, say at least for the wider subclass of (strict) *local* minimizers. This issue is highly nontrivial, a complete answer has not been found, yet (see for instance [12],[43],[46]). We shall discuss about this topic in more details in Section 2.5.

2.3. **Further properties.** Several alternative characterizations of Γ-convergence are available, new definitions are needed in order to introduce them.

Definition 2.12. *Given a sequence $\mathscr{F}_j : X \to \overline{\mathbb{R}}$ define*

$$\Gamma\text{-}\liminf_j \mathscr{F}_j(x) = \inf\{\liminf_j \mathscr{F}_j(x_j) : \, x_j \xrightarrow{d} x\}, \tag{2.7}$$

$$\Gamma\text{-}\limsup_j \mathscr{F}_j(x) = \inf\{\limsup_j \mathscr{F}_j(x_j) : \, x_j \xrightarrow{d} x\}. \tag{2.8}$$

Example 2.13. *Let us consider the case of constant functions, i.e. $\mathscr{F}_j(x) = \alpha_j$ for all $x \in X$. Then, for all $x \in X$, $\Gamma\text{-}\liminf_j \mathscr{F}_j(x) = \liminf_j \alpha_j$ and $\Gamma\text{-}\limsup_j \mathscr{F}_j(x) = \limsup_j \alpha_j$.*

Remark 2.14. *If d and d' are two metrics on X with the topology induced by d finer than that induced by d', i.e. d-convergent sequences in X are also d'-convergent, then*

$$\Gamma(d')\text{-}\liminf_j \mathscr{F}_j \leq \Gamma(d)\text{-}\liminf_j \mathscr{F}_j, \quad \Gamma(d')\text{-}\limsup_j \mathscr{F}_j \leq \Gamma(d)\text{-}\liminf_j \mathscr{F}_j.$$

A direct check shows that both $\Gamma\text{-}\liminf_j \mathscr{F}_j$ and $\Gamma\text{-}\limsup_j \mathscr{F}_j$ are d-lower semicontinuous functions. With this remark at hand it is elementary to show the next result (see [9, Theorem 1.17] and [25, Proposition 4.15]).

Theorem 2.15 (Equivalent definitions)**.** *The following conditions are equivalent*

(i) $\mathscr{F} = \Gamma\text{-}\lim_j \mathscr{F}_j$;

(ii) $\mathscr{F} = \Gamma\text{-}\liminf_j \mathscr{F}_j = \Gamma\text{-}\limsup_j \mathscr{F}_j$;

(iii) *$\mathscr{F}$ satisfies inequality (LB) inequality, and for all $x \in X$ and $\delta > 0$*

$$\mathscr{F}(x) + \delta \geq \limsup_j \mathscr{F}_j(y_j) \tag{2.9}$$

for some sequence $y_j \xrightarrow{d} x$;

(iv) *$\mathscr{F}$ satisfies inequality (LB) inequality, and (2.9) holds on a dense set in X;*

(v) *for all $x \in X$*

$$\mathscr{F}(x) = \sup_{\delta>0} \liminf_j \inf\{\mathscr{F}_j(y) : \, d(x,y) < \delta\}$$
$$= \sup_{\delta>0} \limsup_j \inf\{\mathscr{F}_j(y) : \, d(x,y) < \delta\};$$

(vi) *$\chi_{\mathrm{epi}(\mathscr{F})} = \Gamma\text{-}\lim_j \chi_{\mathrm{epi}(\mathscr{F}_j)}$ in the product metric of $X \times \mathbb{R}$. Here,*

$$\mathrm{epi}(\mathscr{G}) = \{(x,t) \in X \times \mathbb{R} : \, \mathscr{G}(x) \leq t\},$$

is the epigraph of $\mathscr{G} : X \to \overline{\mathbb{R}}$, and χ_E is the indicator function of the set E, i.e., $\chi_E(x) = 0$ if $x \in E$ and $+\infty$ if $x \notin E$.

The last but one item, if properly reformulated in terms of neighbourhoods, is the definition of Γ-convergence in the general setting of a topological space. The last item above instead motivates the terminology *epi-convergence* that is used in some literature, it is related to Kuratowski set convergence (see for instance [7]).

Another characterization in terms of the convergence of Moreau-Yosida approximations is given in [10, Section 1.4]. Here, we shall deal only with (equi-coercive) convex functions on a reflexive Banach space, for those a particularly neat characterization holds true.

Let us first discuss some topological properties of the theory. A very useful feature is that Γ-convergence always occurs upon the extraction of subsequences provided separability of X is assumed (see [25, Theorem 8.5]).

Proposition 2.16 (Compactness)**.** *If (X, d) is a separable metric space, then any sequence $(\mathscr{F}_j)_{j\in\mathbb{N}}$ contains a Γ-convergent subsequence.*

Combining the latter result with the following statement provides an abstract criterion to check Γ-convergence (see [25, Proposition 8.3]).

Proposition 2.17 (Urysohn property)**.** *A sequence $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges to some function $\mathscr{F}$ if and only if every subsequence $(\mathscr{F}_{j_k})_{k\in\mathbb{N}}$ contains a further subsequence which Γ-converges to $\mathscr{F}$.*

In view of Propositions 2.16 and 2.17, we can characterize Γ-convergence for (equi-coercive) convex functions by means of the convergence of linearly perturbed minimum problems following Marcellini [49] (see also [8]).

Proposition 2.18. *Let X be a (real) reflexive and separable Banach space, denote by $\mathscr{C}_\psi$ the class of convex and lower semicontinuous functions $\mathscr{F} : X \to (-\infty, +\infty]$ not identically $+\infty$ (i.e. proper) and satisfying*

$$\psi(x) \le \mathscr{F}(x) \quad \text{for all } x \in X, \tag{2.10}$$

for a proper, convex and lower semicontinuous function ψ super linear at infinity, i.e.

$$\lim_{\|x\|\to+\infty} \frac{\psi(x)}{\|x\|} = +\infty.$$

If $(\mathscr{F}_j)_{j\in\mathbb{N}} \subset \mathscr{C}_\psi$ then

$$\mathscr{F} = \Gamma\text{-}\lim_j \mathscr{F}_j \Longleftrightarrow \lim_j \min_X \{\mathscr{F}_j + \mathscr{G}\} = \min_X \{\mathscr{F} + \mathscr{G}\} \tag{2.11}$$

for all $\mathscr{G} : X \to \mathbb{R}$ *linear and continuous, and the* Γ*-limit* $\mathscr{F}$ *belongs to* $\mathscr{C}_\psi$ *as well.*

Before giving the proof it is convenient to rephrase (2.11) in terms of the pointwise convergence of Fenchel conjugates. Recall that for a proper convex function $\mathscr{H} : X \to (-\infty, +\infty]$, the Fenchel conjugate is the convex function $\mathscr{H}^* : X^* \to (-\infty, +\infty]$ given by

$$\mathscr{H}^*(z) := \sup_X \{\langle z, x\rangle - \mathscr{H}(x)\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $X^* \times X$. Note that

$$\mathscr{H}^*(-z) = -\inf_X \{\langle z, x\rangle + \mathscr{H}(x)\},$$

thus if $\mathscr{H} \in \mathscr{C}_\psi$ the existence of a minimizer for the problem above is guaranteed by the super linearity of ψ and it holds

$$\mathscr{H}^*(-z) = -\min_X \{\langle z, x\rangle + \mathscr{H}(x)\} \leq \psi^*(-z).$$

Then, clearly, (2.11) rewrites as

$$\mathscr{F} = \Gamma\text{-}\lim_j \mathscr{F}_j \iff \lim_j \mathscr{F}_j^*(z) = \mathscr{F}^*(z) \quad \text{for all } z \in X^* \tag{2.12}$$

Proof. Condition (2.10) and the super linearity of ψ imply the equicoercivity of $(\mathscr{F}_j + \mathscr{G})_{j\in\mathbb{N}}$, so that the direct implication of the thesis follows from property (ii) of Proposition 2.8 and Theorem 2.11.

To prove the opposite implication we fix a subsequence $(\mathscr{F}_{j_k})_{k\in\mathbb{N}}$ and use Proposition 2.16 to extract a further subsequence $(\mathscr{F}_{j_{k_h}})_{h\in\mathbb{N}}$ Γ-converging to some $\mathscr{H}$ lower semicontinuous and convex (see Proposition 2.19 below for the latter property).

On one hand, by arguing as above, $(\mathscr{F}^*_{j_{k_h}})_{h\in\mathbb{N}}$ converges pointwise to $\mathscr{H}^*$, on the other hand by assumption $(\mathscr{F}^*_{j_{k_h}})_{h\in\mathbb{N}}$ converges to $\mathscr{F}^*$, then $\mathscr{H}^* = \mathscr{F}^*$. In turn, from this, from the reflexivity of X, and from the convexity and the lower semicontinuity of $\mathscr{F}$, we infer equality $\mathscr{H} = \mathscr{F}$. Hence, the Γ-convergence of the whole sequence $(\mathscr{F}_j)_{j\in\mathbb{N}}$ to $\mathscr{F}$ follows from Proposition 2.17. □

Examples 2.4-2.6 shows that Γ-limits and pointwise limits are usually unrelated. However, some pointwise properties are preserved along the Γ-limit process (see [25, Chapter 11]).

Proposition 2.19. *Let* X *be a Banach space. If* $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ*-converges to* $\mathscr{F}$*, and if* $\mathscr{F}_j$ *is either convex or even or positively homogeneous of degree* α*, then* $\mathscr{F}$ *is respectively convex, even, positively homogeneous of degree* α*.*

In addition, if X *is a Hilbert space and each* $\mathscr{F}_j$ *is a quadratic form, then* $\mathscr{F}$ *is a quadratic form as well.*

We remark that the natural setting for many applications is that of a Banach space X endowed with the weak topology $\sigma(X, X^*)$. For, norm bounded sets are pre-compact in that topology, thus coercivity can be ensured simply by imposing suitable growth conditions on the energy. In that framework, a notion of sequential Γ-convergence one can be introduced exactly according to Definition 2.3 simply testing (LB) and (UB) inequalities on weakly convergent sequences. The drawback is that in such a case many of the properties we have quoted so far no longer satisfied, for instance the limit function is only sequentially weak lower semicontinuous but not lower semicontinuous in general. Despite this, the standard metric space setting is recovered in view of the ensuing result (see [25, Proposition 8.7, Corollary 8.8, Proposition 8.10]).

Proposition 2.20. *If X is a Banach space with separable dual X^* there exists a metric d on X inducing on each norm bounded set the topology $\sigma(X, X^*)$.*

In particular, let $\mathscr{F}_j : X \to \overline{\mathbb{R}}$ for which there exists $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(t) \uparrow +\infty$ as $t \uparrow +\infty$, and $\mathscr{F}_j(x) \geq \psi(\|x\|)$ for all $j \in \mathbb{N}$, then

$$\Gamma(\sigma(X, X^*))\text{-}\lim_j \mathscr{F}_j = \Gamma(d)\text{-}\lim_j \mathscr{F}_j.$$

In what follows, given a Banach space X, $\Gamma(X)$, $\Gamma(w\text{-}X)$ shall denote Γ-limits in the strong and weak topology of X, respectively.

Eventually, we provide some examples that will be of interest in the subsequent sections and for which the result above is instrumental (see Section 3).

Example 2.21. *Consider the quadratic forms $\mathscr{F}_j : L^2(\Omega, \mathbb{R}^m) \to [0, +\infty)$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, defined as*

$$\mathscr{F}_j(u) := \int_\Omega \langle a_j(x)u(x), u(x)\rangle \, dx,$$

with $a_j \in L^\infty(\Omega, \mathrm{Sym}(m))$ and satisfying $\alpha \, \mathrm{Id}_m \leq a_j(x) \leq \beta \, \mathrm{Id}_m$ $\mathcal{L}^n$ a.e. on Ω. Here, $\mathrm{Sym}(m)$ denotes the set of $m \times m$ symmetric matrices, and Id_m is the $m \times m$ identity matrix.

The growth assumptions on a_j's guarantee that (up to subsequences not relabeled) $(a_j)_{j\in\mathbb{N}}$ and $(a_j^{-1})_{j\in\mathbb{N}}$ converge weakly in $L^\infty(\Omega, \mathrm{Sym}(m))$ to matrices a and b^{-1}, respectively. It is then easy to check that the pointwise limit and the $\Gamma(L^2)$-limit of $(\mathscr{F}_j)_{j\in\mathbb{N}}$ is*

$$\mathscr{F}(u) := \int_\Omega \langle a(x)u(x), u(x)\rangle \, dx.$$

Instead, we claim that the $\Gamma(w\text{-}L^2)$*-limit of* $(\mathscr{F}_j)_{j\in\mathbb{N}}$ *is given by*

$$\mathscr{G}(u) := \int_\Omega \langle b(x)u(x), u(x)\rangle\, dx.$$

Note that Proposition 2.20 applies thanks to the separability and reflexivity of L^2 *and the assumptions on* a_j*'s.*

To give evidence to our claim, fix $f \in L^2(\Omega, \mathbb{R}^m)$ *and consider the perturbed functionals*

$$\mathscr{G}_j(u) := \mathscr{F}_j(u) - 2\int_\Omega \langle f(x), u(x)\rangle\, dx.$$

The asymptotic behaviour of the related minimizers w_j *is easily determined. For, by the Dirichlet principle* $w_j = a_j^{-1} f$*, and thus* $(w_j)_{j\in\mathbb{N}}$ *converges weakly to* $b^{-1}f$*. Supposing that* $\Gamma(w\text{-}L^2)\lim_j \mathscr{F}_j$ *exists (this is always true up to subsequences thanks to Proposition 2.16), item (ii) in Theorem 2.8 implies that*

$$\Gamma(w\text{-}L^2)\lim_j \mathscr{G}_j = \Gamma(w\text{-}L^2)\lim_j \mathscr{F}_j - 2\int_\Omega \langle f(x), u(x)\rangle\, dx.$$

In particular, for all $f \in L^2(\Omega, \mathbb{R}^m)$*, the right hand side above is a linearly perturbed quadratic form (cp. with Proposition 2.19) with minimizer* $b^{-1}f$*. In conclusion, the quadratic form* $\mathscr{G}$ *identifies the* Γ*-limit of* $(\mathscr{F}_j)_{j\in\mathbb{N}}$ *thanks to Propositions 2.17 and 2.18.*

Alternatively, we prove the claim directly from the definition by checking inequalities (UB) and (LB) inequalities. Fix $u \in L^2(\Omega, \mathbb{R}^m)$*, for the former a simple computation shows that a recovery sequence is given by* $u_j := a_j^{-1} b u$*. For,* $(u_j)_{j\in\mathbb{N}}$ *converges to* u *weakly in* $L^2(\Omega, \mathbb{R}^m)$*, and moreover*

$$\mathscr{F}_j(u_j) = \int_\Omega \langle b(x)u(x), a_j^{-1}(x) b(x) u(x)\rangle\, dx, \tag{2.13}$$

so that $\lim_j \mathscr{F}_j(u_j) = \mathscr{G}(u)$*. Furthermore, to prove (LB) inequality, consider the bilinear forms related to* $\mathscr{F}_j$ *and* $\mathscr{G}$*, given respectively by*

$$\mathcal{B}_j(v, w) := \int_\Omega \langle a_j(x)v(x), w(x)\rangle\, dx, \quad \mathcal{B}(v, w) := \int_\Omega \langle b(x)v(x), w(x)\rangle\, dx.$$

Note that by the parallelogram law and the polarization identity we have

$$\mathscr{F}_j(v) + \mathscr{F}_j(u_j) \geq 2\mathcal{B}_j(u_j, v) = 2\mathcal{B}(u, v).$$

In turn, from this we get for every sequence $(v_j)_{j\in\mathbb{N}}$ *converging weakly in* $L^2(\Omega, \mathbb{R}^m)$ *to* u

$$\liminf_j \mathscr{F}_j(v_j) \geq \lim_j 2\mathcal{B}(u, v_j) - \lim_j \mathscr{F}_j(u_j) = 2\mathcal{B}(u, u) - \mathscr{G}(u) = \mathscr{G}(u). \tag{2.14}$$

In the periodic setting the limits can be calculated explicitly. For, let $B \in L^\infty(\Omega \times \mathbb{R}^n, \mathrm{Sym}(m))$*, with* $B(x, \cdot)$ $[0,1]^n$*-periodic for* $\mathcal{L}^n$ *a.e. in* Ω*, and such that*

$$\alpha\,\mathrm{Id}_m \le B(x,y) \quad \mathcal{L}^{n\times n} \text{ a.e. in } \Omega \times \mathbb{R}^n.$$

Symmetric matrices are then defined as

$$a_j(y) := \fint_{\varepsilon_j([y/\varepsilon_j]+[0,1]^n)} B\left(x, \frac{y}{\varepsilon_j}\right) dx,$$

where $[t]$ *stands for the integer part of the real number* t*, and*

$$a(y) := \int_{[0,1]^n} a(x,y)\,dy, \quad b(y) := \left(\int_{[0,1]^n} a^{-1}(x,y)\,dy\right)^{-1}.$$

Under the quoted assumptions on B*, one can show that* $(a_j)_{j\in\mathbb{N}}$*,* $(a_j^{-1})_{j\in\mathbb{N}}$ *converge to* a *and* b^{-1} *weakly** $L^\infty(\Omega, \mathrm{Sym}(m))$*, respectively (for a proof see* [52, Proposition 3.1]*).*

Example 2.22. *We build upon Example 2.21 and show that a similar result holds in the one-dimensional case for quadratic forms defined on Sobolev spaces. A detailed study of this subject in any dimension shall be outlined in Section 3.*

We keep using the notation introduced in Example 2.21, restricting the analysis to the one-dimensional case $n = 1$*. Define* $\mathscr{H}$*,* $\mathscr{H}_j$: $W^{1,2}((0,L);\mathbb{R}^m) \to \mathbb{R}$ *respectively by*

$$\mathscr{H}_j(u) := \mathscr{F}_j(u'), \text{ and } \mathscr{H}(u) := \mathscr{G}(u').$$

We claim that $\Gamma(w\text{-}W^{1,2})\text{-}\lim_j \mathscr{H}_j = \mathscr{H}$ *on* $W^{1,2}((0,L);\mathbb{R}^m)$*. Indeed, (LB) inequality follows directly from the analogous inequality for the* $\mathscr{F}_j$*'s. Instead, to show (UB) inequality for a given* u *in* $W^{1,2}((0,L);\mathbb{R}^m)$*, it is sufficient to take the sequence*

$$u_j(x) := u(0) + \int_0^x a_j^{-1}(y) b(y) u'(y)\,dy,$$

a direct computation provides the conclusion.

Moreover, the same result is obtained if considering the strong L^2 *topology, having extended the functionals* $\mathscr{H}_j$ *to* $+\infty$ *on* $L^2 \setminus W^{1,2}$*. This easily follows from Rellich-Kondrakov compact embedding theorem from which the equi-coercivity of the family* $(\mathscr{H}_j)_{j\in\mathbb{N}}$ *is deduced.*

Finally, note that the problem is compatible with the addition of boundary conditions. For instance, if we consider the restrictions of $\mathscr{H}_j$ *to* $W_0^{1,2}((0,L);\mathbb{R}^m)$ *(still denoted by* $\mathscr{H}_j$*), (LB) inequality is satisfied all the more so, and for every function* u *a recovery sequence is* *given by*

$$u_j(x) := \int_0^x a_j^{-1}(y)b(y)u'(y)\,dy - x \fint_0^L a_j^{-1}(y)b(y)u'(y)\,dy.$$

Example 2.23. *We analyze the analogous problem studied in Examples 2.21 and 2.22 for nonlocal quadratic forms defined on fractional Sobolev spaces. We shall prove that the answer is rather easy in any dimension.*

Let $\Delta := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ be the diagonal set in $\mathbb{R}^n \times \mathbb{R}^n$; given measurable kernels $K_j : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to [0,+\infty)$ satisfying

$$\alpha|x-y|^{-(n+2s)} \leq K_j(x,y) \leq \beta|x-y|^{-(n+2s)} \tag{2.15}$$

for all admissible (x,y), for some $0 < \alpha \leq \beta$ and $s \in (0,1)$, define the energies $\mathscr{K}_j : L^2(\Omega) \to [0,+\infty]$ by

$$\mathscr{K}_j(u) := \int_{\Omega\times\Omega} K_j(x,y)|u(x)-u(y)|^2 dxdy \quad u \in W^{s,2}(\Omega),$$

$+\infty$ otherwise. Here, $W^{s,2}(\Omega)$ is the fractional Sobolev space of order s (see Section 5 for more details) on the regular open set $\Omega \subseteq \mathbb{R}^n$.

With fixed $\delta > 0$, let $\Delta_\delta := \{(x,y) \in \Omega\times\Omega : \mathrm{dist}((x,y),\Delta) \leq \delta\}$, we claim that if $K_j \to K$ weak $L^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \triangle_\delta)$ for all $\delta > 0$, then*

$$\Gamma(L^2)\text{-}\lim_j \mathscr{K}_j(u) = \mathscr{K}(u) := \int_{\Omega\times\Omega} K(x,y)|u(x)-u(y)|^2 dxdy \quad u \in W^{s,2}(\Omega),$$

$+\infty$ otherwise. Note that the convergence assumption on $(K_j)_{j\in\mathbb{N}}$ is always satisfied up to subsequences thanks to the growth conditions in (2.15) and a diagonal argument.

In addition, the estimates in (2.15), together with the compact embedding of $W^{s,2}(\Omega)$ into $L^2(\Omega)$ and Proposition 2.20, yield the same Γ-convergence result with respect to the weak $W^{s,2}$-topology.

To enforce inequality

$$\Gamma(L^2)\text{-}\liminf_j \mathscr{K}_j(u) \geq \mathscr{K}(u),$$

first notice that given a sequence $(u_j)_{j\in\mathbb{N}}$ strongly converging in $L^2(\Omega)$ to some u with $\sup_j \mathscr{K}_j(u_j) < +\infty$, then actually $u \in W^{s,2}(\Omega)$. In addition, for all $h \in \mathbb{N}$ the truncations $u_j^h := u_j \wedge h \vee (-h)$ converge in $L^2(\Omega)$ to $u^h := u \wedge h \vee (-h)$, and being the energies decreasing by truncations, we infer for all $\delta > 0$ and $h \in \mathbb{N}$

$$\liminf_j \mathscr{K}_j(u_j) \geq \liminf_j \int_{\Omega\times\Omega\setminus\triangle_\delta} K_j(x,y)|u_j^h(x) - u_j^h(y)|^2 dxdy$$

$$= \int_{\Omega\times\Omega\setminus\triangle_\delta} K(x,y)|u^h(x)-u^h(y)|^2 dxdy,$$

where the last equality is a consequence of the convergences $K_j \to K$ *weak** $L^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\triangle_\delta)$ *and* $u^h_j \to u^h$ *in* $L^2(\Omega)$. *The conclusion then follows by letting* $\delta \downarrow 0^+$ *and* $h \uparrow +\infty$ *in the latter estimate thanks to Lebesgue dominated convergence theorem.*

Eventually, the estimate

$$\Gamma(L^2)\text{-}\limsup_j \mathscr{K}_j(u) \le \mathscr{K}(u)$$

follows easily by taking the constant sequence equal to the function u *itself if the latter is Lipschitz continuous. Indeed, denoting by* $L>0$ *the Lipschitz constant of* u, *from* (2.15) *we infer for all* $j\in\mathbb{N}$

$$0 \le K_j(x,y)|u(x)-u(y)|^2 \le \beta L^2 |x-y|^{-(n+2(s-1))}.$$

Since the function on the right hand side above belongs to $L^1(\Omega\times\Omega)$, *being* $s\in(0,1)$, *we deduce*

$$\lim_j \mathscr{K}_j(u) = \mathscr{K}(u) \tag{2.16}$$

by Lebesgue dominated convergence theorem. In general, to conclude we use the lower semicontinuity of $\Gamma\text{-}\limsup_j \mathscr{K}_j$ *and the continuity of* $\mathscr{K}$, *together with the density of Lipschitz functions in the strong topology of* $W^{s,2}(\Omega)$. *More precisely, being* Ω *a regular open set, we can find a sequence* $(u_k)_{k\in\mathbb{N}}$ *of Lipschitz functions converging strongly to* u *in* $W^{s,2}(\Omega)$, *and infer*

$$\Gamma(L^2)\text{-}\limsup_j \mathscr{K}_j(u) \le \liminf_k \Big(\Gamma(L^2)\text{-}\limsup_j \mathscr{K}_j(u_k)\Big) \overset{(2.16)}{=} \lim_k \mathscr{K}(u_k) = \mathscr{K}(u).$$

2.4. **Asymptotic Γ-development.** Theorem 2.11 together with the upper bound inequality (UB) yield that *every* minimum point of the Γ-limit is the limit point of an asymptotically minimizing sequence, but in general it is *not* the limit of a sequence of minimum points. The following simple example clarifies the stage.

Example 2.24. *The sequence* $\mathscr{F}_j(x) = x^2/j$ Γ*-converges to the constant* 0, *thanks to local uniform convergence on* $\mathbb{R}$. *On the other hand, being* $\mathrm{argmin}_{\mathbb{R}}\mathscr{F}_j = \{0\}$ *the only minimizer of the limit reached by those of* $\mathscr{F}_j$*'s is* $x=0$.

In particular, this shows that in principle only some among the minimizers of the Γ-limit are limits of the minimizers of the approximating functions. The identification of those peculiar minima is particularly significant when analyzing static mechanical models obtained as variational limits of approximating ones.

A selection criterion designed expressly for this issue has been proposed by Anzellotti and Baldo [5]. The underlying idea is to define a *development by* Γ*-convergence* in such a way that it corresponds to a Taylor formula for the minimum values of the $\mathscr{F}_j$'s. Note that the zero order approximation has already been identified in (2.6). We quote the result of interest here referring to [5, Theorem 1.2] for a proof (see also [9, Theorem 1.47]).

Theorem 2.25 (Asymptotic Γ-development)**.** *Let $\mathscr{F}_j$, $\mathscr{F} : X \to \overline{\mathbb{R}}$ be such that $\Gamma(d)\text{-}\lim_j \mathscr{F}_j = F$, with $\mathscr{F}_j$ lower semicontinuous and d-equicoercive on X.*

Let $\alpha_j \downarrow 0^+$, and $\mathscr{G}_j : X \to \overline{\mathbb{R}}$ be given by

$$\mathscr{G}_j(x) := \frac{\mathscr{F}_j(x) - \min_X \mathscr{F}}{\alpha_j}.$$

If d' is a metric not weaker than d, the $\mathscr{G}_j$'s are d'-equicoercive on X and $(\mathscr{G}_j)_{j\in\mathbb{N}}$ $\Gamma(d')$-converges to $\mathscr{G}$, with $\mathscr{G} \neq +\infty$, then each sequence $(x_j)_{j\in\mathbb{N}}$, with x_j minimum point of $\mathscr{F}_j$, d'-converges (up to subsequences) to a minimum point of $\mathscr{F}$ and $\mathscr{G}$, and in addition

$$\min_X \mathscr{F}_j = \min_X \mathscr{F} + \alpha_j \min_X \mathscr{G} + o(\alpha_j) \qquad \text{as } j \uparrow +\infty. \tag{2.17}$$

The function $\mathscr{G}$ is called a *higher* order Γ-limit of $(\mathscr{F}_j)_{j\in\mathbb{N}}$, and the notation $\mathscr{F}_j =_\Gamma \mathscr{F} + \alpha_j \mathscr{G}$ is employed in literature to emphasize that. We remark that the latter equality has the *only* meaning that $\Gamma\text{-}\lim_j \mathscr{F}_j = \mathscr{F}$ and $\Gamma\text{-}\lim_j \mathscr{G}_j = \mathscr{G}$, in a way that the expansion (2.17) holds true. In addition, notice that the domain of $\mathscr{G}$ is a subset of the set of minima of $\mathscr{F}$, that is

$$\mathscr{G}(x) \in \mathbb{R} \Longrightarrow \mathscr{F}(x) = \min_X \mathscr{F}.$$

In particular, the limits of minimizers of the $\mathscr{F}_j$'s minimize both $\mathscr{F}$ and $\mathscr{G}$, in this way a selection criterion for the minimizers of $\mathscr{F}$ is established.

Remark 2.26. *The asymptotic* Γ*-development can be iterated to get a better degree of accuracy in the expansion* (2.17) *and further pieces of information on the limits of minimizers of the $\mathscr{F}_j$'s. More precisely, given vanishing sequences $(\alpha_j^{(i)})_{j\in\mathbb{N}}$, with $\alpha_j^{(i)} \ll \alpha_j^{(i-1)}$, and functionals $\mathscr{F}^{(i)} : X \to \overline{\mathbb{R}}$, $1 \leq i \leq k$, equality*

$$\mathscr{F}_j =_\Gamma \mathscr{F}^{(0)} + \alpha_j^{(1)} \mathscr{F}^{(1)} + \alpha_j^{(2)} \mathscr{F}^{(2)} + \ldots + \alpha_j^{(k)} \mathscr{F}^{(k)}$$

means that $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ*-converges to* $\mathscr{F}^{(0)}$ *and that if*

$$\mathscr{F}_j^{(i)}(x) := \frac{\mathscr{F}_j^{(i-1)}(x) - \min_X \mathscr{F}^{(i-1)}}{\alpha_j^{(i)}},$$

then $\mathscr{F}^{(i)} = \Gamma\text{-}\lim_j \mathscr{F}_j^{(i)}$*. In particular, the Taylor expansion*

$$\min_X \mathscr{F}_j = \min_X \mathscr{F}^{(0)} + \alpha_j^{(1)} \min_X \mathscr{F}^{(1)} + \ldots + \alpha_j^{(k)} \min_X \mathscr{F}^{(k)} + o(\alpha_j^{(k)}) \quad \text{as } j \uparrow +\infty$$

is justified. Moreover, $\mathscr{F}^{(i)}(x) \in \mathbb{R}$ *implies that* x *minimizes* $\mathscr{F}^{(i-1)}$*, so that if* x_j *is a minimizer of* $\mathscr{F}_j$ *over* X *and* x *a cluster point of the sequence* $(x_j)_{j\in\mathbb{N}}$*, then* x *minimizes each one of the* $\mathscr{F}^{(i)}$*'s,* $0 \le i \le k$*.*

The most significant hypotheses to be satisfied to apply Theorem 2.25 are related to the identification of the metric d' and to that of the scaling α_j not to get trivial limits. For what the second issue is concerned, it is again an intrinsic feature of the problem under investigation and so there is no a priori ansatz on it.

In this respect, note that the Γ-limit, $\mathscr{H}$, of

$$\mathscr{H}_j(x) := \frac{\mathscr{F}_j(x) - \min_X \mathscr{F}}{\beta_j}$$

might be non-trivial for some intermediate scaling $(\beta_j)_{j\in\mathbb{N}}$, i.e. $\alpha_j \ll \beta_j \ll 1$. Then, necessarily, $\min_X \mathscr{H} = 0$, and the scale β_j plays no role in the minimum value expansion (2.17).

Remark 2.27. *Let us emphasize that the procedure outlined in Theorem 2.25 makes sense only for (local) minimum points. More precisely, there is no sensible analogous development in the neighbourhood of any point which is not a local minimizer of the zeroth order* Γ*-limit. In what follows I shall resume the contents of a discussion with G. Dal Maso, I. Fonseca and G. Leoni at CNA in Pittsburgh.*

Keeping the notation introduced in Theorem 2.25, fix any positive, vanishing and decreasing sequence (the latter condition is imposed only for the sake of simplicity) $(\alpha_j)_{j\in\mathbb{N}}$ *and a point* z *in the domain of* $\mathscr{F}$*. Set by analogy*

$$\mathscr{G}_j(x) := \frac{\mathscr{F}_j(x) - \mathscr{F}(z)}{\alpha_j},$$

we shall prove that if z *is not a local minimizer of* $\mathscr{F}$ *then*

$$\Gamma\text{-}\lim_j \mathscr{G}_j(z) = -\infty. \tag{2.18}$$

In passing, we note that $\Gamma\text{-}\lim_j \mathscr{G}_j(x) \in \{\pm\infty\}$*, if* $x \neq z$*, and that (LB) inequality is trivially satisfied.*

Assume that z is not a local minimizer of $\mathscr{F}$, then find a sequence of points z_k converging to z and such that $\mathscr{F}(z_k) < \mathscr{F}(z)$. By lower semicontinuity of $\mathscr{F}$, the positive sequence $(\mathscr{F}(z) - \mathscr{F}(z_k))_{k\in\mathbb{N}}$ is infinitesimal. For all j sufficiently big we can find $k_j \in \mathbb{N}$ such that

$$\mathscr{F}(z_{k_j}) \leq \mathscr{F}(z) - 2\alpha_j,$$

and consequently define a subsequence $k_j \uparrow +\infty$.

For every $j \in \mathbb{N}$, consider a recovery sequence $(z_j^h)_{h\in\mathbb{N}}$ for z_{k_j} and select $m_j \in \mathbb{N}$ such that

$$\mathscr{F}_h(z_j^h) \leq \mathscr{F}(z_{k_j}) + \alpha_j \quad \textit{and} \quad d(z_j^h, z_{k_j}) \leq 1/j \qquad \textit{for all } h \geq m_j.$$

From the choices above we infer

$$\mathscr{F}_h(z_j^h) - \mathscr{F}(z) \leq \mathscr{F}(z_{k_j}) - \mathscr{F}(z) + \alpha_j \leq -\alpha_j,$$

as well as

$$d(z_j^h, z) \leq 1/j + d(z_{k_j}, z) \quad \textit{if } h \geq m_j.$$

Furthermore, take a subsequence $h_j \uparrow +\infty$, $h_j \geq m_j$, such that $\alpha_j/\alpha_{h_j} \uparrow +\infty$ as $j \uparrow +\infty$. In conclusion, setting $x_j := z_j^{h_j}$, $(x_j)_{j\in\mathbb{N}}$ converges to z and

$$\mathscr{G}_{h_j}(x_j) \leq -\frac{\alpha_j}{\alpha_{h_j}} \Longrightarrow \lim_j \mathscr{G}_{h_j}(x_j) = -\infty.$$

Finally, since the previous argument can be repeated for every extracted subsequence $(\mathscr{G}_{j_n})_{n\in\mathbb{N}}$ of $(\mathscr{G}_j)_{j\in\mathbb{N}}$, Urysohn property gives (UB) inequality, so that (2.18) follows at once.

The systematic use of Γ-development envisages some drawbacks that shall be outlined below working out some examples. We refer to the paper by Braides and Truskinovsky [13] for more in-depth insights complemented with the discussion of several mechanical models and exhaustive interpretations of the phenomena occurring.

First of all a trivial remark: there is no reason for $\min_X \mathscr{F}_j$ to be an analytic function (even if the development is finite), so that even having at disposal the asymptotic expansion of $\min_X \mathscr{F}_j$ at every order may not provide an exact description of $\min_X \mathscr{F}_j$ itself. The next example suggested in [6] clarifies the stage.

Example 2.28. *Consider the functionals $\mathscr{F}_j : L^2(-1,1) \to [0,+\infty)$ defined as*

$$\mathscr{F}_j(u) := \int_{-1}^{1} \left(|u|^2 + \varepsilon_j^2|u'|^2\right) dx, \quad u \in W^{1,2}(-1,1), \ \textit{and } u(-1) = u(1) = 1,$$

$+\infty$ *otherwise. It is easy to infer that for all* $u \in L^2(-1,1)$

$$\Gamma(L^2)\text{-}\lim_j \mathscr{F}_j(u) =: \mathscr{F}^{(0)}(u) = \int_{-1}^{1} |u|^2 dx,$$

being inequality (LB) inequality trivial, and inequality (UB) deduced from the density of $C_c^\infty(-1,1)$ *functions in* $L^2(-1,1)$ *in the usual metric.*

It is evident that the unique minimizer of $\mathscr{F}^{(0)}$ *is the null function; moreover solving directly the Euler-Lagrange equation gives that the minimizer of* $\mathscr{F}_j$ *is the function*

$$u_j(x) = \frac{\cosh(\varepsilon_j^{-1}x)}{\cosh(\varepsilon_j^{-1})}.$$

Eventually, a direct calculation leads to

$$\min_{L^2} \mathscr{F}_j = \mathscr{F}_j(u_j) = \varepsilon_j \frac{\cosh(2\varepsilon_j^{-1})}{\cosh^2(\varepsilon_j^{-1})} = 2\varepsilon_j + \omega_j \quad \text{as } j \uparrow +\infty,$$

with ω_j *an infinitesimal such that* $\omega_j = o(\varepsilon_j^k)$ *for all* $k \in \mathbb{N}$.

Let us now describe "locking" of minimizers and "chocking" of the Γ-development according to the terminology introduced in [13].

The term "locking" refers to the fact that the domains of the higher-order Γ-limits are subsets (ordered by inclusions) of the minimizers of the zeroth order one, so that they are locked from the very first step. In particular, when the class of minimizers is exhausted and reduced to a singleton, the successive Γ-developments do not act as a selection criterion, despite more and more precise pieces of information are required for the determination of higher order Γ-limits. This is the case envisaged in Example 2.28, we discuss a similar example below.

"Choking" of the Γ-development simply refers to the non-existence of higher-order Γ-limits.

We work out two examples taken from [13].

Example 2.29. *Consider* $\mathscr{F}_j : W^{1,2}(0,1) \to [0,+\infty)$ *defined as*

$$\mathscr{F}_j(u) := \int_0^1 \left(|u'|^2 + \varepsilon_j^2 |u|^2\right) dx, \quad \text{if } u(0)=0,\, u(1)=1$$

$+\infty$ *otherwise, then the* $\Gamma(L^2)$*-limit of* $(\mathscr{F}_j)_{j\in\mathbb{N}}$ *is given by*

$$\mathscr{F}(u) := \int_0^1 |u'|^2 dx, \quad \text{if } u(0)=0,\, u(1)=1$$

$+\infty$ *otherwise. Jensen inequality yields that the unique minimizer of* $\mathscr{F}$ *is the identity function. In turn, from this we infer that the first relevant scaling is* ε_j^2. *Indeed, setting*

$$\mathscr{G}_j(u) := \frac{\mathscr{F}_j(u) - \min_{W^{1,2}} \mathscr{F}}{\varepsilon_j^2}$$

then $\mathscr{G}_j(u) \geq \int_0^1 |u|^2 dx$, *and it is then easy to conclude that* $(\mathscr{G}_j)_{j\in\mathbb{N}}$ $\Gamma(L^2)$*-converges to*

$$\mathscr{G}(u) := \int_0^1 |u|^2 dx \quad \textit{if } u(x) = x,$$

$+\infty$ *otherwise. Finally, we can compare* $u(x) = x$ *with the minimizer* u_j *of* $\mathscr{F}_j$ *given by*

$$u_j(x) = \frac{\sinh(\varepsilon_j x)}{\sinh \varepsilon_j} = x + \frac{\varepsilon_j^2}{6}(x^3 - x) + o(\varepsilon_j^2) \quad \textit{as } j \uparrow +\infty.$$

as follows by solving the Euler-Lagrange equations.

Example 2.30. *We consider the setting of Example 2.22 in the periodic one-dimensional case. In particular, let* $\mathscr{F}_j : W^{1,2}(0,1) \to [0,+\infty]$ *be*

$$\mathscr{F}_j(u) := \int_0^1 a\left(\frac{x}{\varepsilon_j}\right) |u'|^2 dx, \quad u(0) = 0,\, u(1) = 1$$

$+\infty$ *otherwise, with* $a : \mathbb{R} \to \mathbb{R}$ *a* 1*-periodic function satisfying* $0 < \alpha \leq a(x) \leq \beta$ $\mathcal{L}^1$ *a.e.. Setting*

$$\frac{1}{\underline{a}} = \int_0^1 \frac{1}{a(x)} dx, \textit{ and } \frac{1}{a_j} = \int_0^1 \frac{1}{a(x/\varepsilon_j)} dx = \varepsilon_j \int_0^{1/\varepsilon_j} \frac{1}{a(x)} dx,$$

from Example 2.22 we infer easily that

$$\Gamma(L^2)\text{-}\lim_j \mathscr{F}_j(u) =: \mathscr{F}(u) = \underline{a} \int_0^1 |u'|^2 dx \quad u(0) = 0,\, u(1) = 1$$

$+\infty$ *otherwise. Being the functionals* $\mathscr{F}_j$ *and* $\mathscr{F}$ *strictly convex, the analysis of the corresponding Euler-Lagrange equations implies that*

$$\min_{W^{1,2}} \mathscr{F}_j - \min_{W^{1,2}} \mathscr{F} = a_j - \underline{a} = \underline{a}\, a_j\, \varepsilon_j \int_{[1/\varepsilon_j]}^{1/\varepsilon_j} \left(\frac{1}{\underline{a}} - \frac{1}{a(x)}\right) dx. \tag{2.19}$$

The last equality is obtained by the periodicity of $a(\cdot)$ *through a change of variable. Thus, the first relevant scaling is* ε_j, *and since the limiting* *behaviour of*

$$\frac{1}{\varepsilon_j}(\min_{L^2}\mathscr{F}_j - \min_{L^2}\mathscr{F}) = \underline{a}\,a_j \int_{[1/\varepsilon_j]}^{1/\varepsilon_j} \left(\frac{1}{\underline{a}} - \frac{1}{a(x)}\right) dx$$

depends on the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$*, higher order* Γ*-limits do not exist in general, and formula* (2.17) *cannot be improved besides the zeroth order expansion.*

The issue is even more subtle (and interesting for what the analysis of models is concerned) in case parametrized families of functionals are taken into account, i.e. $(\mathscr{F}_j^\lambda)_{j\in\mathbb{N}}$, where λ is a secondary parameter related to some continuous perturbation which do not alter Γ-convergence. For instance, in the analysis of variational models for mechanical systems, λ either can be an imposed displacement on the boundary of the domain or can represent the geometry of the domain itself (such as the length of a bar), or can model distributed forces or can be used to fix volume fractions.

Even assuming that $(\mathscr{F}_j^\lambda)_{j\in\mathbb{N}}$ Γ-converges to some $\mathscr{F}^\lambda$ for all λ, only the pointwise convergence of the minimum functions $\lambda \to \min_X \mathscr{F}_j^\lambda$ to $\lambda \to \min_X \mathscr{F}^\lambda$ can be inferred. Uniform convergence, or better continuous convergence, that is $\lim_j \min_X \mathscr{F}_j^{\lambda_j} = \min_X \mathscr{F}^\lambda$ for all sequences $\lambda_j \to \lambda_0$, is not guaranteed a priori, so that for a certain parameter λ_0 the families $(\min_X \mathscr{F}_j^{\lambda_j})_{j\in\mathbb{N}}$ might have distinct limits for different sequences $\lambda_j \to \lambda_0$. In such a case, we call λ_0 a *singular point* for the Γ-development.

This phenomenon is particularly interesting when studying physical models for which the Γ-limit represents only an approximation, the true model being related to $\mathscr{F}_j^\lambda$ for j big but finite. In such a case, if λ_0 is a singular point, the Taylor expansion obtained by computing the Γ-limit at fixed λ, that is

$$\min_X \mathscr{F}_j^\lambda = \min_X \mathscr{F}^\lambda + \omega_j^\lambda, \tag{2.20}$$

with ω_j^λ infinitesimal as $j \uparrow +\infty$, is not accurate to describe the behaviour of minimum problems. One reason being that the left hand side above is a continuous function of λ at λ_0, while the right hand side is not. In addition, the quantity

$$\limsup_{\lambda\to\lambda_0} |\min_X \mathscr{F}_j^\lambda - \min_X \mathscr{F}^\lambda|$$

might not be infinitesimal as $j \uparrow +\infty$ contrary to formula (2.20) computed for $\lambda = \lambda_0$ (see Example 2.31 below).

Non-uniformity phenomena at singular points λ_0 are associated to non-uniqueness of the Γ-development at order zero for families $(\mathscr{F}_j^{\lambda_j})_{j\in\mathbb{N}}$ depending on the sequence $(\lambda_j)_{j\in\mathbb{N}}$ converging to λ_0. Hence, a "table" of all the possible Γ-limits of $(\mathscr{F}_j^{\lambda_j})_{j\in\mathbb{N}}$ is necessary in order to bookkeep the behaviours close to singular points.

Clearly, analogous comments can be repeated for higher order developments.

We discuss non-uniformity in the ensuing example.

Example 2.31. *Let $\lambda \geq 0$, and consider the one-dimensional homogenization problem related to $\mathscr{F}_j^\lambda : W^{1,2}(0,1) \to [0,+\infty]$ defined as*

$$\mathscr{F}_j^\lambda(u) := \int_0^1 a\left(\frac{\lambda x}{\varepsilon_j}\right)|u'|^2 dx \qquad \text{if } u(0)=0,\ u(1)=1$$

$+\infty$ otherwise, with $a(\cdot)$ is a continuous 1-periodic function satisfying $0<\alpha\leq a(x)\leq\beta$ for all $x\in\mathbb{R}$. By means of Example 2.22 the corresponding $\Gamma(L^2)$-limit is given, for $\lambda>0$, by

$$\mathscr{F}^\lambda(u) := \underline{a}\int_0^1 |u'|^2 dx \qquad \text{if } u(0)=0,\ u(1)=1$$

$+\infty$ otherwise, where $\underline{a}$ is defined in Example 2.30 above, and by

$$\mathscr{F}^0(u) := a(0)\int_0^1 |u'|^2 dx \qquad \text{if } u(0)=0,\ u(1)=1$$

$+\infty$ otherwise.

In particular, arguing as in in Example 2.30 we infer that $\min_{W^{1,2}}\mathscr{F}^\lambda = \underline{a}$ and $\min_{W^{1,2}}\mathscr{F}_j^\lambda = a_j^\lambda$ for $\lambda>0$, with

$$\frac{1}{a_j^\lambda} = \int_0^1 \frac{1}{a(\lambda x/\varepsilon_j)}dx \to \frac{1}{a(0)} \qquad \text{as } \lambda\to 0^+.$$

In conclusion, the function $\lambda \to \min_{W^{1,2}}\mathscr{F}^\lambda$ is not right continuous at $\lambda=0$, being

$$\lim_{\lambda\to0^+}\min_{W^{1,2}}\mathscr{F}^\lambda = \underline{a} \neq a(0) = \min_{W^{1,2}}\mathscr{F}^0,$$

and

$$\lim_{\lambda\to0^+}|\min_{W^{1,2}}\mathscr{F}_j^\lambda - \min_{W^{1,2}}\mathscr{F}^\lambda| = |a(0)-\underline{a}|>0.$$

To fill out the table of Γ-limits, note that for any infinitesimal sequence $(\lambda_j)_{j\in\mathbb{N}}$ two possibilities occur when studying the asymptotics of $(\mathscr{F}_j^{\lambda_j})_{j\in\mathbb{N}}$: either $\lambda_j = O(\varepsilon_j)$ or $\varepsilon_j = o(\lambda_j)$ as $j\uparrow+\infty$. We have *(i)* $\lambda_j = O(\varepsilon_j)$*: if* $\lambda_j/\varepsilon_j \to k \in [0,+\infty)$*, then by continuity of* $a(\cdot)$ *we conclude*

$$\Gamma(L^2)\text{-}\lim_j \mathscr{F}_j^{\lambda_j}(u) := \int_0^1 a(kx)|u'|^2dx \qquad \textit{if } u(0)=0,\, u(1)=1$$

$+\infty$ *otherwise,*

(ii) $\varepsilon_j = o(\lambda_j)$*: Example 2.22 gives*

$$\Gamma(L^2)\text{-}\lim_j \mathscr{F}_j^{\lambda_j}(u) := \underline{a}\int_0^1 |u'|^2dx \qquad \textit{if } u(0)=0,\, u(1)=1$$

$+\infty$ *otherwise.*

Instead, for every $\lambda > 0$ *we have that the* Γ*-limit is always as in item (ii) above.*

Eventually, thanks to Example 2.30 we infer that all $\lambda > 0$ *are singular points at scale* ε_j*.*

A nontrivial application of the Γ-development shall be provided in Section 4.

2.5. **Convergence of local minimizers.** The discussion in the previous sections highlights that (strict) local minimality is a property not preserved in the Γ-limit process. More precisely, given $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converging to $\mathscr{F}$, simple counterexamples show that the following two issues cannot hold true in general:

(A) if x_j (asymptotically) locally minimize $\mathscr{F}_j$ and $(x_j)_{j\in\mathbb{N}}$ converges to x, then x locally minimize $\mathscr{F}$. For instance, the energies $\mathscr{F}_j(x) = x + \sin(jx)$ Γ-converge to $\mathscr{F}(x) = x-1$, by Example 2.6 and Theorem 2.8, that has no local minima though each point is approximated by local minimizers of $\mathscr{F}_j$;

(B) if x locally minimizes $\mathscr{F}$, then there exists $(x_j)_{j\in\mathbb{N}}$ converging to x with x_j asymptotically locally minimizing $\mathscr{F}_j$, provided, of course, the $\mathscr{F}_j$'s are equicoercive (cp. with Example 2.24 in subsection 2.4).

For what the latter item is concerned, we have already analyzed the corresponding case for global minimizers in Section 2.4. The issue in this case is that the Γ-limit might have several (local) minimizers unrelated to (local) minimizers of the approximating energies. Strict local minimizers deserve a separate discussion since they can be regarded as strict global ones by restricting the analysis to a suitable neighbourhood of the relevant point (cp. with [46] for applications). Hence, for those points item (B) is actually fulfilled thanks to Theorem 2.11. In addition, the discussion of subsection 2.4 is appropriate.

On the contrary, the main reason for the failure of the validity of the former item above is the presence of many oscillations in the approximating energy landscapes which are responsible for the existence of many local minimizers in principle unrelated to local minimizers of the limit. This phenomenon is more difficult to be handled as we shall discuss in what follows.

A possible clue to avoid such degenerate behaviours is to choose 'qualified' local minima and look for an intermediate notion between strict local minimality and local minimality stable under Γ-convergence (see subsection 3.4 for a different approach). Recently, Braides and Larsen [12] have proposed to select local minima according to an appropriate notion of stability.

Definition 2.32. *Let $\mathscr{F} : X \to [0,+\infty]$ and $\varepsilon > 0$, a point $\bar{x}$ is ε-stable for $\mathscr{F}$ if for every $\varphi \in C^0([0,1],X)$ with $\varphi(0) = \bar{x}$ and $\mathscr{F}(\varphi(1)) < \mathscr{F}(\bar{x})$ we have*

$$\sup_{0\le s<t\le 1} \big(\mathscr{F}(\varphi(t)) - \mathscr{F}(\varphi(s))\big) \ge \varepsilon. \tag{2.21}$$

A point $\bar{x}$ is stable for $\mathscr{F}$ if it is ε-stable for some $\varepsilon > 0$.

The basic idea behind the notion above is that lower energy states are accessible for an ε-stable point only if crossing an energy barrier of height ε.

Simple examples show that in general neither local minimizers are ε-stable, let for instance $\mathscr{F}(x) = ((x-1)\vee 0)^2$ for $x \ge 0$, $\mathscr{F}(x) = -\mathscr{F}(-x)$ if $x < 0$ and $\bar{x} = 0$; nor ε-stable points are necessarily local minimizers, let for instance

$$\mathscr{F}(x) = \begin{cases} 0 & x = 0 \\ \sin^2(1/x) - x^2 & x \ne 0, \end{cases} \tag{2.22}$$

then $\bar{x} = 0$ is not a local minimizer but it is 1-stable.

Instead, strict local minimizers are stable in case the relevant function enjoys some natural requirements.

Proposition 2.33. *Let $\bar{x}$ be a strict local minimizer of a d-coercive and d-lower semicontinuous function $\mathscr{F} : X \to \overline{\mathbb{R}}$, then $\bar{x}$ is stable.*

Proof. Assume that $\mathscr{F}(\bar{x}) < \mathscr{F}(x)$ for every point $x \in \overline{B_r(\bar{x})} \setminus \{\bar{x}\}$, for some radius $r > 0$, then $\varepsilon = \inf_{\partial B_r(\bar{x})} \mathscr{F} - \mathscr{F}(\bar{x}) > 0$ thanks to Theorem 2.2. We claim that $\bar{x}$ is ε-stable. Suppose not, then a path $\varphi \in C^0([0,1],X)$ exists with $\varphi(0) = \bar{x}$, $\mathscr{F}(\varphi(1)) < \mathscr{F}(\bar{x})$ and

$$\sup_{0\le s<t\le 1} \big(\mathscr{F}(\varphi(t)) - \mathscr{F}(\varphi(s))\big) < \varepsilon.$$

In particular, from this we infer $\mathscr{F}(\varphi(t)) < \mathscr{F}(\bar{x}) + \varepsilon$ for all $t \in [0,1]$, so that $\varphi(t) \in B_r(\bar{x})$ for all $t \in [0,1]$. Indeed, $\varphi(0) = \bar{x}$ and $\varphi([0,1])$ is a connected set (being φ continuous) with values into $X \setminus \partial B_r(\bar{x})$. Hence, $\varphi(1) \in B_r(\bar{x})$ and $\mathscr{F}(\varphi(1)) < \mathscr{F}(\bar{x})$, a contradiction. □

The first naive attempt to couple stability with Γ-convergence is contained in the following definition.

Definition 2.34. *$(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges stably to $\mathscr{F}$ if*

(i) $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges to $\mathscr{F}$;
(ii) if $(x_j)_{j\in\mathbb{N}}$ is converging to x, x_j an ε-stable local minimizer of $\mathscr{F}_j$ for some $\varepsilon > 0$ and for all j big enough, then x is an ε-stable local minimizer of $\mathscr{F}$;
(iii) if x is a strict local minimizer for $\mathscr{F}$, there exist $\varepsilon > 0$ and $(x_j)_{j\in\mathbb{N}}$ converging to x, with x_j an ε-stable local minimizer of $\mathscr{F}_j$ for all j big enough.

Remark 2.35. *If $\mathscr{F}$ is a constant function and $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges to $\mathscr{F}$, then actually $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges stably to $\mathscr{F}$.*

We warn the reader that we have slightly departed from the terminology used in [12], there condition (ii) above is substituted by the weaker requirement

(ii′) if $(x_j)_{j\in\mathbb{N}}$ is converging to x, x_j an ε-stable point for $\mathscr{F}_j$ for some $\varepsilon > 0$ and for all j big enough, then x is an ε-stable point for $\mathscr{F}$.

Item (iii) is naturally formulated for strict local minima in view of the discussion in subsection 2.4. It is guaranteed by mild assumptions on the approximating sequence following an argument very similar to that exploited in Proposition 2.33. Proposition 2.36 below can be compared with [46, Theorem 4.1] in which, given a strict local minimizer of the Γ-limit, the existence of local minimizers of the approximating functions converging to such a point was established.

Proposition 2.36. *Let $\mathscr{F}_j : X \to \overline{\mathbb{R}}$ be d-lower semicontinuous functions, with $(\mathscr{F}_j)_{j\in\mathbb{N}}$ d-equicoercive and Γ-converging to some $\mathscr{F}$.*

If $\bar{x}$ is a strict local minimizer of $\mathscr{F}$, then there exist $\varepsilon > 0$ and $(x_j)_{j\in\mathbb{N}}$ converging to $\bar{x}$, with x_j an ε-stable local minimizer of $\mathscr{F}_j$ for all j big enough.

Proof. Assume that $\mathscr{F}(\bar{x}) < \mathscr{F}(x)$ for every point $x \in \overline{B_r(\bar{x})} \setminus \{\bar{x}\}$, for some radius $r > 0$, then $2\varepsilon = \inf_{\partial B_r(\bar{x})} \mathscr{F} - \mathscr{F}(\bar{x}) > 0$. We claim that $x_j \in \mathrm{argmin}_{\overline{B_r(\bar{x})}} \mathscr{F}_j$ is ε-stable for j big enough (the existence of x_j is ensured by the d-coercivity hypothesis on $\mathscr{F}_j$).

Let us first show that $(x_j)_{j\in\mathbb{N}}$ d-converges to $\bar{x}$. To this aim consider $(y_j)_{j\in\mathbb{N}}$ a recovery sequence for $\mathscr{F}(\bar{x})$, then

$$\mathscr{F}(\bar{x}) = \lim_j \mathscr{F}_j(y_j) \geq \limsup_j \mathscr{F}_j(x_j).$$

On the other hand, by d-equicoercivity there exists a subsequence $(x_{j_k})_{k\in\mathbb{N}}$ converging to some $x \in \overline{B_r(\bar{x})}$, so that

$$\mathscr{F}(x) \leq \liminf_k \mathscr{F}_{j_k}(x_{j_k}) \leq \lim_j \mathscr{F}_j(y_j) = \mathscr{F}(\bar{x}).$$

By strict minimality $\bar{x} = x$, and then by Urysohn property the whole sequence $(x_j)_{j\in\mathbb{N}}$ converges to x.

Therefore, since $\inf_{\partial B_r(\bar{x})} \mathscr{F}_j - \mathscr{F}_j(x_j) \to \inf_{\partial B_r(\bar{x})} \mathscr{F} - \mathscr{F}(\bar{x})$ thanks to Theorem 2.11, we infer $\inf_{\partial B_r(\bar{x})} \mathscr{F}_j - \mathscr{F}_j(x_j) > \varepsilon$ for j sufficiently big. Then, if x_j was not ε-stable for $\mathscr{F}_j$ there would be paths $\varphi_j \in C^0([0,1], X)$ with $\varphi_j(0) = x_j$, $\mathscr{F}_j(\varphi_j(1)) < \mathscr{F}_j(x_j)$ and

$$\sup_{0\leq s<t\leq 1} \big(\mathscr{F}_j(\varphi_j(t)) - \mathscr{F}_j(\varphi_j(s))\big) < \varepsilon.$$

Hence, $\mathscr{F}_j(\varphi_j(t)) < \mathscr{F}_j(x_j) + \varepsilon$, and thus by continuity $\varphi_j(t) \in B_r(\bar{x})$ for all $t \in [0,1]$. In conclusion, $\varphi_j(1) \in B_r(\bar{x})$ and $\mathscr{F}_j(\varphi_j(1)) < \mathscr{F}_j(x_j)$, a contradiction. □

The main drawbacks of Definition 2.34 pertain item (ii) there, and can be highlighted when comparing it to the basic properties of Γ-convergence (see Theorem 2.8):

(i) given $\mathscr{F}$ d-lower semicontinuous, the constant sequence $\mathscr{F}_j = \mathscr{F}$ in general does not Γ-converge stably to $\mathscr{F}$ itself. Consider for instance

$$\mathscr{F}(x) = \begin{cases} \cos(1/x) & x > 0 \\ x - 1 & x \leq 0, \end{cases} \tag{2.23}$$

then $x_j = (\pi j)^{-1}$ are 2-stable local minimizers, but $\bar{x} = 0$ is neither stable nor a local minimizer (see also (2.22) for an example of locally minimizing 1-stable points converging to a 1-stable point that is not a local minimizer);

(ii) $\mathscr{F}_j(x) = \sin(jx)$ Γ-converge stably to $\mathscr{F}(x) = -1$ (cp. with Example 2.6 and Remark 2.35). Despite this, $\mathscr{G}_j(x) = \mathscr{F}_j(x) + x^2$ Γ-converge to $-1 + x^2$, but not stably since each point different from $\bar{x} = 0$ is the limit of strict local minimizers of the $\mathscr{G}_j$'s though unstable and not locally minimizing.

The first topic is analogous to the fact that a constant sequence $\mathscr{F}_j = \mathscr{F}$ does not Γ-converge to $\mathscr{F}$ itself, unless $\mathscr{F}$ is d-lower semicontinuous. Instead, the second counterexample shows that stable Γ-convergence is not preserved for the addition of (d-coercive) continuous perturbations contrary to Γ-convergence.

More stringent requirements than those in Definition 2.34 are introduced below.

Definition 2.37. $(\mathscr{F}_j)_{j\in\mathbb{N}}$ *Γ-converges strongly-stably to* $\mathscr{F}$ *if*

(i) $\mathscr{F}$ *and each* $\mathscr{F}_j$ *are d-lower semicontinuous;*
(ii) each stable point for $\mathscr{F}$ *is a local minimizer;*
(iii) $(\mathscr{F}_j)_{j\in\mathbb{N}}$ *Γ-converges stably to* $\mathscr{F}$*;*
(iv) if $\varphi\in C^0([0,1],X)$ *with* $\varphi(0)=\bar{x}$ *and* $(x_j)_{j\in\mathbb{N}}$ *converges to* $\bar{x}$, *then there exist paths* φ_j, $\psi_j\in C^0([0,1],X)$, *with* $\psi_j(0)=x_j$, $\varphi_j(0)=\psi_j(1)$, *and a partition* $0=\tau_1^j<\tau_2^j<\ldots<\tau_{n_j}^j=1$ *of* $[0,1]$ *with* $\lim_j\max_i|\tau_{i+1}^j-\tau_i^j|=0$, *such that*

$$\lim_j \sup_{0\leq t_1<t_2\leq 1}\big(\mathscr{F}_j(\psi_j(t_2))-\mathscr{F}_j(\psi_j(t_1))\big)=0, \tag{2.24}$$

$$\lim_j \sup_{t\in[0,1]} \mathrm{dist}(\varphi_j(t),\varphi(t))=0, \tag{2.25}$$

$$\lim_j \max_i \left|\mathscr{F}_j(\varphi_j(\tau_i^j))-\mathscr{F}(\varphi(\tau_i^j))\right|=0, \tag{2.26}$$

and for some positive infinitesimal β_j *it holds for all* $i\in\{1,\ldots,n_j-1\}$.

$$\sup_{t,s\in[\tau_i^j,\tau_{i+1}^j]}|\mathscr{F}_j(\varphi_j(t))-\mathscr{F}_j(\varphi_j(s))|\leq\beta_j+\left|\mathscr{F}_j(\varphi_j(\tau_{i+1}^j))-\mathscr{F}_j(\varphi_j(\tau_i^j))\right|. \tag{2.27}$$

Few remarks are in order: first, properties in items (i) and (ii) identify the class of functions of interest as to rule out examples as those in (2.22) and (2.23). For what item (iv) is concerned, in view of (2.24), $\mathscr{F}_j$ is almost decreasing (and it might be discontinuous) along ψ_j (the latter remark is important for applications in fracture mechanics as in [47]); on the points of the partition τ_i^j the $\mathscr{F}_j$-energy of φ_j is close to the $\mathscr{F}$-energy of φ by (2.26) (uniformly in i). Note that the paths φ_j are uniformly close to φ thanks to (2.25). In (2.27) the oscillation of $\mathscr{F}_j$ along φ_j on each interval of the partition is controlled only in terms of the oscillation at the end-points up to an infinitesimal error. Hence, by combining (2.27) with (2.26) we get that for some positive infinitesimal δ_j and for all $i\in\{1,\ldots,j-1\}$ it holds

$$\sup_{t,s\in[\tau_i^j,\tau_{i+1}^j]}|\mathscr{F}_j(\varphi_j(t))-\mathscr{F}_j(\varphi_j(s))|\leq\delta_j+\left|\mathscr{F}(\varphi(\tau_{i+1}^j))-\mathscr{F}(\varphi(\tau_i^j))\right|.$$

Note also that (2.27) can be equivalently formulated as

$$\mathscr{F}_j(\varphi_j(\tau^j_{i+1}))\wedge\mathscr{F}_j(\varphi_j(\tau^j_i))-\beta_j \le \mathscr{F}_j(\varphi_j(t)) \le \mathscr{F}_j(\varphi_j(\tau^j_{i+1}))\vee\mathscr{F}_j(\varphi_j(\tau^j_i))+\beta_j. \tag{2.28}$$

Finally, the path ψ_j is needed to link the path φ_j, for which the conditions above are satisfied, to the point of interest x_j (recall that $\psi_j(0) = x_j$ and $\psi_j(1) = \varphi_j(0)$).

All in all, the conditions in item (iv) ensure that if the path φ is an 'ε-slide' for $\mathscr{F}$ between $\varphi(0)$ and $\varphi(1)$ (according to the terminology in [47]), i.e.

$$\mathscr{F}(\varphi(1)) < \mathscr{F}(\varphi(0)) \quad \text{and} \quad \sup_{0\le s<t\le 1}(\mathscr{F}(\varphi(t)) - \mathscr{F}(\varphi(s))) < \varepsilon,$$

then one can find close to φ an $(\varepsilon-\delta)$-slide for $\mathscr{F}_j$ between $\varphi_j(0)$ and $\varphi_j(1)$ for j big enough and $\delta \in (0,\varepsilon)$ (cp. with the proof of (2.29) in Proposition 2.38 below).

Let us now prove that strong-stable Γ-convergence enforces some stability with respect to the addition of (suitable) continuous perturbations.

Proposition 2.38. *Suppose $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges strongly-stably to $\mathscr{F}$. Then, for all continuous perturbations $\mathscr{G}$ such that $(\mathscr{F}_j+\mathscr{G})_{j\in\mathbb{N}}$ is d-equicoercive, $(\mathscr{F}_j+\mathscr{G})_{j\in\mathbb{N}}$ satisfies (i) and (iii) in Definition 2.34 and item (ii′) right after.*

Proof. Items (i) and (iii) in Definition 2.34 are satisfied thanks to Theorem 2.8 and Proposition 2.36, respectively. To conclude let $(x_j)_{j\in\mathbb{N}}$ be converging to $\bar{x}$, with x_j ε-stable for $\mathscr{F}_j+\mathscr{G}$ for all j big enough, then in particular $(\varepsilon-\delta)$-stable for all $\delta \in (0,\varepsilon)$. We shall show that $\bar{x}$ is $(\varepsilon-\delta)$-stable for those δ.

Suppose towards contradiction that $\bar{x}$ is un-$(\varepsilon-\delta)$-stable for some δ, i.e. we can find $\varphi \in C^0([0,1],X)$ with $\varphi(0) = \bar{x}$, $(\mathscr{F}+\mathscr{G})(\varphi(1)) < (\mathscr{F}+\mathscr{G})(\bar{x})$ and

$$\sup_{0\le s<t\le 1}\big((\mathscr{F}+\mathscr{G})(\varphi(t)) - (\mathscr{F}+\mathscr{G})(\varphi(s))\big) < \varepsilon-\delta.$$

Choose φ_j and ψ_j as in item (iv) of Definition 2.37, and set $\phi_j(t) = \psi_j(2t)$ for $t \in [0,1/2]$, $\phi_j(t) = \varphi_j(2t-1)$ for $t \in [1/2,1]$. We claim that items (i)-(iii) above, the continuity of $\mathscr{G}$ and elementary computations give for j big enough

$$(\mathscr{F}_j+\mathscr{G})(\phi_j(t_2)) - (\mathscr{F}_j+\mathscr{G})(\phi_j(t_1)) \le \varepsilon-\delta+o(1) \tag{2.29}$$

for all t_1, $t_2 \in [0,1]$, with $t_1 < t_2$, and moreover

$$(\mathscr{F}_j+\mathscr{G})(\phi_j(1))-(\mathscr{F}_j+\mathscr{G})(\phi_j(0)) \le (\mathscr{F}+\mathscr{G})(\varphi(1))-(\mathscr{F}+\mathscr{G})(\bar{x})+o(1). \tag{2.30}$$

Given this for granted, x_j is not ε-stable for $\mathscr{F}_j + \mathscr{G}$ for j sufficiently large, a contradiction.

Let us now prove (2.29). It is sufficient to show that

$$\mathscr{F}_j(\phi_j(t_2)) - \mathscr{F}_j(\phi_j(t_1)) \leq \mathscr{F}(\varphi(\tau_k^j)) - \mathscr{F}(\varphi(\tau_i^j)) + o(1), \qquad (2.31)$$

for some $\tau_i^j < \tau_k^j$, thanks to condition (2.25), $\lim_j \max_i |\tau_{i+1}^j - \tau_i^j| = 0$ and the continuity of $\mathscr{G}$.

First, choose t_1, $t_2 \in [0, 1/2]$, then by (2.24) we get

$$\mathscr{F}_j(\phi_j(t_2)) - \mathscr{F}_j(\phi_j(t_1)) = \mathscr{F}_j(\psi_j(2t_2)) - \mathscr{F}_j(\psi_j(2t_1)) \leq o(1).$$

If, instead, t_1, $t_2 \in [1/2, 1]$ then by (2.26) and (2.27)

$$\begin{aligned}\mathscr{F}_j(\phi_j(t_2)) - \mathscr{F}_j(\phi_j(t_1)) &= \mathscr{F}_j(\varphi_j(2t_2 - 1)) - \mathscr{F}_j(\varphi_j(2t_1 - 1)) \\ &\leq \mathscr{F}(\varphi(\tau_k^j)) - \mathscr{F}(\varphi(\tau_i^j)) + o(1),\end{aligned}$$

for some $\tau_i^j < \tau_k^j$. Finally, let $0 < t_1 < 1/2 < t_2$, then

$$\mathscr{F}_j(\phi_j(t_2)) - \mathscr{F}_j(\phi_j(t_1)) = \mathscr{F}_j(\varphi_j(2t_2 - 1)) - \mathscr{F}_j(\psi_j(2t_1)).$$

To estimate the last terms note that by (2.26) and (2.27), if $(2t_2 - 1) \in [\tau_i^j, \tau_{i+1}^j]$, we have

$$\begin{aligned}\mathscr{F}_j(\varphi_j(2t_2 - 1)) &\leq \mathscr{F}_j(\varphi_j(\tau_i^j)) \vee \mathscr{F}_j(\varphi_j(\tau_{i+1}^j)) + o(1) \\ &\leq \mathscr{F}(\varphi(\tau_i^j)) \vee \mathscr{F}(\varphi(\tau_{i+1}^j)) + o(1),\end{aligned}$$

while

$$\mathscr{F}_j(\psi_j(2t_1)) \geq \mathscr{F}_j(\psi_j(1)) + o(1) = \mathscr{F}_j(\varphi_j(0)) + o(1) \geq \mathscr{F}(\varphi(0)) + o(1).$$

In conclusion, (2.31) holds in this case as well. Similarly, one can prove (2.30). $\square$

Remark 2.39. *It is evident that if the sequence $(\mathscr{F}_j)_{j\in\mathbb{N}}$ is equicoercive itself, then the conclusions of Proposition 2.38 still hold substituting the strong-stable Γ-convergence assumption with (i), (ii), (iv) in Definition 2.37 together with standard Γ-convergence. For, stable Γ-convergence in item (iii) of Definition 2.37 is a consequence of Proposition 2.36 and of the argument exploited in Proposition 2.38 to prove $(\varepsilon - \delta)$-stability.*

Remark 2.40. *Let us now show by an example that, in general, strong-stable Γ-convergence of $(\mathscr{F}_j + \mathscr{G})_{j\in\mathbb{N}}$ to $\mathscr{F} + \mathscr{G}$ fails since condition (ii) in Definition 2.34 is violated. We slightly modify example* (2.22) *in order to gain coercivity.*

Let $\mathscr{F}(x) = \sin^2(1/x)$ for $x \neq 0$ and $\mathscr{F}(0) = 0$, note that each local minimizer of $\mathscr{F}$ it is actually a global one. Then, clearly, the constant sequence equal to $\mathscr{F}$ Γ-converges strongly-stably to $\mathscr{F}$ itself.

Despite this, setting $\mathscr{G}(x) = (1-x^2)^2 - 1$, $\mathscr{F}+\mathscr{G}$ is coercive and do not Γ-converge stably to itself. For, the origin is not a local minimizer of $\mathscr{F}+\mathscr{G}$, though it is 1-stable and limit of local minimizers of $\mathscr{F}+\mathscr{G}$.

Eventually, we end this section by pointing out that non-trivial examples of strong-stable convergence are discussed in the original paper [12].

3. G-convergence

Given a bounded open set Ω of $\mathbb{R}^n$ we denote by $\mathcal{M}(\Omega,\alpha,\beta)$ the set of $n\times n$ symmetric matrix fields B with Borel measurable coefficients satisfying

$$\alpha \mathrm{Id}_n \leq B(x) \leq \beta \mathrm{Id}_n \quad \text{for } \mathcal{L}^n \text{ a.e. } x\in\Omega, \tag{3.1}$$

where Id_n is the $n\times n$ identity matrix. Let $B\in\mathcal{M}(\Omega,\alpha,\beta)$ and $f\in H^{-1}(\Omega)$, then consider the variational problem

$$-\mathrm{div}(B\nabla w) = f \quad \mathcal{L}^n \text{ a.e. } \Omega, \text{ with } w\in W^{1,2}_0(\Omega), \tag{3.2}$$

for which Dirichlet principle supplies a solution (see [14, Theorem V.6]).

In several applied fields it is often interesting to analyze parameter depending problems as that above with $(B_j)_{j\in\mathbb{N}}\subset\mathcal{M}(\Omega,\alpha,\beta)$. This happens when looking for physical properties of multi-phase media, for instance mixing periodically two different materials, in this case $B_j(x) = a(x/\varepsilon_j)\,\mathrm{Id}_n$, $a:\mathbb{R}^n\to\{\alpha,\beta\}$ 1-periodic, $0<\alpha<\beta$, and $\varepsilon_j>0$. Thermal or electrical conductivities, constitutive relations and many others, are prominent examples all obeying to PDEs as that in (3.2). In a sample much bigger than the separate components, several physical properties of the microscopically heterogeneous medium (i.e. ε_j fixed) behave macroscopically like those of a homogeneous one (i.e. $\varepsilon_j\downarrow 0$ as $j\uparrow+\infty$) which then furnishes a convenient approximation of the real one for (numerical) analysis purposes. Mathematically, we are led to determine the asymptotic behaviour as $j\uparrow+\infty$ of the solutions, u_j, of families of problems as those in (3.2) related to B_j. Note that thanks to (3.1) the sequence $(u_j)_{j\in\mathbb{N}}$ converges weakly in $W^{1,2}$ to some $u\in W^{1,2}_0(\Omega)$ (up to subsequences not relabeled in what follows). Thus, the issue is to understand whether u solves a similar problem or not, and in case it does, the form of the matrix B in the latter. Clearly, it would be desirable to accomplish this task without requiring restrictive convergence hypotheses on $(B_j)_{j\in\mathbb{N}}$, such as strong convergence in $L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^{n\times n})$, that are not ensured by the compactness properties of the problem itself. Indeed, the natural convergences for $(B_j)_{j\in\mathbb{N}}$ and $(u_j)_{j\in\mathbb{N}}$ following from the ellipticity bounds are weak*

$L^\infty(\Omega,\mathbb{R}^{n\times n})$ and weak $W^{1,2}(\Omega)$, respectively. Note also that the sequence $(B_j\nabla u_j)_{j\in\mathbb{N}}$ has no obvious convergence properties a priori.

To solve this problem, at least in an abstract fashion, necessary preliminaries are in order. We quote the needed results without going into the details being well-established in literature (see for instance [25, Chapter 12]).

3.1. **Quadratic forms and linear operators.** The functional analytic setting of interest here is that of positive quadratic forms and linear operators on a separable (real) Hilbert space X endowed with a norm $\|\cdot\|$ induced by a scalar product $\langle\cdot,\cdot\rangle$. In what follows we identify X with its dual space X^*.

Definition 3.1. *$\mathscr{F}:X\to[0,+\infty]$ is a quadratic form if there exist a linear subspace Y of X and a bilinear symmetric form $\mathcal{B}:Y\times Y\to\mathbb{R}$ such that*

$$\mathscr{F}(x)=\begin{cases}\mathcal{B}(x,x) & x\in Y\\ +\infty & x\neq Y\end{cases}$$

The domain of $\mathscr{F}$ is the linear subspace $D_{\mathscr{F}}=\{x\in X:\mathscr{F}(x)<+\infty\}$, and the bilinear form associated with $\mathscr{F}$ is $\mathcal{B}|_{D_{\mathscr{F}}\times D_{\mathscr{F}}}$.

A useful algebraic characterization of quadratic forms is contained in the statement below (cp. with [25, Proposition 11.9]).

Proposition 3.2. *$\mathscr{F}:X\to[0,+\infty]$ is a quadratic form if and only if*

(i) $\mathscr{F}(tx)\le t^2\mathscr{F}(x)$ for all $x\in X$ and $t>0$;

(ii) $\mathscr{F}(x+y)+\mathscr{F}(x-y)\le 2\mathscr{F}(x)+2\mathscr{F}(y)$ for all x, $y\in X$.

Note that any quadratic form is necessarily convex, so that lower semicontinuity can be intended with respect to either the strong or weak topology indistinctly being the two properties equivalent in that case (see for instance [25, Proposition 1.18]).

In view of this last remark, of the proposition just stated and of Proposition 2.19, we infer the ensuing closure property of quadratic forms.

Corollary 3.3. *The class of lower semicontinuous quadratic forms is closed under Γ-convergence.*

Next, we introduce linear operators related to a quadratic form $\mathscr{F}$. To this aim fix $x\in D_{\mathscr{F}}$ we look for the representation

$$\mathcal{B}(x,z)=\langle y,z\rangle \ \text{ for all } z\in D_{\mathscr{F}},\ \text{for some } y\in\overline{D_{\mathscr{F}}} \tag{3.3}$$

and consider the linear subspace $D_A=\{x\in\overline{D_{\mathscr{F}}}:\text{(3.3) holds}\}$. Define the linear operator $A:D_A\to\overline{D_{\mathscr{F}}}$ as $Ax=y$ for $x\in D_A$. Note that, since $D_{\mathscr{F}}$ is dense in $\overline{D_{\mathscr{F}}}$, the uniqueness of y is guaranteed. It is easy to infer that A is a (not necessarily bounded) positive and symmetric operator, that is respectively

$$\langle Ax, x\rangle \geq 0\ , \qquad \text{and} \qquad \langle Ax, z\rangle = \langle x, Az\rangle \ \text{ for all } x, z \in D_A.$$

The operator A can be characterized variationally by using a 'partial' version of the Dirichlet principle.

Proposition 3.4. *Let $\mathscr{F} : X \to [0, +\infty]$ be a quadratic form, $x \in D_{\mathscr{F}}$, and $y \in X$. Let A be the linear operator related to $\mathscr{F}$ and denote by O the orthogonal projection from X onto $\overline{D_{\mathscr{F}}}$.*

The following conditions are equivalent:

(i) $x \in D_A$ and $Ax = Oy$;
(ii) $\mathscr{F}(z) - 2\langle y, z\rangle \geq \mathscr{F}(x) - 2\langle y, x\rangle$ for all $z \in X$.

The missing condition to ensure the existence of solutions to equation (3.3) for all points is the lower semicontinuity of $\mathscr{F}$, that for quadratic forms can be characterized as follows.

Proposition 3.5. *The quadratic form $\mathscr{F}$ is lower semicontinuous on X if and only if $(D_{\mathscr{F}}, \|\cdot\|_{\mathscr{F}})$ is Hilbert, where $\|\cdot\|_{\mathscr{F}} = \|\cdot\| + \mathscr{F}(\cdot)$.*

As a consequence of Proposition 3.5 we have,

Corollary 3.6. *If $\mathscr{F}$ is lower semicontinuous on X, then D_A is dense in $(D_{\mathscr{F}}, \|\cdot\|_{\mathscr{F}})$. In particular, D_A is dense in $\overline{D_{\mathscr{F}}}$.*

Moreover, if $A : D_A \to \overline{D_A}$ is the operator associated to $\mathscr{F}$, then $\mathscr{F}$ itself is the lower semicontinuous envelope of the quadratic form

$$\mathscr{Q}(x) = \begin{cases} \langle Ax, x\rangle & x \in \overline{D_A} \\ +\infty & \textit{otherwise in } X. \end{cases}$$

Let us now recall the definition of the adjoint of a linear operator $L : D_L \to V$ densely defined on a subspace D_L of a Hilbert space V. To this aim let,

$$D_{L^*} = \{z \in V : \langle Lx, z\rangle = \langle x, y\rangle \ \text{ for all } x \in D_L,\ \text{for some } y \in V\}.$$

Since D_L is dense in V, the uniqueness of the element y above is guaranteed; hence, we define a linear operator $L^* : D_{L^*} \to V$ as $L^*z = y$, for $z \in D_{L^*}$. L^* is called the *adjoint* of L; L is said to be *self-adjoint* if and only if $D_L = D_{L^*}$ and $L = L^*$.

In particular, if $A : D_A \to \overline{D_{\mathscr{F}}}$ is the operator associated to a lower semicontinuous quadratic form $\mathscr{F}$ on X, D_A is dense in $\overline{D_{\mathscr{F}}}$ in view of Corollary 3.6. Hence, the adjoint operator $A^* : D_{A^*} \to \overline{D_{\mathscr{F}}}$ is well defined on $\overline{D_{\mathscr{F}}}$. In addition, $D_A \subseteq D_{A^*}$ being A symmetric. Actually, there is a bijection between lower semicontinuous quadratic forms and self-adjoint operators.

Theorem 3.7. *If $\mathscr{F}$ is a lower semicontinuous quadratic form on X the operator $A : D_A \to \overline{D_{\mathscr{F}}}$ associated to $\mathscr{F}$ is self-adjoint.*

Vice versa, let $D_A \subseteq X$ be a subspace, $A : D_A \to \overline{D_A}$ be a positive, self-adjoint linear operator. Then, the quadratic form

$$\mathscr{F}(x) = \begin{cases} \sup\limits_{D_A} \langle Ay, 2x - y\rangle & x \in \overline{D_A} \\ +\infty & \text{otherwise in } X, \end{cases}$$

is lower semicontinuous and the operator associated to $\mathscr{F}$ is A itself.

3.2. ***G*-convergence for self-adjoint operators.** Given $\lambda > 0$ set

$$Q_\lambda(X) = \{\mathscr{F} : X \to [0, +\infty] \text{ lower semicontinuous quadratic form}, \mathscr{F}(x) \geq \lambda\|x\|^2 \; \forall\, x \in X\},$$

and

$$P_\lambda(X) = \{A : D_A \to \overline{D_A},\, A \text{ self-adjoint}, \langle Ax, x\rangle \geq \lambda\|x\|^2 \; \forall\, x \in D_A\}.$$

From Theorem 3.7 it follows that $\mathscr{F} \in Q_\lambda(X)$ if and only if the associated operator $A \in P_\lambda(X)$. Following Spagnolo [67], [68] we introduce a weak notion of operatorial convergence on $P_\lambda(X)$.

Definition 3.8. *Let A, A_j be in $P_\lambda(X)$, for all $j \in \mathbb{N}$, and O and O_j denote the orthogonal projections onto $\overline{D_A}$ and $\overline{D_{A_j}}$, respectively. We say that $(A_j)_{j\in\mathbb{N}}$ G-converges weakly (strongly) to A if*

$$A_j^{-1} O_j z \to A^{-1} O z \quad \text{weakly (strongly) for all } z \in X.$$

If $D_A = D_{A_j} = X$, weak (strong) G-convergence is nothing but the weak (strong) convergence of the solutions of the equations $A_j x = z$ to that of $Ax = z$ for all $z \in X$. More generally, the following result holds true (cp. with [2, Proposition 2.3]).

Proposition 3.9. *Let $A, (A_j)_{j\in\mathbb{N}} \subset Q_\lambda(X)$, then*

(i) $(A_j)_{j\in\mathbb{N}}$ G-converges weakly (strongly) to A;
(ii) for all $(z_j)_{j\in\mathbb{N}} \subset X$ converging strongly to z in X, then $(A_j^{-1} O_j z_j)_{j\in\mathbb{N}}$ converges weakly (strongly) to $A^{-1} O z$ in X;

In addition, if $D_{A_j} = D_A = X$, then (i) and (ii) are equivalent to

(iii) $\Gamma\text{-}\lim_j \chi_{\mathrm{graph} A_j} = \chi_{\mathrm{graph} A}$ in the product w-$X \times s$-X topology.

Proof. The implication $(ii) \Rightarrow (i)$ is trivial, so we deal only with $(i) \Rightarrow (ii)$. To this aim set $x_j = A_j^{-1} O_j (z_j - z) \in X$, and note that

$$\lambda\|x_j\|^2 \leq \langle A_j x_j, x_j\rangle = \langle O_j(z_j - z), x_j\rangle \leq \|z_j - z\|\|x_j\|, \tag{3.4}$$

so that $(x_j)_{j\in\mathbb{N}}$ converges strongly to $\underline{0}$ in X. In particular, since $(A_j^{-1}O_jz)_{j\in\mathbb{N}}$ converges weakly (strongly) to $A^{-1}Oz$ in X by hypothesis, the conclusion follows at once.

Let us assume that $D_A = D_{A_j} = X$ and prove equivalence $(ii) \Leftrightarrow (iii)$. To show $(ii) \Rightarrow (iii)$ fix $(x,z) \in \text{graph}A$, i.e., $z = Ax$ with $x \in D_A$, then setting $x_j = A_j^{-1}z$, the sequence $(x_j)_{j\in\mathbb{N}}$ converges weakly to $A^{-1}z$, that is to x. In addition, the sequence $(x_j, A_jx_j) = (x_j, z) \in \text{graph}A_j$ and converges in w-$X \times s$-X to (x,z), so that $\Gamma\text{-}\limsup_j \chi_{\text{graph}A_j} \leq \chi_{\text{graph}A}$.

To show $\Gamma\text{-}\liminf_j \chi_{\text{graph}A_j} \geq \chi_{\text{graph}A}$, let $(x_j, z_j) \in \text{graph}A_j$ be given converging in w-$X \times s$-X to some (x,z). Let $y_j = A_j^{-1}z$, by hypothesis $(y_j)_{j\in\mathbb{N}}$ converges weakly to $A^{-1}z$. In addition

$$\lambda\|x_j - y_j\|^2 \leq \langle A_j(x_j - y_j), x_j - y_j\rangle \leq \|z_j - z\|\|x_j - y_j\|,$$

so that $(x_j - y_j)_{j\in\mathbb{N}}$ converges strongly to $\underline{0}$, and then $x = A^{-1}z$.

Vice versa, suppose (iii) hold true. If $(z_j)_{j\in\mathbb{N}}$ converges strongly to z set $x_j = A_j^{-1}z_j$ and $x = A^{-1}z$. Then, $((x_j, z_j))_{j\in\mathbb{N}} \in \text{graph}A_j$ and arguing as in (3.4) shows that $(x_j)_{j\in\mathbb{N}}$ is norm bounded. Hence, a subsequence $(x_{j_k})_{k\in\mathbb{N}}$ weakly converges to some $y \in X$. Item (iii) then implies that $(y,z) \in \text{graph}A$, i.e. $z = Ay$, hence $y = x$ and by Urysohn property the whole sequence $(x_j)_{j\in\mathbb{N}}$ converges weakly to x. □

Remark 3.10. *Item (iii) can be taken as a definition of G-convergence in case of multiple-valued mappings $A : X \to \mathcal{P}(X)$, $\mathcal{P}(X)$ being the power set of X. Actually, it is the general abstract definition of G-convergence (see for instance* [60]*).*

Thanks to Theorem 3.7 we can link the Γ-convergence of equicoercive quadratic forms to the G-convergence of the corresponding self-adjoint linear operators. This result has been originally proved in [32] and [68], and later extended to families of strictly convex equi-coercive functions in [8].

Theorem 3.11. *Let A, A_j be in $P_\lambda(X)$, and $\mathscr{F}$, $\mathscr{F}_j$ be the corresponding quadratic forms in $Q_\lambda(X)$. Then, the following conditions are equivalent:*

(a) $(A_j)_{j\in\mathbb{N}}$ G-converges to A in the weak topology of X,
(b) $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges to $\mathscr{F}$ in the weak topology of X,
(c) $\lim_j \min_X\big(\mathscr{F}_j(x) - \langle z, x\rangle\big) = \min_X\big(\mathscr{F}(x) - \langle z, x\rangle\big)$ for all $z \in X$.

Moreover, the following conditions are equivalent:

(d) $(A_j)_{j\in\mathbb{N}}$ G-converges to A in the strong topology of X,
(e) $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges to $\mathscr{F}$ in the weak and strong topology of X.

A sketch of the proof of Theorem 3.11 has been supplied in the Introduction in a simplified setting, we do not give full details here since this result is well-known (see [25, Theorem 13.5]). Instead, we shall draw the attention on a more recent abstract characterization for bounded operators proposed by Mielke [52, Propositions 2.4 and 2.5].

Proposition 3.12. *Let $\mathscr{F}$, $\mathscr{F}_j \in Q_\lambda(X)$, and A, $A_j \in P_\lambda(X)$ be the corresponding operators, respectively. Assume, in addition, that $D_A = \overline{D_{\mathscr{F}}}$, $D_{A_j} = \overline{D_{\mathscr{F}_j}}$, and that $A : D_A \to D_A$, $A_j : D_{A_j} \to D_{A_j}$ are bounded.*

Then, $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges to $\mathscr{F}$ in the weak topology of X, if and only if there exists a sequence of linear and continuous operators $\Lambda_j : D_{\mathscr{F}} \to D_{\mathscr{F}_j}$ such that

- *(i) if $x_j \in D_{\mathscr{F}_j}$ and $(x_j)_{j\in\mathbb{N}}$ weakly converges to x in X, then $(\Lambda_j^* A_j x_j)_{j\in\mathbb{N}}$ weakly converges to Ax in $D_{\mathscr{F}}$;*
- *(ii) for all $x \in D_{\mathscr{F}}$, $(\Lambda_j x)_{j\in\mathbb{N}}$ weakly converges to x in X;*
- *(iii) if $x_j \in D_{\mathscr{F}_j}$ and $(x_j)_{j\in\mathbb{N}}$ weakly converges to $x \notin D_{\mathscr{F}}$, then $\mathscr{F}_j(x_j) \uparrow +\infty$.*

Proof. First, note that by assumption $D_A = D_{\mathscr{F}} = \overline{D_{\mathscr{F}}}$, $D_{A_j} = D_{\mathscr{F}_j} = \overline{D_{\mathscr{F}_j}}$.

We start with proving the 'only if' part. Note that $(\Lambda_j x)_{j\in\mathbb{N}}$ is a recovery sequence for any $x \in D_{\mathscr{F}}$. Indeed, $(\Lambda_j x)_{j\in\mathbb{N}}$ weakly converges to x by item (ii), and by item (i) above

$$\mathscr{F}_j(\Lambda_j x) = \langle \Lambda_j^* A_j \Lambda_j x, x\rangle \to \langle Ax, x\rangle = \mathscr{F}(x). \tag{3.5}$$

To check (LB) inequality, assume that a sequence $(x_j)_{j\in\mathbb{N}}$ weakly converging to some point $x \in D_{\mathscr{F}}$ is given. Then, use item (i) and (3.5) to infer the conclusion,

$$\begin{aligned}\mathscr{F}_j(x_j) = \mathscr{F}_j(x_j - \Lambda_j x) + 2\langle A_j x_j, \Lambda_j x\rangle - \mathscr{F}_j(\Lambda_j x)&\\ \geq 2\langle \Lambda_j^* A_j x_j, x\rangle - \mathscr{F}_j(\Lambda_j x),&\end{aligned}$$

so that

$$\liminf_j \mathscr{F}_j(x_j) \geq \langle Ax, x\rangle = \mathscr{F}(x).$$

Eventually, condition (iii) takes care of the case $x \notin D_{\mathscr{F}}$.

To prove the opposite implication we consider the bounded linear operators $\Lambda_j = A_j^{-1} O_j A : D_{\mathscr{F}} \to D_{\mathscr{F}_j}$, where O_j denotes the orthogonal projection onto $D_{\mathscr{F}_j}$. By the very definition of Λ_j and in view of Proposition 3.4, the point $\Lambda_j x$, $x \in D_{\mathscr{F}}$, is the minimizer of $\mathscr{F}_j(z) - 2\langle Ax, z\rangle$. Since

$$\mathscr{F}_j(\Lambda_j x) - 2\langle Ax, \Lambda_j x\rangle \leq \mathscr{F}_j(\underline{0}) - 2\langle Ax, \underline{0}\rangle = 0,$$

and $\mathscr{F}_j \in Q_\lambda(X)$, $(\|\Lambda_j x\|)_{j\in\mathbb{N}}$ is bounded so that for a subsequence $(\Lambda_{j_k}x)_{k\in\mathbb{N}}$ weakly converges to $\widetilde{x}$. By Theorem 2.11 $\widetilde{x}$ is a minimizer of $\min_X(\mathscr{F}(z)-2\langle Ax,z\rangle)$. Being the minimizer of such a problem unique $\widetilde{x}=x$, and the whole sequence $(\Lambda_j x)_{j\in\mathbb{N}}$ weakly converges to x and (ii) is established. In particular, we have also proved that $\lim_j \mathscr{F}_j(\Lambda_j x) = \mathscr{F}(x)$, for all $x \in X$.

Furthermore, if $x_j \in D_{\mathscr{F}_j}$ and $(x_j)_{j\in\mathbb{N}}$ weakly converges to x in X, being O_j self-adjoint, a direct computation shows that $\Lambda_j^* A_j = AO$ on $D_{\mathscr{F}_j}$, so that

$$\langle \Lambda_j^* A_j x_j, z\rangle = \langle AOx_j, z\rangle \to \langle Ax, z\rangle,$$

and item (i) then follows at once.

Finally, (iii) is guaranteed by the fact that $(\mathscr{F}_j)_{j\in\mathbb{N}} \subset Q_\lambda(X)$ and the Γ-convergence assumption. □

The proof highlights that there is always a canonical way to construct recovery sequences by solving suitable minimization problems (this is the essence of the equivalence of items (b) and (c) in Theorem 3.11), though being not necessarily the only one.

The previous characterization is particularly useful when dealing with convergence issues for solutions of linear and nonlinear mechanical evolutionary systems where two families of energies, kinetic and potential, are involved. A compatibility condition is needed to correctly describe the asymptotic behaviour of the equilibria of the corresponding mechanical systems: the existence of a family of joint recovery operators as that in Proposition 3.12 (see [52] and the references therein).

Remark 3.13. *Example 2.21 can be rephrased in light of Proposition 3.12. For, a family of 'recovery' operators* $\Lambda_j : L^2(\Omega,\mathbb{R}^m) \to L^2(\Omega,\mathbb{R}^m)$ *satisfying the assumptions of the latter is given by* $\Lambda_j(u) = a_j^{-1}bu$*, as it follows from* (2.13) *and* (2.14)*.*

Analogously, Example 2.22 can be reinterpreted in terms of Proposition 3.12. For instance, when zero Dirichlet boundary conditions are imposed, consider the recovery operators $\widehat{\Lambda}_j : W_0^{1,2}((0,L);\mathbb{R}^m) \to W_0^{1,2}((0,L);\mathbb{R}^m)$ *given by*

$$\widehat{\Lambda}_j(u) = \int_0^x \Lambda_j(u')(y)\,dy - x \fint_0^L \Lambda_j(u')(y)\,dy,$$

with Λ_j *defined above.*

We are now ready to analyze the problem motivating this study and quoted in the introduction of the section. For a sequence $(B_j)_{j\in\mathbb{N}} \subset$

$\mathcal{M}(\Omega, \alpha, \beta)$ consider the corresponding quadratic forms on $L^2(\Omega)$

$$\mathscr{F}_j(u) = \begin{cases} \int_\Omega \langle B_j \nabla u, \nabla u \rangle \, dx & u \in W_0^{1,2}(\Omega) \\ +\infty & \text{otherwise in } L^2(\Omega). \end{cases} \tag{3.6}$$

The associated operators A_j can be expressed in terms of the distributional divergence on the sets

$$D_j = \{u \in W_0^{1,2}(\Omega) : \, -\mathrm{div}(B_j \nabla u) \in L^2(\Omega)\}$$

as $A_j u = -\mathrm{div}(B \nabla u)$. Note that the symmetric matrix B_j, the quadratic form $\mathscr{F}_j$ and the operator A_j are in one-to-one correspondence, hence in what follows we shall refer to G-convergence for $(B_j)_{j \in \mathbb{N}}$ meaning the G-convergence for the sequence of operators $(A_j)_{j \in \mathbb{N}}$.

We are in the setting of Theorem 3.11, thus the asymptotic behaviour of the solutions of

$$-\mathrm{div}(B_j \nabla u_j) = f, \quad \text{for } f \in H^{-1}(\Omega),$$

is characterized in terms of the G-convergence of $(A_j)_{j \in \mathbb{N}}$ in the weak topology of $L^2(\Omega)$, or equivalently via the Γ-convergence in the weak topology of $L^2(\Omega)$ of the associated quadratic forms $(F_{B_j})_{j \in \mathbb{N}}$.

Note that G-convergence is always ensured up to subsequences. In this respect, we mention the ensuing compactness result due to Spagnolo [67].

Theorem 3.14. *For any sequence $(B_j)_{j \in \mathbb{N}} \subset \mathcal{M}(\Omega, \alpha, \beta)$ there exists a matrix field $B \in \mathcal{M}(\Omega, \alpha, \beta)$ and a subsequence (not relabeled) such that $(B_j)_{j \in \mathbb{N}}$ G-converges weakly to B.*

Actually, a more refined analysis shows that the same result holds true in the strong $L^2(\Omega)$ and weak $W^{1,2}(\Omega)$ topology (see [25, Theorem 3.12, Example 3.13]). In addition, following the celebrated Div-Curl lemma, it is also possible to show that the sequence $(B_j \nabla u_j)_{j \in \mathbb{N}}$ converges weakly in $L^2(\Omega; \mathbb{R}^n)$ to the corresponding quantity for the G-limit (see [70, Lemma 10.3]).

The one-dimensional case, considered in Example 2.22, is particularly simple since G-convergence can be characterized in terms of the weak*-convergence in L^∞ of the inverse coefficients. More generally, a similar statement holds for isotropic operators as shown in [51, Theorem 2.2].

Proposition 3.15. *Let $(B_j)_{j \in \mathbb{N}} \in \mathcal{M}(\Omega, \alpha, \beta)$ be defined as $B_j = \varphi_j \mathrm{Id}_n$, with $\varphi_j(x) = \Pi_{i=1}^n b_j^i(x_i)$, then $(B_j)_{j \in \mathbb{N}}$ G-converges weakly to the diagonal matrix $B = \mathrm{diag}(b^i(x)) \in \mathcal{M}(\Omega, \alpha, \beta)$ if and only if*

$$\big((\varphi_j - b^i)/b_j^i\big)_{j \in \mathbb{N}} \text{ converge to } 0 \text{ weak}^* L^\infty(\Omega) \text{ for all } i \in \{1, \dots, n\}.$$

In general, it is not elementary to find an explicit formula for the G-limit, even in the simplest case, as the next density theorem confirms (see [51, Theorems 3.1, 3.4]).

Theorem 3.16. *There exists a dimensional constant $C \geq 1$ such that for any $B \in \mathcal{M}(\Omega, \alpha, \beta)$ one can find $(\varphi_j)_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that the sequence $(B_j)_{j\in\mathbb{N}}$, with $B_j = \varphi_j \mathrm{Id}_n$, belongs to $\mathcal{M}(\Omega, \alpha/C, C\beta)$ and G-converges weakly to B.*

For periodic coefficients the G-limit matrix B can be identified via a suitable minimization problem (see [32, Theorem 4.2]). Such a result builds upon the localization methods of Γ-convergence, with which far reaching extensions for nonlinear homogenization problems can be obtained as well (see for instance [11, Theorems 14.5, 14.7], and compare with [70] for a different approach).

Proposition 3.17. *Given $\widehat{B} \in \mathcal{M}(\Omega, \alpha, \beta)$ let $B_j(x) = \widehat{B}(x/\varepsilon_j)$, then $(B_j)_{j\in\mathbb{N}}$ G-converges weakly to B defined as*

$$\langle B(x)\xi, \xi\rangle = \inf\left\{\int_{(0,1)^n} \langle \widehat{B}(x)\nabla u, \nabla u\rangle\, dx : \, u - \langle x, \xi\rangle \in W^{1,2}_{\#}((0,1)^n)\right\},$$

where $W^{1,2}_{\#}((0,1)^n)$ denotes the subspace of $W^{1,2}((0,1)^n)$ functions that are $(0,1)^n$-periodic.

Let us conclude the section by remarking that in this notes we have only scratched the surface of the vast fields or research which are nowadays G-convergence and homogenization. We refer to the books [11], [20], [44], [60], [70] for in-depth accounts of those theories supplemented with several applications and detailed references. Many links with physical models and further problems in the theory of composites can be found in the book by Milton [56].

3.3. **G-convergence for non-self-adjoint linear operators.** In this paragraph we deal with non-self-adjoint operators, or more precisely, given $f \in H^{-1}(\Omega)$, with non-variational PDEs of the form

$$-\mathrm{div}(B\nabla w) = f \quad \mathcal{L}^n \text{ a.e. } \Omega, \text{ with } w \in W^{1,2}_0(\Omega). \tag{3.7}$$

The important fact is that we do *not* require the matrix field $B : \Omega \to \mathbb{R}^{n\times n}$ to be symmetric $\mathcal{L}^n$ a.e. in Ω. As a consequence, the corresponding operator is not self-adjoint and the variational structure of the problem is lost. The appropriate operatorial notion of convergence in this case has been introduced by Tartar [69, 70] and Murat [59] and called H-convergence.

Let us first fix the functional framework under analysis: we consider the subset $\widetilde{\mathcal{M}}(\Omega, \alpha, \beta)$ of $n \times n$ matrices B with Borel measurable coefficients satisfying

$$\alpha\,\mathrm{Id}_n \leq B(x), \quad \beta^{-1}\mathrm{Id}_n \leq B^{-1}(x) \quad \text{for } \mathcal{L}^n \text{ a.e. } x \in \Omega,$$

where Id_n is the $n \times n$ identity matrix, and $0 < \alpha \leq \beta$. The second condition easily yields that $\|B\|_{L^\infty(\Omega,\mathbb{R}^{n\times n})} \leq \beta$. Lax-Milgram theorem implies the existence of solutions to the problem above under the quoted assumptions (see [14, Corollary V.8]).

Definition 3.18. *A sequence $(B_j)_{j\in\mathbb{N}} \subset \widetilde{\mathcal{M}}(\alpha, \beta, \Omega)$ H-converges to $B \in \widetilde{\mathcal{M}}(\alpha, \beta, \Omega)$ if for all $f \in H^{-1}(\Omega)$ the solutions $u_j \in W^{1,2}_0(\Omega)$ of* (3.7) *satisfy*

(i) $(u_j)_{j\in\mathbb{N}}$ converges weakly to u in $W^{1,2}(\Omega)$,
(ii) $(B_j\nabla u_j)_{j\in\mathbb{N}}$ converges weakly to $B\nabla u$ in $L^2(\Omega;\mathbb{R}^m)$.

Therefore, keeping the notation in Definition 3.18 above, we get that $u \in W^{1,2}_0(\Omega)$ is the solution of

$$-\mathrm{div}(B\nabla w) = f \quad \mathcal{L}^n \text{ a.e. } \Omega, \text{ with } w \in W^{1,2}_0(\Omega).$$

Remark 3.19. *Comparing Definitions 3.8 and 3.18 few remarks are in order. First, the extra condition (ii) in Definition 3.18 is necessary to determine uniquely the limit since*

$$\mathrm{div}(B(x)\nabla w(x)) = \mathrm{div}\left((B(x) + B')\nabla w(x)\right) \quad \mathcal{L}^n \text{ a.e. } \Omega,\ w \in W^{1,2}_0(\Omega)$$

for any skew-symmetric matrix B' with sufficiently small norm to retain the ellipticity bounds.

In addition, as already noted, the convergence of $(B_j\nabla u_j)_{j\in\mathbb{N}}$ to $B\nabla u$ is guaranteed in the symmetric case. In particular, H-convergence reduces to G-convergence in that setting (see [70, Lemma 10.3]*).*

Rather than overviewing the theory of H-convergence, for which we quote the references [69, 70, 59], we discuss some recent new insights by Ansini and Zeppieri [4] with which H-convergence for elliptic PDEs is recasted into the framework of G- or Γ-convergence building upon a variational principle introduced by Cherkaev and Gibiansky [19]. Actually, the developments of the latter proposed by Milton [55] are of interest here.

The starting point is to consider the coupled system

$$\begin{cases} -\mathrm{div}(B\nabla u) = f_1 & u \in W^{1,2}_0(\Omega) \\ -\mathrm{div}(B^t\nabla v) = f_2 & v \in W^{1,2}_0(\Omega) \end{cases} \tag{3.8}$$

which can be rewritten as

$$\begin{cases} -\mathrm{div} j_u = f_1 & u \in W_0^{1,2}(\Omega) \\ -\mathrm{div} j_v = f_2 & v \in W_0^{1,2}(\Omega) \end{cases} \tag{3.9}$$

by introducing the fields

$$\begin{cases} j_u = B\nabla u \\ j_v = B^t \nabla v. \end{cases} \tag{3.10}$$

In applications to electrostatics u and v are potentials with associated electric fields $-\nabla u$ and $-\nabla v$, B is the conductivity tensor, j_u and j_v the current fields. Then, (3.10) represents the constitutive laws.

Denoting the symmetric and antisymmetric part of a matrix with the superscripts s and a, respectively, and setting $\phi = u+v$, $\psi = u-v$, the system in (3.10) can be equivalently expressed as

$$\begin{cases} B^s\nabla\phi + B^a\nabla\psi = j_u + j_v \\ -B^a\nabla\phi - B^s\nabla\psi = j_u - j_v \end{cases} \tag{3.11}$$

that is in more compact form

$$\Theta_B \begin{pmatrix} \nabla\phi \\ \nabla\psi \end{pmatrix} = \begin{pmatrix} j_u + j_v \\ j_u - j_v \end{pmatrix}, \text{ where } \Theta_B = \begin{pmatrix} B^s & B^a \\ -B^a & -B^s \end{pmatrix}.$$

The matrix field $\Theta_B \in L^\infty(\Omega, \mathbb{R}^{2n\times 2n})$ is symmetric but not positive definite. Being B^s positive definite $\mathcal{L}^n$ a.e., B defines the convex-concave quadratic form $\mathscr{Q}_B : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to [0,+\infty]$ given by

$$\mathscr{Q}_B(\phi,\psi) = \frac{1}{2}\int_\Omega \Theta_B \begin{pmatrix} \nabla\phi \\ \nabla\psi \end{pmatrix} : \begin{pmatrix} \nabla\phi \\ \nabla\psi \end{pmatrix} dx,$$

where the symbol : denotes the scalar product in $\mathbb{R}^{2n}$. By introducing the linear functional $\mathscr{L}$ on $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$

$$\mathscr{L}(\phi,\psi) = \int_\Omega \begin{pmatrix} j_u + j_v \\ j_u - j_v \end{pmatrix} : \begin{pmatrix} \nabla\phi \\ \nabla\psi \end{pmatrix} dx,$$

the saddle point variational principle

$$\inf_{\phi\in W_0^{1,2}(\Omega)} \sup_{\psi\in W_0^{1,2}(\Omega)} (\mathscr{Q}_B - \mathscr{L}) = \sup_{\psi\in W_0^{1,2}(\Omega)} \inf_{\phi\in W_0^{1,2}(\Omega)} (\mathscr{Q}_B - \mathscr{L}) \tag{3.12}$$

identifies the solutions of (3.11).

A partial Legendre transform can be employed to obtain a convex quadratic form, and a corresponding minimization principle. This amounts to rewriting (3.11) as

$$\begin{cases}(B^s)^{-1}(j_u+j_v)-(B^s)^{-1}B^a\nabla\psi=\nabla\phi\\ -B^a(B^s)^{-1}(j_u+j_v)+(B^s-B^a(B^s)^{-1}B^a)\nabla\psi=j_u-j_v\end{cases}\tag{3.13}$$

that is

$$\Sigma_B\begin{pmatrix} j_u+j_v\\ \nabla\psi\end{pmatrix}=\begin{pmatrix}\nabla\phi\\ j_u-j_v\end{pmatrix},$$

where

$$\Sigma_B=\begin{pmatrix}(B^s)^{-1} & -(B^s)^{-1}B^a\\ B^a(B^s)^{-1} & B^s-B^a(B^s)^{-1}B^a\end{pmatrix}.\tag{3.14}$$

The very definition of Σ_B yields the ensuing properties (see [4, Section 3.1.1] for the proofs).

Proposition 3.20. *If $B\in\widetilde{\mathcal{M}}(\Omega,\alpha,\beta)$, the matrix field $\Sigma_B\in L^\infty(\Omega,\mathbb{R}^{2n\times 2n})$ satisfies for $\mathcal{L}^n$ a.e. $x\in\Omega$*

(i) Σ_B *is symmetric, and coercive, i.e.* $\lambda\mathrm{Id}_{2n}\leq\Sigma_B$ *for some* $\lambda>0$*;*
(ii) $\det\Sigma_B=1$
(iii) $\Sigma_B^{-1}=J\Sigma_B J$, *where if* 0_n *is the null* $n\times n$ *matrix, then*

$$J=\begin{pmatrix}0_n & \mathrm{Id}_n\\ \mathrm{Id}_n & 0_n\end{pmatrix}.$$

It is now possible to turn the minimax principle (3.12) leading to (3.8) into a minimization one. Let $\Sigma_B\in L^\infty(\Omega,\mathbb{R}^{2n\times 2n})$ be as in (3.14) for some $B\in\widetilde{\mathcal{M}}(\Omega,\alpha,\beta)$, fix h in $L^2(\Omega;\mathbb{R}^n)$, $k\in W^{1,2}(\Omega)$ and $f\in H^{-1}(\Omega)$, then define the linearly perturbed quadratic form

$$\mathscr{F}_B^{f,h,k}(j,\psi)=\frac{1}{2}\int_\Omega\Sigma_B\begin{pmatrix} j\\ \nabla\psi\end{pmatrix}:\begin{pmatrix} j\\ \nabla\psi\end{pmatrix}dx-\int_\Omega(2\langle j,k\rangle+\langle\nabla\psi,h\rangle)\,dx,$$

on the convex set

$$K_f=\{(j,\psi)\in L^2(\Omega;\mathbb{R}^n)\times W_0^{1,2}(\Omega),\ \mathrm{div}j=f\}$$

and $+\infty$ otherwise on $L^2(\Omega;\mathbb{R}^n)\times W^{1,2}(\Omega)$.

Elementary computations shows that the Euler-Lagrange equation satisfied by minimizers of $\mathscr{F}_B^{f,h,k}$ is exactly the system in (3.8) with $\psi=u-v$, $f_1=-(f+\mathrm{div}h)/2$, $f_2=-(f-\mathrm{div}h)/2$, and $k=0$ (see [4, Section 3.2]).

We are now able to resume the contents of [4, Theorems 4.1 and 4.5] that interpret H-convergence as Γ-convergence of suitable quadratic forms. The equivalences below rely upon Theorem 3.11 and some technical properties of H-convergence.

Theorem 3.21. *Let B, $(B_j)_{j\in\mathbb{N}}\subset\widetilde{\mathcal{M}}(\Omega,\alpha,\beta)$, $h\in L^2(\Omega;\mathbb{R}^n)$ and $k\in W_0^{1,2}(\Omega)$, then the following are equivalent*

(a) $(B_j)_{j\in\mathbb{N}}$ *H-converges to* B;
(b) $(\mathscr{F}^{f,h,k}_{B_j})_{j\in\mathbb{N}}$ Γ*-converges to* $\mathscr{F}^{f,h,k}_{B}$ *in the weak topology of* $L^2(\Omega;\mathbb{R}^n)\times W^{1,2}(\Omega)$ *for every* $f\in H^{-1}(\Omega)$.

3.4. **An extension to nonlinear elliptic equations.** In this final section we describe a contribution by Ambrosetti and Sbordone [2] aimed at obtaining convergence of solutions of Euler-Lagrange equations related to non-convex energies. The significant outcome of their analysis is that the solutions of the approximating equations are, in principle, neither *global* nor *local* minimizers of the energies under study. Despite this, their asymptotic behaviour can still be studied via Γ-convergence methods.

We shall consider the Hilbertian setting for the sake of simplicity, in the original paper [2] the results are stated in the broader framework of reflexive and separable Banach spaces. First, we introduce a weak notion of subdifferentiability.

Definition 3.22. *Let* $\mathscr{F}:X\to\overline{\mathbb{R}}$, *put* $D_{\mathscr{F}}:=\{x\in X:\,\mathscr{F}(x)\in\mathbb{R}\}$. *For* $x\in D_{\mathscr{F}}$ *consider (the possibly empty set)*

$$\partial^-\mathscr{F}(x):=\left\{z\in X:\,\liminf_{y\to x}\frac{\mathscr{F}(y)-\mathscr{F}(x)-\langle z,y-x\rangle}{\|y-x\|}\geq 0\right\},$$

and for $x\in X\setminus D_{\mathscr{F}}$ *set* $\partial^-\mathscr{F}(x)=\emptyset$.

If $\partial^-\mathscr{F}(x)\neq\emptyset$, *we say that* $\mathscr{F}$ *is subdifferentiable at* x, *and* $z\in\partial^-\mathscr{F}(x)$ *is said a subdifferential of* $\mathscr{F}$ *at* x.

Simple properties coming from the very definition are subsumed below (cp. with [2, Lemma 3.2, Proposition 3.3]).

Lemma 3.23. *Let* $\mathscr{F}:X\to\overline{\mathbb{R}}$ *and* $x\in X$. *Then,*

(i) *if* x *is a minimum point of* $\mathscr{F}$, *then* $\underline{0}\in\partial^-\mathscr{F}(x)$;
(ii) *if* $\mathscr{F}$ *is convex, then* $\partial^-\mathscr{F}(x)$ *coincides with the subdifferential of* $\mathscr{F}$ *in the sense of convex analysis;*
(iii) *if* $\mathscr{F}=\mathscr{F}_1+\mathscr{F}_2$ *with* $\mathscr{F}_2$ *differentiable, then*

$$\partial^-\mathscr{F}(x)=\partial^-\mathscr{F}_1(x)+\mathscr{F}_2'(x).$$

(iv) *if* $\mathscr{F}=\mathscr{F}_1+\mathscr{F}_2$, *with* $\mathscr{F}_1$ *convex and* $\mathscr{F}_2$ *differentiable, such that*

$$\langle\mathscr{F}_2'(y_1)-\mathscr{F}_2'(y_2),y_1-y_2\rangle\geq\langle\varphi'(x-y_1)-\varphi'(x-y_2),y_1-y_2\rangle$$

for all x, y_1, $y_2\in X$ *and for some differentiable function* $\varphi:X\to\mathbb{R}$ *with* $\varphi(\underline{0})=\varphi'(\underline{0})=0$, *then*

$$\partial^-\mathscr{F}(x)=\{z\in X:\,\mathscr{F}(y)-\langle z,y-x\rangle+\varphi(y-x)\geq\mathscr{F}(x),\ \forall y\in X\}.$$

Semiconvex functions, also known as λ-convex, i.e. functions $\mathscr{F}: X \to \mathbb{R}$ such that $\mathscr{F} + \lambda\|\cdot\|^2$ is convex for some constant $\lambda \geq 0$, satisfy the assumptions in (iv) above with $\varphi(x) = -\lambda\|x\|^2$.

More generally, in what follows we shall consider the class $Q_\varphi(X)$ of functions satisfying for all $x \in X$ for which $\partial^-\mathscr{F}(x) \neq \emptyset$ equality

$$\partial^-\mathscr{F}(x) = \{z \in X : \mathscr{F}(y) - \langle z, y - x\rangle + \varphi(y - x) \geq \mathscr{F}(x), \ \forall y \in X\}, \tag{3.15}$$

with $\varphi : X \to [0, +\infty)$ weakly continuous and differentiable, such that $\varphi^{-1}(0) = \{\underline{0}\}$, $\varphi' : X \to X$ is continuous from the weak topology of X to the strong topology of X and $\varphi'(\underline{0}) = \underline{0}$.

The crucial feature of such a class of functions is that if $\mathscr{F} \in Q_\varphi(X)$ and $z \in \partial^-\mathscr{F}(x)$, then x is a global minimizer of the perturbed functionals

$$\mathscr{G}(y) = \mathscr{F}(y) - \langle z, y - x\rangle + (1 + \delta)\varphi(y - x),$$

for all $\delta \geq 0$, and actually the *unique* minimizer if $\delta > 0$. Γ-convergence machinery can then be exploited. In the next theorem G-convergence for multiple-valued mappings is intended according to Remark 3.10.

Theorem 3.24. *Let $\mathscr{F}$, $\mathscr{F}_j : X \to \mathbb{R} \cup \{+\infty\}$ be weakly lower semicontinuous satisfying*

(i) $\mathscr{F}$ and each $\mathscr{F}_j \in Q_\varphi(X)$ for some φ as above;
(ii) $\mathscr{F}_j(x) \geq \psi(\|x\|)$ for all $x \in X$ and for some $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(t) \uparrow +\infty$ as $t \uparrow +\infty$, and $\sup_j \mathscr{F}_j(\bar{x}) < +\infty$ for some $\bar{x} \in X$.

If $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges weakly to $\mathscr{F}$, then $(\partial^-\mathscr{F}_j)_{j\in\mathbb{N}}$ G-converges weakly to $\partial^-\mathscr{F}$.

Proof. Let $z_j \in \partial^-\mathscr{F}_j(x_j)$, with z_j converging strongly to z in X and x_j weakly to x in X. Consider the functions

$$\mathscr{G}_j(y) := \mathscr{F}_j(y) - \langle z_j, y - x_j\rangle + \varphi(y - x_j),$$

assumption (i) for $\mathscr{F}_j$ then rewrites as

$$\mathscr{G}_j(x_j) \leq \mathscr{G}_j(y) \quad \text{for all } y \in X.$$

Moreover, by hypotheses it follows that

$$\mathscr{G}(y) = \Gamma(w\text{-}X)\text{-}\lim_j \mathscr{G}_j(y) = \mathscr{F}(y) - \langle z, y - x\rangle + \varphi(y - x),$$

and thus x turns out to minimize $\mathscr{G}$ on X by Theorem 2.11. Noting that $\mathscr{G}(x) = \mathscr{F}(x)$ we get

$$\mathscr{F}(y) - \langle z, y - x\rangle + \varphi(y - x) \geq \mathscr{F}(x) \quad \text{for all } y \in X,$$

then assumption $\varphi'(\underline{0}) = \underline{0}$ implies that $z \in \partial^-\mathscr{F}(x)$.

Vice versa, suppose that $z \in \partial^- \mathscr{F}(x)$. With fixed $\delta > 0$, assumption (i) for $\mathscr{F}$ implies that for all $y \in X \setminus \{x\}$

$$\mathscr{G}(x) < \mathscr{G}(y) := \mathscr{F}(y) - \langle z, y - x\rangle + (1+\delta)\varphi(y-x).$$

Let now

$$\mathscr{G}_j(y) := \mathscr{F}_j(y) - \langle z, y - x\rangle + (1+\delta)\varphi(y-x),$$

if $x_j \in \operatorname{argmin}_X \mathscr{G}_j$ then (ii) and the positivity of φ give

$$\psi(\|x_j\|) - \|z\|\|x_j - x\| \le \mathscr{G}_j(x_j) \le \mathscr{G}_j(\bar{x}) \Longrightarrow \sup_j \|x_j\| < +\infty.$$

Since $\Gamma(w\text{-}X)\text{-}\lim_j \mathscr{G}_j = \mathscr{G}$, and being x the unique minimizer of $\mathscr{G}$, Theorem 2.11 implies that the whole sequence $(x_j)_{j\in\mathbb{N}}$ converges weakly to x by Urysohn property.

Eventually, arguing as before $\underline{0} \in \partial^-\mathscr{G}_j(x_j) = \partial^-\mathscr{F}_j(x_j) - z - \varphi'(x_j - x)$. Setting $z_j = z + \varphi'(x_j - x) \in \partial^-\mathscr{F}_j(x_j)$, the weak-strong continuity of φ', $\varphi'(\underline{0}) = \underline{0}$ and the weak convergence of $(x_j)_{j\in\mathbb{N}}$ to x give the conclusion. □

Remark 3.25. *Following* [35, Section 4] *it is possible to show that the class $Q_\varphi(X)$ is closed under Γ-convergence, so that assumption $\mathscr{F} \in Q_\varphi(X)$ in Theorem 3.24 above is actually not needed.*

A basic example in which the assumptions of Theorem 3.24 are satisfied is given by

$$\mathscr{F}_j(x) = \frac{1}{2}\langle A_j x, x\rangle + b(x),$$

with $(A_j)_{j\in\mathbb{N}}$ G-converging weakly in X, and $b : X \to \mathbb{R}$ differentiable, bounded from below, weakly continuous and satisfying for all x, $y \in X$

$$|\langle b'(x) - b'(y), x - y\rangle| \le \|x - y\|^2.$$

The latter condition guarantees the equi-generalized subdifferentiability hypothesis in (i) Theorem 3.24 with $\varphi = \|\cdot\|^2/2$ (see [2, Proposition 3.3, Corollary 4.4]). In general, the corresponding Euler-Lagrange equations

$$A_j x + b'(x) = 0$$

have multiple solutions, the generic one denoted by x_j, that are obtained via topological methods (see [2, Remark 4.5]). Hence, x_j is not necessarily a minimizer of $\mathscr{F}_j$. Despite this, the stability brought into the problem by condition $\mathscr{F}_j \in Q_\varphi(X)$ yields that x_j is actually the only global minimizer of a suitable perturbed function, so that variational properties of the cluster points of the sequence $(x_j)_{j\in\mathbb{N}}$ can be determined via Γ-convergence.

In this respect, we mention that Theorem 3.24 is the forerunner of a broad theory developed in a series of papers by De Giorgi, Degiovanni, Marino and Tosques (see [35] and the references therein). Generalized notions of convexity for functions and of monotonicity for subdifferentials, departing strictly from the standard ones and including that in (3.15) presented here, were introduced and proved to guarantee the convergence of Euler-Lagrange equations given the Γ-convergence of the corresponding energies. In this way, several variational problems with lack of convexity can be analyzed via those methods.

The initial interest in those researches relied in the investigation of the connection between Γ-convergence of sequence of energies and the convergence of solutions of the corresponding evolution equations. Under the mentioned generalized convexity for energies or monotonicity for subdifferentials, the convergence of gradient-flows, suitably reformulated, follows (see [35] again). In parallel, a non-variational theory for operators which are not subdifferentials is developed, with that some hyperbolic problems fall into the analysis as well.

Recently, Sandier and Serfaty have proposed a general scheme to study convergence issues for gradient flows (working also in a metric space setting), furnishing several interesting applications as well (see [63], [65] and the references therein).

Along the same paths, let us mention that far reaching generalizations of the ideas presented in Theorem 3.24 have been lately developed by Mielke, Roubíček and Stefanelli [54] to analyze relaxation and convergence problems for energetic solutions of rate-independent evolutionary mechanical systems.

4. Linearized elasticity as Γ-limit of finite elasticity

Let Ω be a bounded open subset of $\mathbb{R}^n$, and consider an homogeneous hyperelastic body for which Ω represents the reference configuration. Let

$$\mathscr{E}(v) = \int_\Omega W(\nabla v)dx$$

be the stored energy functional, where $v : \Omega \to \mathbb{R}^n$ is the deformation, ∇v the deformation gradient, and the energy density $W : \mathbb{R}^{n\times n} \to [0, +\infty]$ satisfies

(W1) $W(RF) = W(F)$ for all $F \in \mathbb{R}^{n\times n}$ and $R \in SO(n)$, that is W is *frame indifferent*;
(W2) W is of class $C^2(U)$ and $\sup_U W < +\infty$, where U is a neighbourhood of $SO(n)$;
(W3) $W(F) = 0$ for all $F \in SO(n)$;

(W4) $W(F) \geq \operatorname{dist}^2(F, SO(n))$ for all $F \in \mathbb{R}^{n\times n}$;
(W5) $W(F) = +\infty$ if $\det F \leq 0$.

Thanks to assumption (W4), admissible deformations belong to the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^n)$. External loads are represented by a continuous linear functional $\mathscr{L} : W^{1,2}(\Omega, \mathbb{R}^n) \to \mathbb{R}$, we may also fix boundary displacements $h \in W^{1,2}(\Omega, \mathbb{R}^n)$ by imposing a Dirichlet boundary condition.

We are interested in studying the behaviour of the stable states of the system in case of small deformation regime. Thus, with fixed a sequence $\varepsilon_j \downarrow 0$, we are led to the variational problem

$$m_j := \min_{u\in h+W_0^{1,2}(\Omega,\mathbb{R}^n)} \int_\Omega W(\mathrm{Id}_n + \varepsilon_j \nabla u)dx - \varepsilon_j \mathscr{L}(x + \varepsilon_j u). \tag{4.1}$$

In what follows we shall characterize the behaviour of the minimimum problems above by resorting to the Γ-convergence properties of the associated rescaled energies $\mathscr{F}_j : X \to [0, +\infty]$

$$\mathscr{F}_j(u) = \frac{1}{\varepsilon_j^2} \int_\Omega W(\mathrm{Id}_n + \varepsilon_j \nabla u)dx - \mathscr{L}(u), \tag{4.2}$$

with $X := h + W_0^{1,2}(\Omega, \mathbb{R}^n)$ the space of admissible displacements.

The rescaling ε_j^{-2} is necessary to avoid trivial results. Indeed, denoting by $\mathscr{E}_j$ the energies to be minimized in (4.1), it is easy to see that $\Gamma\text{-}\lim_j \mathscr{E}_j(u) = 0$ and that $\Gamma\text{-}\lim_j {\varepsilon_j}^{-1}\mathscr{E}_j(u) = -\mathscr{L}(x)$ for all $u \in X$ in the weak $W^{1,2}$ topology in view of assumptions (W2)-(W4). Thus, the energies

$$\mathscr{F}_j(u) = {\varepsilon_j}^{-1}({\varepsilon_j}^{-1}\mathscr{E}_j(u) + \mathscr{L}(x))$$

are introduced in the spirit of the asymptotic Γ-development (cp. with subsection 2.4).

Further notation is needed: given a positive definite symmetric fourth order tensor Θ the associated quadratic form is denoted by $\Theta[\cdot, \cdot]$, i.e. for every $n \times n$ matrix A if $\Theta = (\theta_{ijhk})_{i,j,h,k=1}^n$

$$\Theta[A, A] = \sum_{i,j,h,k=1}^n \theta_{ijhk} a_{ij} a_{hk}.$$

Note that being Θ a symmetric tensor

$$\Theta[A, A] = \Theta\left[\frac{A + A^t}{2}, \frac{A + A^t}{2}\right].$$

Theorem 4.1. *Assume that $W : \mathbb{R}^{n\times n} \to [0, +\infty]$ satisfies properties (W1)-(W5) above, then*

(i) *Compactness: if* $\sup_j \mathscr{F}_j(u_j) < +\infty$*, then* $\sup_j \|u_j\|_{W^{1,2}(\Omega)} < +\infty$*;*
(ii) Γ*-convergence:* $\Gamma\text{-}\lim_j \mathscr{F}_j = \mathscr{F}$ *in the weak* $W^{1,2}$ *topology, where* $\mathscr{F} : X \to [0,+\infty]$ *is given by*
$$\mathscr{F}(u) = \frac{1}{2}\int_\Omega D^2W(\mathrm{Id}_n)[e(u), e(u)]dx - \mathscr{L}(u),$$
with $e(u) = (\nabla u + \nabla^t u)/2$ *the linearized strain of* u*.*

Before going into the proof, we note that Theorem 4.1 together with Theorem 2.11 ensure the convergence of global minimizers of $\mathscr{F}_j$ to that of $\mathscr{F}$ together with the Taylor expansion
$$m_j = -\varepsilon_j \mathscr{L}(x) + \varepsilon_j{}^2 \min_{u\in X} \mathscr{F} + o(\varepsilon_j{}^2).$$
Thus, Theorem 4.1 provides a rigorous variational argument for the derivation of linearized elasticity from finite elasticity, contrary to the usual Taylor expansion which corresponds to checking (UB) inequality only for Lipschitz displacements (see the proof of Theorem 4.1).

Let us also mention that the one stated above is only a prototype result which can be improved into many directions. First, (qualified) non-homogeneous materials can be considered, Dirichlet boundary conditions can be imposed only on a subset of the boundary of Ω, the growth conditions in (W4) can be relaxed in order to include the analysis of a large class of compressible rubber-like materials, and the convergence of recovery sequences can be improved to strong $W^{1,2}(\Omega,\mathbb{R}^n)$ and not only weak (for all these generalizations see [3]). In addition, families of multiwell energies can be also dealt with (cp. with [64]).

In the sequel we shall prove Theorem 4.1 neglecting the continuous perturbation $\mathscr{L}$ since it affects neither the Γ-convergence nor the equi-coerciveness properties of $(\mathscr{F}_j)_{j\in\mathbb{N}}$ thanks to (W4). With a slight abuse of notation we keep denoting by $\mathscr{F}_j$ the first term on the right hand side in (4.2).

The equi-coerciveness properties of $(\mathscr{F}_j)_{j\in\mathbb{N}}$ rely on a key result by Friesecke, James and Müller [41] in which the L^2-distance of a gradient field from $SO(n)$ is controlled via the L^2-distance from a single rotation modulo a universal constant prefactor.

Theorem 4.2 (Geometric Rigidity)**.** *There exists a constant* $C = C(\Omega) > 0$ *such that for every field* $v \in W^{1,2}(\Omega,\mathbb{R}^n)$ *there exists a constant rotation* $R \in SO(n)$ *satisfying*
$$\int_\Omega |\nabla v - R|^2 dx \le C\int_\Omega \mathrm{dist}^2(\nabla v, SO(n))dx.$$

The geometric rigidity estimate has been employed as an essential ingredient for the variational derivation of several low dimensional theories from three-dimensional elasticity models (see, for instance, [41] and [42] and the references therein).

Lemma 4.3. *Under the assumptions of Theorem 4.1 there exists a constant $C = C(\Omega) > 0$ such that if $u \in X$ and $j \in \mathbb{N}$ are fixed, there is a rotation $R_j \in SO(n)$ for which*

$$|\mathrm{Id}_n - R_j|^2 \leq C{\varepsilon_j}^2 \left(\mathscr{F}_j(u) + \int_{\partial\Omega} |h|^2 d\mathcal{H}^{n-1}\right).$$

Proof. Given $u \in X$, let $v_j = x + \varepsilon_j u$, then by Theorem 4.2

$$\int_\Omega |\nabla v_j - R_j|^2 dx \leq C \int_\Omega d^2(\nabla v_j, SO(n))dx.$$

for some $R_j \in SO(n)$; in turn, assumption (W4) implies

$$\int_\Omega |\nabla v_j - R_j|^2 dx \leq C \int_\Omega W(\nabla v_j)dx \leq C{\varepsilon_j}^2 \mathscr{F}_j(u). \tag{4.3}$$

Poincaré-Wirtinger inequality and the continuity of the trace operator yield

$$\int_{\partial\Omega} |v_j - R_j x - a_j|^2 d\mathcal{H}^{n-1} \leq C \int_\Omega |v_j - R_j x - a_j|^2 dx \leq C \int_\Omega |\nabla v_j - R_j|^2 dx \tag{4.4}$$

if a_j denotes the mean value of $v_j - R_j x$ on Ω. Then, recalling that $u - h \in W_0^{1,2}(\Omega, \mathbb{R}^n)$, we have $v_j = x + \varepsilon_j h$ on $\partial\Omega$, and thus

$$\int_{\partial\Omega} |(\mathrm{Id}_n - R_j)x - a_j|^2 d\mathcal{H}^{n-1} \leq 2 \int_{\partial\Omega} |v_j - R_j x - a_j|^2 d\mathcal{H}^{n-1} + 2\varepsilon_j^2 \int_{\partial\Omega} |h|^2 d\mathcal{H}^{n-1}. \tag{4.5}$$

[27, Lemma 3.3] ensures that for some constant $C = C(\Omega) > 0$

$$|\mathrm{Id}_n - R_j|^2 \leq C \min_{z \in \mathbb{R}^n} \int_{\partial\Omega} |(\mathrm{Id}_n - R_j)x - z|^2 d\mathcal{H}^{n-1}, \tag{4.6}$$

by collecting estimates (4.3)-(4.6) the desired result follows at once. □

We are now ready to show Theorem 4.1, we follow the approach in [3] (the result was first proved under more stringent hypothesis in [27]).

Proof of Theorem 4.1. Let us first prove that the family $(\mathscr{F}_j)_{j\in\mathbb{N}}$ is equi-coercive in the weak $W^{1,2}$ topology. To this aim suppose a sequence $(u_j)_{j\in\mathbb{N}}$ is given such that $\sup_j \mathscr{F}_j(u_j) < +\infty$, and let $v_j = x + \varepsilon_j u_j$. Then, by (W4) and Lemma 4.3, for some $R_j \in SO(n)$ we have

$$\int_\Omega |\nabla u_j|^2 dx \leq \frac{2}{\varepsilon_j^2}\left(\int_\Omega |\nabla v_j - R_j|^2 dx + \mathcal{L}^n(\Omega)|\mathrm{Id}_n - R_j|^2\right) \leq C\left(\mathscr{F}_j(u_j) + \int_{\partial\Omega}|h|^2 d\mathcal{H}^{n-1}\right). \quad (4.7)$$

Moreover, since $u_j - h \in W_0^{1,2}(\Omega,\mathbb{R}^n)$, Poincaré inequality yields that

$$\|u_j\|_{L^2(\Omega)} \leq C\|\nabla u_j\|_{L^2(\Omega,\mathbb{R}^n)}$$

for some constant $C > 0$, and the equi-coerciveness thus follows straightforwardly from (4.7).

Let us now prove the Γ-convergent statement. To this aim we shall show separately the inequalities $\Gamma\text{-}\liminf_j \mathscr{F}_j(u) \geq \mathscr{F}(u)$ and $\Gamma\text{-}\limsup_j \mathscr{F}_j(u) \leq \mathscr{F}(u)$ (cp. with Theorem 2.15).

We begin with the former inequality. Given a sequence $(u_j)_{j\in\mathbb{N}}$ converging weakly in $W^{1,2}$ to a map u we may assume (up to extracting a subsequence not relabeled for convenience) that

$$\liminf_j \mathscr{F}_j(u_j) = \lim_j \mathscr{F}_j(u_j) < +\infty.$$

In particular, $(u_j)_{j\in\mathbb{N}} \subseteq X$ and then by weak $W^{1,2}$ convergence $u \in X$. Set

$$\Omega_j = \{x \in \Omega : |\nabla u_j(x)| \leq \varepsilon_j^{-1/2}\}$$

and note that $(\mathbf{1}_{\Omega_j}\nabla u_j)_{j\in\mathbb{N}}$ converges to ∇u weakly in L^2 since by Chebyshev inequality and (4.7)

$$\mathcal{L}^n(\Omega_j^c) \leq \varepsilon_j \int_\Omega |\nabla u_j|^2 dx \leq C\varepsilon_j\left(\mathscr{F}_j(u_j) + \int_{\partial\Omega}|h|^2 d\mathcal{H}^{n-1}\right) \leq C\varepsilon_j.$$

Next, we use (W2)-(W4) to infer the existence of an increasing function $\eta : [0,+\infty) \to [0,+\infty)$ with $\eta(t) \downarrow 0^+$ as $t \downarrow 0^+$ and such that for all $F \in \mathbb{R}^{n\times n}$

$$W(\mathrm{Id}_n + F) \geq \frac{1}{2}\partial^2 W(\mathrm{Id}_n)[F,F] - \eta(|F|)|F|^2.$$

In turn, from this and the very definition of Ω_j we find

$$\mathscr{F}_j(u_j) \geq \frac{1}{\varepsilon_j^2}\int_{\Omega_j} W(\mathrm{Id}_n + \varepsilon_j \nabla u_j)dx$$

$$\geq \int_{\Omega_j} \left(\frac{1}{2} \partial^2 W(\mathrm{Id}_n)[\nabla u_j, \nabla u_j] - \eta(\varepsilon_j |\nabla u_j|) |\nabla u_j|^2 \right) dx$$
$$\geq \frac{1}{2} \int_\Omega \partial^2 W(\mathrm{Id}_n)[\mathbf{1}_{\Omega_j} \nabla u_j, \mathbf{1}_{\Omega_j} \nabla u_j] dx - \eta(\varepsilon_j^{1/2}) \int_\Omega |\nabla u_j|^2 dx.$$

In particular, by taking into account the weak L^2 lower semicontinuity of the quadratic form associated to $\partial^2 W(\mathrm{Id}_n)$ and being $(u_j)_{j\in\mathbb{N}}$ bounded in the $W^{1,2}$ norm we infer

$$\liminf_j \mathscr{F}_j(u_j) \geq \liminf_j \frac{1}{2} \int_\Omega \partial^2 W(\mathrm{Id}_n)[\mathbf{1}_{\Omega_j} \nabla u_j, \mathbf{1}_{\Omega_j} \nabla u_j] dx$$
$$\geq \frac{1}{2} \int_\Omega \partial^2 W(\mathrm{Id}_n)[\nabla u, \nabla u] dx = \frac{1}{2} \int_\Omega \partial^2 W(\mathrm{Id}_n)[e(u), e(u)] dx = \mathscr{F}(u).$$

In particular, we conclude that $\Gamma\text{-}\liminf_j \mathscr{F}_j(u) \geq \mathscr{F}(u)$.

To prove the opposite inequality we first exhibit a recovery sequence for displacements $u \in W^{1,\infty}(\Omega, \mathbb{R}^n) \cap X$. In this case set $u_j = u$ and notice that Taylor expansion around the identity matrix is allowed since ∇u is bounded $\mathcal{L}^n$ a.e. on Ω. Hence, by taking into account that W is C^2 close to Id_n, $W(\mathrm{Id}_n) = 0$ and $\partial_F W(\mathrm{Id}_n) = 0$, we find

$$\lim_j \frac{1}{{\varepsilon_j}^2} W(\mathrm{Id}_n + \varepsilon_j \nabla u) = \frac{1}{2} \partial^2 W(\mathrm{Id}_n)[e(u), e(u)] \quad \mathcal{L}^n \text{ a.e. in } \Omega.$$

In addition, assumption (W2) guarantees that for some absolute positive constant C

$$W(\mathrm{Id}_n + \varepsilon_j \nabla u) \leq C {\varepsilon_j}^2 |\nabla u|^2 \quad \mathcal{L}^n \text{ a.e. } \Omega,$$

for all j sufficiently big; hence, Lebesgue dominated convergence theorem implies

$$\Gamma\text{-}\limsup_j \mathscr{F}_j(u) \leq \limsup_j \mathscr{F}_j(u_j) \leq \int_\Omega \lim_j \frac{1}{{\varepsilon_j}^2} W(\mathrm{Id}_n + \varepsilon_j \nabla u) dx = \mathscr{F}(u). \tag{4.8}$$

If u is any map in X, we use the density of $C_c^\infty(\Omega, \mathbb{R}^n)$ in $W_0^{1,2}(\Omega, \mathbb{R}^n)$ to get functions $u_k \in h + C_c^\infty(\Omega, \mathbb{R}^n)$ converging to u strongly in $W^{1,2}(\Omega, \mathbb{R}^n)$. Thus, (4.8), the lower semicontinuity of Γ-limsup and the continuity of $\mathscr{F}$ give the conclusion,

$$\Gamma\text{-}\limsup_j \mathscr{F}_j(u) \leq \liminf_k (\Gamma\text{-}\limsup_j \mathscr{F}_j(u_k)) \overset{(4.8)}{\leq} \liminf_k \mathscr{F}(u_k) = \mathscr{F}(u).$$

□

5. Obstacle problems for nonlocal energies

In this section we deal with the homogenization of nonlocal energies in periodically perforated domains. In its simplest form the problem amounts in characterizing the asymptotic behaviour as $j \uparrow +\infty$ of the solutions of

$$\begin{cases} -\triangle^s u = f \qquad u = 0 \ \mathcal{L}^n \text{ a.e. on } T_j \cap \Omega \\ u \in W_0^{s,2}(\Omega) \end{cases} \tag{5.1}$$

where $f \in L^2(\Omega)$, $-\triangle^s$ is the fractional Laplacian of order $s \in (1/2, 1)$ and $W_0^{s,2}$ is the Sobolev-Slobodeckij space with null trace at the boundary (see afterwards for the precise definitions, if $s \in (0, 1/2]$ the problem is formulated slightly differently). Given a compact set T with positive Lebesgue measure, the set of 'holes', or perforations, is defined as $T_j = \cup_{\mathtt{i} \in \mathbb{Z}^n}(\varepsilon_j \mathtt{i} + \lambda_j T)$, where ε_j is the size of the lattice $\varepsilon_j \mathbb{Z}^n$ on the vertices of which the holes are centered, and $\lambda_j \in (0, \varepsilon_j)$ is a vanishing scaling parameter.

The interest in the problem is motivated by several applications in different fields, running from the classical Signorini's problem in contact mechanics and diffusion through semi-permeablke membranes (corresponding to the case $s = 1/2$, see [37]), to stock-option pricing models in Finance (see the papers [16], [17], [39], [40] for an overview).

The asymptotic analysis has been performed first by Caffarelli and Mellet [16], [17] by means of an extension formula by Caffarelli and Silvestre [18] with which the solution to the nonlocal equation in $\mathbb{R}^n$ is interpreted as the boundary trace of the solution of a degenerate but local elliptic equation in the higher dimensional half-space $\mathbb{R}^{n+1}_+$.

Instead, Γ-convergence techniques were applied by the Author first in [38] still exploiting the extension formula by Caffarelli and Silvestre, and then in [39] giving an intrinsic proof at the level of the nonlocal energies.

Indeed, problem (5.1) has a natural variational character as the Euler-Lagrange equation satisfied by minimizers, among functions in $W_0^{s,2}(\Omega)$, of the perturbed quadratic form

$$\mathscr{F}_j(u) = \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy - \int_\Omega f u \, dx + \mathscr{K}_j(u),$$

where $\mathscr{K}_j : W_0^{s,2}(\Omega) \to [0, +\infty]$ equals 0 if $u = 0$ $\mathcal{L}^n$ a.e. on $T_j \cap \Omega$ and $+\infty$ otherwise. With this interpretation at hand (5.1) rewrites as

$$\int_{\Omega\times\Omega} 2\frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy = \int_\Omega f(x)\varphi(x)dx$$

for all $\varphi \in C_c^\infty(\Omega \setminus T_j)$ (see [36] for a survey of fractional Sobolev spaces and different equivalent definitions of the fractional laplacian). Note that being the energy under examination strictly convex the solution of (5.1) is exactly its minimizer.

We shall show that depending on the limit value of the quotient $\lambda_j/\varepsilon_j^{\frac{n}{n-2s}}$ as $j \uparrow +\infty$ (which we shall assume to exist) different phenomena occur. In particular, for $\lambda_j \sim \varepsilon_j^{\frac{n}{n-2s}}$ the *singular* penalization represented by $\mathscr{K}_j$ is substituted in the limit by a *finite* penalization of capacitary nature related to the homogenizing holes.

Heuristically, the motivation for this fact is clear when approximating a nonzero constant function $u = k \in \mathbb{R} \setminus \{0\}$. For, suppose $(u_j)_{j\in\mathbb{N}}$ is converging in $L^2(\Omega)$ to u, and that the corresponding energies are equi-bounded, i.e. $\sup_j \mathscr{F}_j(u_j) < +\infty$. Then, on one hand the approximating functions u_j are close to u in mean, on the other hand they have to make a transition from values almost equal to k to zero around each hole $\varepsilon_j \mathtt{i} + \lambda_j T$ in Ω. The minimal energy paid for each of these transitions is the so-called variational capacity of the relevant hole related to the energy under consideration (see the next section for the definitions). The nontrivial fact that adding up all those contributions gives the optimal energetic configuration is related to the geometric assumption on the placements of the holes and the scaling properties of the quoted capacity, more precisely to the integrability at ∞ of the singular kernel $|\cdot|^{-(n+2s)}$.

Such a behaviour is well-known in literature for energies defined on standard Sobolev spaces, that is for the analogous problem with the Dirichlet energy instead of the fractional seminorm, i.e., for $u \in W_0^{1,2}(\Omega)$

$$\int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx + \mathscr{K}_j(u), \tag{5.2}$$

equivalently from the point of view of PDEs

$$\begin{cases} -\triangle u = f, \quad u = 0 \ \mathcal{L}^n \text{ a.e. on } \Omega \cap T_j \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

The homogenization of (nonlinear) elliptic problems in perforated domains has received much attention since the '70's to today mainly for applications in mechanics (fiber reinforced materials in the antiplane setting for linearly elastic materials) and electrostatics (effective models for composites). Note that the Euler-Lagrange equations of the minimimum problems related to the functionals in (5.2) do not fit the framework of Theorem 3.14 due to the singular constraints imposed via $\mathscr{K}_j$.

Much literature has been devoted to study the asymptotics of the energies in (5.2), and many far reaching generalizations of such a problem, with different approaches starting with the seminal papers by Marchenko and Khruslov [50], Rauch and Taylor [61], [62] and Cioranescu and Murat [21]. Building upon a preliminary contribution by De Giorgi, Dal Maso and Longo [33], abstract Γ-convergence methods were employed by Dal Maso in a series of papers to give a full solution of the problem (see [23],[24]) without any assumption on the sizes, shapes or distributions of the obstacles. The analysis there is based on the abstract localization methods of Γ-convergence and fine tools from potential theory. Here, we shall limit ourselves to a prototype result which can be worked out 'by hands', highlighting at the end of the section many possible generalizations.

The survey [26] provides a very detailed review of all the classical contributions discussing also parallel results for monotone operators as well as further developments (see [10] for a more up-to-date state of the art).

5.1. **Preliminaries and further definitions.** With fixed $s \in (0,1)$, for any $\mathcal{L}^n$-measurable function $w : \Omega \to \mathbb{R}$ and any $\mathcal{L}^{n\times n}$-measurable set $E \subseteq \Omega \times \Omega$ define

$$\mathcal{D}_s(w,E) = \int_E \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}} dxdy\,.$$

In addition, set

$$|w|^2_{W^{s,2}(A)} := \mathcal{D}_s(w, A\times A).$$

The latter quantity is called the *s-fractional seminorm* of w, and

$$W^{s,2}(\Omega) := \Big\{ w \in L^2(\Omega) :\, |w|^2_{W^{s,2}(\Omega)} < +\infty \Big\}$$

is the *Sobolev-Slobodeckij space*, which turns out to be a Hilbert space if endowed with the norm $(\|\cdot\|^2_{L^2(\Omega)} + |\cdot|^2_{W^{s,2}(\Omega)})^{1/2}$ (we refer to the books [1], [71] and to the recent paper [36] as main references for these topics).

$\mathcal{D}_s(w,\cdot)$ is called the *locality defect* of the $W^{s,2}$ seminorm, the terminology being justified because given two disjoint subdomains $A, B \subseteq \Omega$ one gets

$$|w|^2_{W^{s,2}(A\cup B)} = |w|^2_{W^{s,2}(A)} + |w|^2_{W^{s,2}(B)} + 2\mathcal{D}_s(w, A\times B). \tag{5.3}$$

For $s \in (1/2, 1)$, $W^{s,2}_0(\Omega)$ denotes the closure in the norm $W^{s,2}$ topology of $C^\infty_c(\Omega)$; while for $s \in (0, 1/2]$ we recall that traces are not well-defined as $W^{s,2}_0(\Omega) = W^{s,2}(\Omega)$. Hence, with fixed an open subset $\Omega' \Subset \Omega$, for those values of the parameter s we shall impose a null Dirichlet boundary condition by taking $\hat{W}^{s,2}(\Omega)$, the (weakly closed) subspace of functions in $W^{s,2}(\Omega)$ with support in Ω'.

We shall employ only few analytical results concerning the spaces $W^{s,2}$ in what follows. First, we recall the scaled version of the Poincaré-Wirtinger inequality in fractional Sobolev spaces (a consequence of [71, Theorems 2.6.1 and 4.2.3], of a scaling argument and of Hölder inequality)

$$\|u - u_{x+rE}\|^2_{L^2(x+rA,\mathbb{R}^m)} \leq c\, r^{2s} |u|^2_{W^{s,2}(x+rA,\mathbb{R}^m)} \tag{5.4}$$

for any $x \in \mathbb{R}^n$, $r > 0$ and for some $c = c(n, s, E, A) > 0$ with A an open set in $\mathbb{R}^n$ and E a $\mathcal{L}^n$-measurable subset of A.

Second, we quote an elementary bound on singular kernels (cp. with [39, Lemma A.1]): for any $\mathcal{L}^n$-measurable set E and for any point $x \in \mathbb{R}^n$ with $\mathrm{dist}(x, E) > 0$

$$\int_E \frac{1}{|y - x|^{n+2s}}\, dy \leq c\, (\mathrm{dist}(x, E))^{-2s} \tag{5.5}$$

for some positive constant $c = c(n, p, s)$.

The notion of variational capacity for fractional Sobolev spaces and some related properties are instrumental tools in what follows. For $s \in (0, 1)$, and given a set $T \subseteq \mathbb{R}^n$ of positive Lebesgue measure let

$$\mathrm{cap}_s(T) := \inf_{\{A\in\mathcal{A}(\mathbb{R}^n):\, A\supseteq T\}} \inf \left\{ |u|^p_{W^{s,2}(\mathbb{R}^n)} : u \in W^{s,2}(\mathbb{R}^n),\, u \geq 1\, \mathcal{L}^n \text{ a.e. on } A \right\}, \tag{5.6}$$

with the convention $\inf \emptyset = +\infty$. The set function in (5.6) turns out to be a Choquet capacity (see [1, Chapter V]).

In the sequel relative capacities shall be crucial. We introduce two different notions, the first shall be useful in the Γ-liminf inequality, the second in the Γ-limsup inequality, respectively. For every $0 < r \leq R$ set

$$\mathrm{cap}_s(T, B_R; r) = \inf_{\{A\in\mathcal{A}(\mathbb{R}^n):\, A\supseteq T\}} \inf \Big\{ |w|^2_{W^{s,2}(B_R)} : w \in W^{s,2}(\mathbb{R}^n),\, w \geq 1\, \mathcal{L}^n \text{ a.e. on } A,\, w = 0 \text{ on } \mathbb{R}^n \setminus \overline{B}_r, \Big\},$$

and

$$C_s(T, B_r) = \inf_{\{A\in\mathcal{A}(\mathbb{R}^n):\, A\supseteq T\}} \inf \Big\{ |w|^2_{W^{s,2}(\mathbb{R}^n)} : w \in W^{s,2}(\mathbb{R}^n),\, w \geq 1\, \mathcal{L}^n \text{ a.e. on } A,\, w = 0 \text{ on } \mathbb{R}^n \setminus \overline{B}_r, \Big\}.$$

Note that $C_s(T,B_r) \geq \mathrm{cap}_s(T)$ for all r sufficiently big. In [39, Lemma 2.12] it is established that

$$\lim_{r\to+\infty} C_s(T,B_r) = \mathrm{cap}_s(T), \tag{5.7}$$

and also that

$$\mathrm{cap}_s(T) - C_s(T,B_r) \leq \mathrm{cap}_s(T) - \mathrm{cap}_s(T,B_R;r) \leq \frac{c\,r^{2s}}{(R-r)^{2s}} C_s(B_\rho,B_r), \tag{5.8}$$

for some constant $c = c(n,s)$ and for all $0<\rho<r<R$ with $T\subseteq B_\rho$.

Remark 5.1. *If ξ_r is a $(1/r)$-minimizer for $C_s(T,B_r)$ and $R(r)/r \to +\infty$ as $r \uparrow +\infty$, then* (5.8) *yields*

$$\lim_{r\to+\infty} \mathcal{D}_s(\xi_r, B_{R(r)} \times B^c_{R(r)}) = 0.$$

5.2. **Statement of the Main Result.** With fixed a bounded set T with positive Lebesgue measure, for all $j\in\mathbb{N}$ define the *obstacle set* $T_j \subseteq \mathbb{R}^n$ to be $T_j = \cup_{\mathtt{i}\in\mathbb{Z}^n} T_j^{\mathtt{i}}$ with

$$T_j^{\mathtt{i}} := \varepsilon_j \mathtt{i} + \lambda_j T, \qquad \text{and } \lambda_j \in (0,\varepsilon_j). \tag{5.9}$$

Note then that $T_j^{\mathtt{i}} \subseteq Q_j^{\mathtt{i}} := \varepsilon_j(\mathtt{i} + [0,1]^n)$ for all $\mathtt{i}\in\mathbb{Z}^n$ and $j\in\mathbb{N}$ sufficiently big.

Consider the functionals $\mathscr{F}_j : L^2(\Omega) \to [0,+\infty]$ defined by

$$\mathscr{F}_j(u) := \begin{cases} |u|^2_{W^{s,2}(\Omega)} & \text{if } u\in W^{s,2}_0(\Omega),\ u=0\ \mathcal{L}^n \text{ a.e. on } T_j\cap\Omega \\ +\infty & \text{otherwise} \end{cases} \tag{5.10}$$

if $s\in(1/2,1)$, we substitute $W^{s,2}_0(\Omega)$ with $\hat{W}^{s,2}(\Omega)$ for $s\in(0,1/2]$.

We are now able to state the Γ-convergence result.

Theorem 5.2. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open connected set with Lipschitz regular boundary and such that*

$$\vartheta := \lim_j \varepsilon_j^{-n}\lambda_j^{n-2s} \in [0,+\infty].$$

Then, $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges in the strong $L^2(\Omega)$ topology to $\mathscr{F} : L^2(\Omega) \to [0,+\infty]$ defined by

$$\mathscr{F}(u) := |u|^2_{W^{s,2}(\Omega)} + \vartheta\,\mathrm{cap}_s(T)\int_\Omega |u(x)|^2\,dx \tag{5.11}$$

if $u\in W^{s,2}_0(\Omega)$ for $s\in(1/2,1)$ or $u\in\hat{W}^{s,2}(\Omega)$ for $s\in(0,1/2]$, and $+\infty$ otherwise in $L^2(\Omega)$.

The well-known compact embedding of $W^{s,2}(\Omega)$ into $L^2(\Omega)$ on one hand justifies the choice of the L^2 topology, on the other hand, together with of Theorem 5.2, provides the convergence of the solutions of problems (5.1) to the solution of

$$-\triangle^s u + \vartheta\,\mathrm{cap}_s(T)\,u = f, \quad u \in W_0^{s,2}(\Omega),\ s \in (1/2, 1)$$

($u \in \hat{W}^{s,2}(\Omega)$ for $s \in (0, 1/2]$) in view of the continuity of the linear perturbation defined via f. As anticipated, a lower order term substitutes in the limit PDE the Dirichlet condition on the set of holes T_j (cp. the last equation with (5.1)).

Two technical lemmata are instrumental for proving Theorem 5.2. With the help of those results, Γ-convergence can be checked only on sequences of functions that take constant values on suitable annuli surrounding the obstacle sets almost matching the values of the corresponding limit function. Thus, the heuristic argument quoted in the introduction can be rendered rigorous.

We first deal with the unscaled setting in Lemma 5.3 obtaining a preliminary rough estimate, and then turn to the framework of interest in Lemma 5.4.

In Lemma 5.3 a family of annuli around points with integers coordinates and a set of values are assigned. The values of any function u in $W^{s,2}$ are then changed accordingly on those sets. The relevant fact is that the absolute energetic error of the construction can be estimated only by local quantities related to u and to the chosen data (see formula (5.12)). In doing this, long range interaction terms are carefully estimated by distinguishing the zones close to the diagonal set in $\Omega \times \Omega$ and those far from it. We refer to [39, Lemma 3.8] for the proof.

Lemma 5.3. *Set $\mathcal{I} = \{\mathtt{i} \in \mathbb{Z}^n : \mathtt{i} + [0,1]^n \subseteq \Omega\}$, for any $m \in \mathbb{N}$, $m \geq 2$, $\rho \in (0, 1/2)$ and $\mathtt{i} \in \mathcal{I}$ let $A'_{\mathtt{i}} = \mathtt{i} + B_{\rho/m} \setminus \overline{B}_{\rho/m^2}$, $A_{\mathtt{i}} = \mathtt{i} + B_\rho \setminus \overline{B}_{\rho/m^3}$, and $\varphi_{\mathtt{i}}(\cdot) = \varphi(\cdot - \mathtt{i})$, where $\varphi \in C_c^\infty(B_\rho \setminus \overline{B}_{\rho/m^3})$ and $\varphi = 1$ on $B_{\rho/m} \setminus \overline{B}_{\rho/m^2}$.*

Then there exists a constant $c = c(n,s) > 0$ such that for any $u \in W^{s,2}(\Omega)$, and any $\#\mathcal{I}$-tuple of vectors $\{z_{\mathtt{i}}\}_{\mathtt{i}\in\mathcal{I}}$, $z_{\mathtt{i}} \in \mathbb{R}^n$, the function

$$w(x) = \sum_{\mathtt{i}\in\mathcal{I}} \varphi_{\mathtt{i}}(x) z_{\mathtt{i}} + \left(1 - \sum_{\mathtt{i}\in\mathcal{I}} \varphi_{\mathtt{i}}(x)\right) u(x)$$

belongs to $W^{s,2}(\Omega)$, $w = z_{\mathtt{i}}$ on $A'_{\mathtt{i}}$ and $w = u$ on $\Omega \setminus \overline{A}$, with $A := \cup_{\mathtt{i}\in\mathcal{I}} A_{\mathtt{i}}$. In addition, for every measurable set $E \subseteq \Omega \times \Omega$ it holds

$$|\mathcal{D}_s(w, E) - \mathcal{D}_s(u, E)|$$

$$\leq c\left(\mathcal{D}_s(u,\Omega\times A)+m^4\rho^{-2s}\sum_{\mathtt{i}\in\mathcal{I}}\int_{A_{\mathtt{i}}}|u(y)-z_{\mathtt{i}}|^2\,dy\right). \quad (5.12)$$

Next, in Lemma 5.4 we improve upon Lemma 5.3 by choosing suitably the new values in a way that an elementary slicing and averaging argument detects a family of annuli on which the energies do not concentrate. In particular, the final absolute energetic error turns out to be only a small proportion of the energy of the starting function (cp. with (5.15)).

The slicing/averaging argument employed below goes back to De Giorgi [28], it has been largely employed in variational problems of the kind of that of interest here in order to change boundary values controlling the error in energy.

Before starting the proof we fix some notation: fixed $m\in\mathbb{N}$, set

$$\mathcal{I}_j:=\{\mathtt{i}\in\mathbb{Z}^n:\varepsilon_j(\mathtt{i}+[0,1]^n)\subseteq\Omega\},$$

and for all $\mathtt{i}\in\mathcal{I}_j$, $h\in\mathbb{N}$ consider the balls

$$B_j^{\mathtt{i},h}:=\{x\in\mathbb{R}^n:\,|x-\varepsilon_j\mathtt{i}|<m^{-3h}\varepsilon_j\},$$

and the annuli

$$C_j^{\mathtt{i},h}:=\{x\in\mathbb{R}^n:\,m^{-3h-2}\varepsilon_j<|x-\varepsilon_j\mathtt{i}|<m^{-3h-1}\varepsilon_j\}.$$

Note that $C_j^{\mathtt{i},h}\subset B_j^{\mathtt{i},h}\setminus\overline{B}_j^{\mathtt{i},h+1}\subset Q_j^{\mathtt{i}}$.

Lemma 5.4. *Let $(u_j)_{j\in\mathbb{N}}$ be converging to u in $L^2(\Omega)$ with $\sup_j|u_j|_{W^{s,2}(\Omega)}<+\infty$. With fixed m, $N\in\mathbb{N}$, for every $j\in\mathbb{N}$ there exists $h_j\in\{1,\dots,N\}$ and a function $w_j\in W^{s,2}(\Omega)$ such that*

$$w_j=u_j\ \text{ on }\Omega\setminus\cup_{\mathtt{i}\in\mathcal{I}_j}(\overline{B}_j^{\mathtt{i},h_j}\setminus B_j^{\mathtt{i},h_j+1}), \quad (5.13)$$

$$w_j=(u_j)_{C_j^{\mathtt{i},h_j}}\ \text{ on }C_j^{\mathtt{i},h_j}, \quad (5.14)$$

for some $c=c(n,s,m)>0$ it holds for every measurable set E in $\Omega\times\Omega$

$$|\mathcal{D}_s(u_j,E)-\mathcal{D}_s(w_j,E)|\leq\frac{c}{N}|u_j|^2_{W^{s,2}(\Omega)}, \quad (5.15)$$

and the sequences $(w_j)_{j\in\mathbb{N}}$, $(\zeta_j)_{j\in\mathbb{N}}$, with $\zeta_j:=\sum_{\mathtt{i}\in\mathcal{I}_j}(u_j)_{C_j^{\mathtt{i},h_j}}1_{Q_j^{\mathtt{i}}}$, converge to u in $L^2(\Omega)$.

Proof. Given m, $N\in\mathbb{N}$, then for every $j\in\mathbb{N}$ and $h\in\{1,\dots,N\}$ fixed, apply Lemma 5.3 with $(A')^h_{\mathtt{i}}:=C_j^{\mathtt{i},h}$, $A^h_{\mathtt{i}}:=B_j^{\mathtt{i},h}\setminus\overline{B}_j^{\mathtt{i},h+1}$, $z_{\mathtt{i}}=(u_j)_{C_j^{\mathtt{i},h}}$, $\mathtt{i}\in\mathcal{I}_j$. Take note that $\rho=m^{-3h}\varepsilon_j$. If $w_j^{\mathtt{i},h}$ denotes the resulting function and $A^h = \cup_{\mathtt{i}\in\mathcal{I}_j} A^h_{\mathtt{i}}$, then for some constant $c = c(n,s)$ and for any measurable set E in $\Omega\times\Omega$ by (5.12) it holds

$$|\mathcal{D}_s(u_j,E)-\mathcal{D}_s(w_j,E)| \le c\,\mathcal{D}_s(u_j,\Omega\times A^h) + c\,m^4\left(\frac{m^{3h}}{\varepsilon_j}\right)^{2s}\sum_{\mathtt{i}\in\mathcal{I}_j}\int_{A^h_{\mathtt{i}}}|u_j-(u_j)_{C_j^{\mathtt{i},h}}|^2dx.$$

This estimate, together with the scaled Poincarè-Wirtinger inequality (5.4) with $r = m^{-3h}\varepsilon_j$, gives

$$|\mathcal{D}_s(u_j,E)-\mathcal{D}_s(w_j,E)| \le c\left(\mathcal{D}_s(u_j,\Omega\times A^h)+|u_j|^2_{W^{s,2}(A^h)}\right)\le c\,\mathcal{D}_s(u_j,\Omega\times A^h), \quad (5.16)$$

for some $c = c(n,s,m) > 0$. By summing up and averaging on h, being the A^h's disjoint, we find $h_j \in \{1,\dots,N\}$ such that

$$\mathcal{D}_s(u_j,\Omega\times A^{h_j}) \le \frac{1}{N}\mathcal{D}_s(u_j,\Omega\times\cup_h A^h). \quad (5.17)$$

Set $w_j := w_j^{\mathtt{i},h_j}$, then (5.13) and (5.14) are satisfied by construction, and moreover (5.16) and (5.17) imply (5.15).

To prove that $(w_j)_{j\in\mathbb{N}}$ converges to u in $L^2(\Omega)$ we use (5.4), with $r = m^{-3h}\varepsilon_j$, and the very definition of w_j as convex combination of u_j and the mean value $(u_j)_{C_j^{i,h_j}}$ on $B_j^{\mathtt{i},h_j}\setminus\overline{B}_j^{\mathtt{i},h_j+1}$ to get

$$\begin{aligned}\|u_j-w_j\|^2_{L^2(\Omega)} &= \|u_j-w_j\|^2_{L^2(A^{h_j})}\\ &= \sum_{\mathtt{i}\in\mathcal{I}_j}\|u_j-w_j\|^2_{L^2(B_j^{\mathtt{i},h_j}\setminus B_j^{\mathtt{i},h_j+1})} \le \sum_{\mathtt{i}\in\mathcal{I}_j}\|u_j-(u_j)_{C_j^{i,h_j}}\|^2_{L^2(B_j^{\mathtt{i},h_j}\setminus B_j^{\mathtt{i},h_j+1})}\\ &\le c\left(\frac{\varepsilon_j}{m^{3h_j}}\right)^{2s}\sum_{\mathtt{i}\in\mathcal{I}_j}|u_j|^2_{W^{s,2}(B_j^{\mathtt{i},h_j}\setminus B_j^{\mathtt{i},h_j+1})}\le c\varepsilon_j^{2s}|u_j|^2_{W^{s,2}(\Omega)},\end{aligned}$$

where $c = c(n,s,m) > 0$.

Eventually, let us show the convergence of $(\zeta_j)_{j\in\mathbb{N}}$ to u in $L^2(\Omega)$. To this aim we prove that $(\zeta_j - u_j)_{j\in\mathbb{N}}$ is infinitesimal in $L^2(\Omega)$.

For, note that (5.4) applied with $r = \varepsilon_j$ which gives for some $c = c(n,s,m,N) > 0$

$$\sum_{\mathtt{i}\in\mathcal{I}_j}\|u_j-(u_j)_{C_j^{\mathtt{i},h_j}}\|^2_{L^2(Q_j^{\mathtt{i}})}\le c\,\varepsilon_j^{2s}|u_j|^2_{W^{s,2}(\Omega)}. \quad (5.18)$$

Given this, the conclusion is a straightforward consequence of the definition of ζ_j and of (5.18), i.e.

$$\|\zeta_j - u_j\|^2_{L^2(\Omega)} = \sum_{\mathtt{i}\in\mathcal{I}_j} \|u_j - (u_j)_{C_j^{\mathtt{i},h_j}}\|^2_{L^2(Q_j^{\mathtt{i}})} + \|u_j\|^2_{L^2(\Omega\setminus\cup_{\mathcal{I}_j} Q_j^{\mathtt{i}})}.$$

□

5.3. **Proof of the Γ-convergence.** We establish the Γ-convergence result contained in Theorem 5.2.

We first show the lower bound inequality in Propositions 5.5 below. In view of Lemma 5.4 we consider only sequences assuming constant values around the obstacles, which are then approximately mean values of the target function close to the $T_j^{\mathtt{i}}$'s (cp. with (ζ_j) in Lemma 5.4).

Then we use a separation of scale argument. Fix a lengthscale $\delta > 0$ and consider the δ-neighbourhood Δ_δ of the diagonal set Δ, i.e.

$$\Delta := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\},$$
$$\Delta_\delta := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \operatorname{dist}((x,y),\Delta) \leq \delta\}.$$

The asymptotic energy contribution on $\Omega \times \Omega \setminus \Delta_\delta$ gives the fractional seminorm of the target function since the kernel is no longer singular there.

Further, to recover the limit capacitary term we evaluate only the self-interaction energies close to each obstacle. This is sufficient because the obstacles shrink at a faster scale than the distances between their centers, so that capacity behaves additively in the limit. Hence, for each $T_j^{\mathtt{i}}$ we solve a local capacitary problem with transition between the local value of the target function around $T_j^{\mathtt{i}}$ to zero. In conclusion, when we evaluate the total contribution of those terms we get a discretization of the integral of the target function squared.

Eventually, in Proposition 5.6 we build a special sequence for which, using the previous arguments, there is actually no loss of energy asymptotically.

Proposition 5.5. *For every $u_j \to u$ in $L^2(\Omega)$ we have*

$$\liminf_j \mathscr{F}_j(u_j) \geq \mathscr{F}(u).$$

Proof. Fix $N \in \mathbb{N}$, $\delta > 0$, and set $m = [1/\delta] \in \mathbb{N}$, $[\cdot]$ denoting the integer part function.

Without loss of generality suppose that $\liminf_j \mathscr{F}_j(u_j) < +\infty$, in a way that $u \in W^{s,2}_0(\Omega)$ for $s \in (1/2, 1)$, or $u \in \hat{W}^{s,2}(\Omega)$ for $s \in (0, 1/2])$, being the latter spaces weakly closed. Consider the sequence $(w_j)_{j\in\mathbb{N}}$ provided by Lemma 5.4, for the sake of notational convenience its dependence on δ, N is not highlighted. Note that whatesover the choices of δ and N are, $(w_j)_{j\in\mathbb{N}}$ converges to u in $L^2(\Omega)$ and for some $c = c(n,s,\delta) > 0$ it holds

$$\Big(1+\frac{c}{N}\Big)\liminf_j \mathscr{F}_j(u_j) \geq \liminf_j \mathscr{F}_j(w_j). \tag{5.19}$$

Since for j sufficiently big $\cup_{\mathtt{i}\in\mathcal{I}_j}(Q_j^{\mathtt{i}}\times Q_j^{\mathtt{i}}) \subseteq \Delta_\delta$, we can argue as follows

$$\begin{aligned}
&\liminf_j \mathscr{F}_j(w_j) \\
&\quad\geq \liminf_j \left(\int_{\Omega\times\Omega\setminus\Delta_\delta} \frac{|w_j(x)-w_j(y)|^2}{|x-y|^{n+2s}}dx\,dy + \sum_{\mathtt{i}\in\mathcal{I}_j} |w_j|^2_{W^{s,2}(Q_j^{\mathtt{i}})}\right) \\
&\quad= \int_{\Omega\times\Omega\setminus\Delta_\delta} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dx\,dy + \liminf_j \sum_{\mathtt{i}\in\mathcal{I}_j} |w_j|^2_{W^{s,2}(Q_j^{\mathtt{i}})}, \quad (5.20)
\end{aligned}$$

thanks to the convergence $|w_j(x)-w_j(y)|^2 \to |u(x)-u(y)|^2$ in $L^1(\Omega)$ and being $|\cdot|^{-(n+2s)} \in L^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\Delta_\delta)$. We claim that

$$\liminf_j \sum_{\mathtt{i}\in\mathcal{I}_j} |w_j|^2_{W^{s,2}(Q_j^{\mathtt{i}})} \geq \vartheta\left(\mathrm{cap}_s(T)-\epsilon_\delta\right)\int_U |u(x)|^2dx, \tag{5.21}$$

with $\epsilon_\delta > 0$ infinitesimal as $\delta \downarrow 0^+$. Given this for granted, by (5.19) inequality (5.20) rewrites as

$$\begin{aligned}
\Big(1+\frac{c}{N}\Big)\liminf_j \mathscr{F}_j(u_j) \geq& \int_{\Omega\times\Omega\setminus\Delta_\delta} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dx\,dy \\
&+ \vartheta\left(\mathrm{cap}_s(T)-\epsilon_\delta\right)\int_\Omega |u(x)|^2dx. \quad (5.22)
\end{aligned}$$

The thesis then follows by passing to the limit first as $N \uparrow +\infty$ and then as $\delta \downarrow 0^+$ in (5.22).

To conclude we are left with proving (5.21). We keep the notation of Lemma 5.4, and further set $B_j^{\mathtt{i}} := \{x\in\mathbb{R}^n : |x-\varepsilon_j\mathtt{i}| < m^{-(3h_j+1)}\varepsilon_j\}$, for all $\mathtt{i}\in\mathcal{I}_j$. Note that $B_j^{\mathtt{i}} \subseteq Q_j^{\mathtt{i}}$. We have

$$\begin{aligned}
&|w_j|^2_{W^{s,2}(Q_j^{\mathtt{i}})} \\
&\geq \inf\left\{|w|^2_{W^{s,2}(B_j^{\mathtt{i}})} : w\in W^{s,2}(\mathbb{R}^n),\ w = (u_j)_{C_j^{\mathtt{i},h_j}} \text{ on } C_j^{\mathtt{i},h_j},\ \tilde{w} = 0 \text{ q.e. on } T_j^{\mathtt{i}}\right\} \\
&= \inf\left\{|w|^2_{W^{s,2}(B_j^{\mathtt{i}})} : w\in W^{s,2}(\mathbb{R}^n),\ w = 0 \text{ on } C_j^{\mathtt{i},h_j},\ \tilde{w} = (u_j)_{C_j^{\mathtt{i},h_j}} \text{ q.e. on } T_j^{\mathtt{i}}\right\} \\
&\qquad\qquad = |(u_j)_{C_j^{\mathtt{i},h_j}}|^2 \mathrm{cap}_s\left(T_j^{\mathtt{i}}, B_j^{\mathtt{i}}; \frac{\varepsilon_j}{m^{3h_j+2}}\right)
\end{aligned}$$

$$= \lambda_j^{n-2s} |(u_j)_{C_j^{\mathtt{i},h_j}}|^2 \mathrm{cap}_s \left(T, B_{\frac{\varepsilon_j}{m^{3h_j+1}\lambda_j}}; \frac{\varepsilon_j}{m^{3h_j+2}\lambda_j} \right). \quad (5.23)$$

The last equality is justified by an elementary translation and scaling argument. Being $h_j \in \{1, \ldots, N\}$, estimate (5.8) gives

$$\mathrm{cap}_s \left(T, B_{\frac{\varepsilon_j}{m^{3h_j+1}\lambda_j}}; \frac{\varepsilon_j}{m^{3h_j+2}\lambda_j} \right) \geq \mathrm{cap}_s(T) - \frac{c}{(m-1)^{2s}} C_s \left(B_1, B_{\frac{\varepsilon_j}{m^{3h_j+2}\lambda_j}} \right).$$

Hence, if $A \in \mathcal{A}(\Omega)$ is such that $A \Subset \Omega$, for j sufficiently big we infer

$$\sum_{\mathtt{i} \in \mathcal{I}_j} |w_j|^2_{W^{s,2}(Q_j^{\mathtt{i}})}$$
$$\geq \left(\mathrm{cap}_s(T) - \frac{c}{(m-1)^{2s}} C_s \left(B_1, B_{\frac{\varepsilon_j}{m^{3h_j+2}\lambda_j}} \right) \right) \lambda_j^{n-2s} \varepsilon_j^{-n} \int_A |\zeta_j(x)|^2 dx.$$

Finally, the thesis follows at once by the convergence of relative capacities to the global one proved in (5.7). the strong convergence of $(\zeta_j)_{j\in\mathbb{N}}$ to u in $L^2(\Omega)$ established in Lemma 5.4, and eventually by letting A increase to Ω and $m \uparrow +\infty$. □

In the next proposition we prove that the lower bound established in Proposition 5.5 is optimal. Thanks to the insight provided by Proposition 5.5 we show that the capacitary contribution is concentrated along the diagonal set Δ and is due to short range interactions. Instead, long range interactions are responsible for the nonlocal term in the limit.

Proposition 5.6. *For every $u \in L^2(\Omega)$ there exists a sequence $(u_j)_{j\in\mathbb{N}}$ such that $u_j \to u$ in $L^2(\Omega)$ and*

$$\limsup_j \mathscr{F}_j(u_j) \leq \mathscr{F}(u).$$

Proof. Without loss of generality we may suppose $\vartheta < +\infty$, since otherwise if $\mathscr{F}(u) < +\infty$ then $u = 0$ $\mathcal{L}^n$ a.e. on Ω and u itself provides a trivial recovery sequence.

A density argument, which holds true thanks to the very definitions of $W_0^{s,2}(\Omega)$ for $s \in (1/2, 1)$, and $\hat{W}^{s,2}(\Omega)$ for $s \in (0, 1/2]$, the continuity of $\mathscr{F}$ and the lower semicontinuity of Γ-$\limsup_j \mathscr{F}_j$ in the strong $W^{s,2}$ topology allow the choice $u \in C_c^\infty(\Omega)$. In addition, we may also assume that u, extended to 0, belongs to $W^{s,2}(\Omega')$ for some bounded open smooth set Ω' with $\Omega \Subset \Omega'$.

Let $(w_j)_{j\in\mathbb{N}}$ be the sequence obtained from u by applying Lemma 5.4 on Ω' with $m = 2$. We keep the notation introduced there and further set

$$\mathcal{I}'_j = \mathcal{I}_j \cup \{\mathtt{i} \in \mathbb{Z}^n : \varepsilon_j(\mathtt{i} + [0,1]^n) \cap \partial\Omega \neq \emptyset\},$$

and for $\mathtt{i} \in \mathcal{I}'_j$

$$B^{\mathtt{i}}_j = B^{\mathtt{i},h_j}_j (= \{x \in \mathbb{R}^n : |x - \varepsilon_j \mathtt{i}| < 2^{-3h_j - 1}\varepsilon_j\}),$$
$$u^{\mathtt{i}}_j = u_{C^{\mathtt{i},h_j}_j}, \quad \Omega_j = \Omega \setminus \left(\cup_{\mathcal{I}'_j} B^{\mathtt{i}}_j\right).$$

Given $N \in \mathbb{N}$, let $\xi \in W^{s,2}(\mathbb{R}^n)$ be such that $\xi = 0$ on $\mathbb{R}^n \setminus \overline{B}_N$, $\xi \geq 1$ $\mathcal{L}^n$ a.e. on T and $|\xi|^2_{W^{s,2}(\mathbb{R}^n)} \leq C_s(T, B_N) + 1/N$. Then, recalling that $\lambda_j \varepsilon_j^{-n/(n-2s)}$ is bounded, define

$$u_j(x) := \begin{cases} w_j(x) & \Omega_j \\ \left(1 - \xi\left(\frac{x - \varepsilon_j \mathtt{i}}{\lambda_j}\right)\right) u^{\mathtt{i}}_j & B^{\mathtt{i}}_j,\ \mathtt{i} \in \mathcal{I}'_j. \end{cases} \tag{5.24}$$

For the sake of notational simplicity we have not highlighted the dependence of the sequence $(u_j)_{j\in\mathbb{N}}$ on the parameter $N \in \mathbb{N}$. Clearly, $(u_j)_{j\in\mathbb{N}}$ converges strongly to u in $L^2(\Omega)$, and moreover it satisfies the obstacle condition by construction. The rest of the proof is devoted to show that $u_j \in W^{s,2}(\Omega)$ with

$$\limsup_j \mathscr{F}_j(u_j) \leq \mathscr{F}(u) + \epsilon_\delta + \epsilon_N,$$

where $\epsilon_\delta \downarrow 0^+$ as $\delta \downarrow 0^+$ and $\epsilon_N \downarrow 0^+$ as $N \uparrow +\infty$.

First, we can reduce the calculations to compute the energy of u_j only on a neighbourhood of the diagonal Δ. For, Lebesgue dominated convergence and the convergence of $(u_j)_{j\in\mathbb{N}}$ to u in $L^2(\Omega)$ give

$$\lim_j \mathcal{D}_s(u_j, (\Omega \times \Omega) \setminus \Delta_\delta) = \mathcal{D}_s(u, (\Omega \times \Omega) \setminus \Delta_\delta).$$

Moreover, since $u_j = w_j$ on Ω_j by (5.15) in Lemma 5.4, we have for some constant $c = c(n, s)$

$$\begin{aligned} &\limsup_j \mathcal{D}_s(u_j, (\Omega_j \times \Omega_j) \cap \Delta_\delta) \\ &\leq \limsup_j \mathcal{D}_s(w_j, (\Omega \times \Omega) \cap \Delta_\delta) \leq \left(1 + \frac{c}{N}\right) \mathcal{D}_s(u, (\Omega \times \Omega) \cap \Delta_\delta) = \epsilon_\delta. \end{aligned} \tag{5.25}$$

The conclusion follows at once provided we show that

$$\begin{aligned} \limsup_j \big(\mathcal{D}_s(u_j, (\Omega \times (\Omega \setminus \overline{\Omega}_j)) \cap \Delta_\delta) + \mathcal{D}_s(u_j, ((\Omega \setminus \overline{\Omega}_j) \times \Omega_j) \cap \Delta_\delta)\big) \\ \leq \vartheta \operatorname{cap}_s(T) \int_\Omega |u|^2 dx + \epsilon_N + \epsilon_\delta. \end{aligned} \tag{5.26}$$

To prove the latter inequality we split the left hand side above as follows:

$$\begin{aligned}&\mathcal{D}_s(u_j,(\Omega\times(\Omega\setminus\overline{\Omega}_j))\cap\Delta_\delta)+\mathcal{D}_s(u_j,((\Omega\setminus\overline{\Omega}_j)\times\Omega_j)\cap\Delta_\delta)\\&\quad\le\sum_{\mathtt{i}\in\mathcal{I}'_j}|u_j|^2_{W^{s,2}(B^{\mathtt{i}}_j)}+\sum_{\{(\mathtt{i},\mathtt{k})\in\mathcal{I}'_j\times\mathcal{I}'_j:\,0<|\varepsilon_j\mathtt{i}-\varepsilon_j\mathtt{k}|<\delta\}}\mathcal{D}_s(u_j,B^{\mathtt{i}}_j\times B^{\mathtt{k}}_j)\\&\quad+2\sum_{\mathtt{i}\in\mathcal{I}'_j}\mathcal{D}_s(u_j,(B^{\mathtt{i}}_j\times\Omega_j)\cap\Delta_\delta)=I^1_j+I^2_j+I^3_j.\end{aligned}$$

We estimate separately each term I^h_j, $h\in\{1,2,3\}$. All the constants c appearing below depend only on n, s and $\|u\|_{W^{1,\infty}(\Omega')}$, so that this dependence will no longer be indicated.

Step 1. Estimate of I^1_j:

$$\limsup_j I^1_j\le\vartheta\left(\mathrm{cap}_s(T)+\epsilon_N\right)\int_\Omega|u|^2dx. \tag{5.27}$$

A straightforward change of variables leads to

$$\begin{aligned}I^1_j&=\lambda_j^{n-2s}\sum_{\mathtt{i}\in\mathcal{I}_j}|u^{\mathtt{i}}_j|^2|\xi|^2_{W^{s,2}(\lambda_j^{-1}(B^{\mathtt{i}}_j-\varepsilon_j\mathtt{i}))}\le\lambda_j^{n-2s}\left(C_s(T,B_N)+\frac{1}{N}\right)\sum_{\mathtt{i}\in\mathcal{I}_j}|u^{\mathtt{i}}_j|^2\\&=\left(C_s(T,B_N)+\frac{1}{N}\right)\lambda_j^{n-2s}\varepsilon_j^{-n}\int_\Omega|\zeta_j|^2dx,\end{aligned}$$

where ζ_j is defined in Lemma 5.4. Arguing as in Proposition 5.5 and using (5.7), we conclude (5.27).

Step 2. Estimate of I^2_j:

$$\limsup_j I^2_j\le\epsilon_\delta+\epsilon_N. \tag{5.28}$$

The very definition of u_j in (5.24) yields for all $(x,y)\in B^{\mathtt{i}}_j\times B^{\mathtt{k}}_j$, $\mathtt{i}\neq\mathtt{k}$ and $\mathtt{i},\mathtt{k}\in\mathcal{I}_j$,

$$u_j(x)-u_j(y)=\left(u^{\mathtt{i}}_j-u^{\mathtt{k}}_j\right)-\xi\left(\lambda_j^{-1}(x-\varepsilon_j\mathtt{i})\right)u^{\mathtt{i}}_j+\xi\left(\lambda_j^{-1}(y-\varepsilon_j\mathtt{k})\right)u^{\mathtt{k}}_j.$$

Hence, we can bound I^2_j as follows

$$\begin{aligned}I^2_j&\le c\sum_{\{(\mathtt{i},\mathtt{k})\in\mathcal{I}'_j\times\mathcal{I}'_j:\,0<|\varepsilon_j\mathtt{i}-\varepsilon_j\mathtt{k}|<\delta\}}\int_{B^{\mathtt{i}}_j\times B^{\mathtt{k}}_j}\frac{|u^{\mathtt{i}}_j-u^{\mathtt{k}}_j|^2}{|x-y|^{n+2s}}dxdy\\&+c\sum_{\{(\mathtt{i},\mathtt{k})\in\mathcal{I}'_j\times\mathcal{I}'_j:\,0<|\varepsilon_j\mathtt{i}-\varepsilon_j\mathtt{k}|<\delta\}}\int_{B^{\mathtt{i}}_j\times B^{\mathtt{k}}_j}\frac{|\xi(\lambda_j^{-1}(x-\varepsilon_j\mathtt{i}))|^2}{|x-y|^{n+2s}}dxdy=:I^{2,1}_j+I^{2,2}_j.\end{aligned}$$

To estimate the right hand side above we note that for any $(x, y) \in B_j^{\mathtt{i}} \times B_j^{\mathtt{k}}$, $\mathtt{i}, \mathtt{k} \in \mathcal{I}'_j$ with $\mathtt{i} \neq \mathtt{k}$

$$\varepsilon_j|\mathtt{i}-\mathtt{k}|/2 \leq |x-y| \leq 2\varepsilon_j|\mathtt{i}-\mathtt{k}|.$$

From this we infer $|u_j^{\mathtt{i}} - u_j^{\mathtt{k}}| \leq 2\|u\|_{W^{1,\infty}(\Omega')}\varepsilon_j|\mathtt{i}-\mathtt{k}|$, and so we deduce

$$\int_{B_j^{\mathtt{i}} \times B_j^{\mathtt{k}}} \frac{|u_j^{\mathtt{i}} - u_j^{\mathtt{k}}|^2}{|x-y|^{n+2s}} dxdy \leq c\, \frac{\varepsilon_j^{n-2(s-1)}}{|\mathtt{i}-\mathtt{k}|^{n+2(s-1)}}. \tag{5.29}$$

In addition, for every fixed $\mathtt{i} \in \mathcal{I}_j$ we have

$$\{\mathtt{k} \in \mathcal{I}'_j :\, 0 < \varepsilon_j|\mathtt{i}-\mathtt{k}|_\infty < \delta\} \subseteq \cup_{h=1}^{[\delta/\varepsilon_j]}\{\mathtt{k} \in \mathcal{I}'_j :\, h \leq |\mathtt{i}-\mathtt{k}|_\infty < h+1\},$$

where $[t]$ denotes the integer part of t. The latter inclusion, (5.29) and the estimate

$$\#\{\mathtt{k} \in \mathcal{I}'_j :\, h \leq |\mathtt{i}-\mathtt{k}|_\infty < h+1\} \leq 2^n h^{n-1}, \tag{5.30}$$

entail

$$\begin{aligned} I_j^{2,1} \leq c \sum_{\mathtt{i}\in\mathcal{I}'_j} \sum_{h=1}^{[\delta/\varepsilon_j]} \sum_{\{\mathtt{k}\in\mathcal{I}'_j :\, h \leq |\mathtt{i}-\mathtt{k}|_\infty < h+1\}} \frac{\varepsilon_j^{n-2(s-1)}}{h^{n+2(s-1)}} \\ \leq c \sum_{h=1}^{[\delta/\varepsilon_j]} \frac{\varepsilon_j^{-2(s-1)}}{h^{1+2(s-1)}} \leq c\, \delta^{2(1-s)}, \end{aligned} \tag{5.31}$$

since $\sum_{h=1}^{M} h^{-(1+\gamma)} \leq 2(M^{-\gamma})/(-\gamma)$, for any $\gamma < 0$ and $M \in \mathbb{N}$.

To deal with $I_j^{2,2}$ we argue as above, so that for every $\mathtt{i} \in \mathcal{I}'_j$ we have

$$\sum_{\{\mathtt{k}\in\mathcal{I}'_j :\, \mathtt{k}\neq\mathtt{i}\}} \int_{B_j^{\mathtt{k}}} \frac{1}{|x-y|^{n+2s}} dy \leq c \sum_{\{\mathtt{k}\in\mathcal{I}'_j :\, \mathtt{k}\neq\mathtt{i}\}} \frac{\varepsilon_j^{-2s}}{|\mathtt{i}-\mathtt{k}|^{n+2s}} \overset{(5.30)}{\leq} c \sum_{h\geq 1} \frac{\varepsilon_j^{-2s}}{h^{1+2s}}.$$

Thus, being $\xi(\lambda_j^{-1}(\cdot - \varepsilon_j\mathtt{i}))$ supported in $B_j^{\mathtt{i}}$, a change of variables yields

$$I_j^{2,2} \leq c\, \lambda_j^n \varepsilon_j^{-n-2s}\|\xi\|^2_{L^2(B_N)} \leq c\, \varepsilon_j^{\frac{4s^2}{n-2s}}\|\xi\|^2_{L^2(B_N)}. \tag{5.32}$$

Clearly, (5.31) and (5.32) imply (5.28).

Step 3. Estimate of I_j^3*:*

$$\limsup_j I_j^3 \leq \epsilon_\delta + \epsilon_N. \tag{5.33}$$

Being $u_j = w_j$ on Ω_j and $\operatorname{spt}(\xi(\lambda_j^{-1}(\cdot - \varepsilon_j\mathtt{i}))) \subseteq B_j^{\mathtt{i}}$, we find

$$I_j^3 \leq c \sum_{\mathtt{i}\in\mathcal{I}'_j} \mathcal{D}_s(\xi(\lambda_j^{-1}(\cdot - \varepsilon_j\mathtt{i})), B_j^{\mathtt{i}} \times (\Omega \setminus B_j^{\mathtt{i}}))$$

$$+ c\sum_{\mathtt{i}\in\mathcal{I}_j'}\int_{B_j^{\mathtt{i}}\times\Omega_j}\frac{|w_j(x)-u_j^{\mathtt{i}}|^2}{|x-y|^{n+2s}}dxdy$$
$$+ c\sum_{\mathtt{i}\in\mathcal{I}_j'}\int_{(B_j^{\mathtt{i}}\times\Omega_j)\cap\Delta_\delta}\frac{|w_j(y)-w_j(x)|^2}{|x-y|^{n+2s}}dxdy = I_j^{3,1}+I_j^{3,2}+I_j^{3,3}.$$

Note that by a change of variables the integral $I_j^{3,1}$ rewrites as

$$I_j^{3,1} \leq c\,\lambda_j^{n-2s}\varepsilon_j^{-n}\mathcal{D}_s\left(\xi, B_{\frac{\varepsilon_j}{8^{h_j}\lambda_j}}\times\left(\mathbb{R}^n\setminus\overline{B}_{\frac{\varepsilon_j}{8^{h_j}\lambda_j}}\right)\right) = \epsilon_N, \qquad (5.34)$$

by Remark 5.1.

To deal with the term $I_j^{3,2}$ we first integrate out y thanks to estimate (5.5), and observe that by construction $w_j|_{C_j^{\mathtt{i},h_j}} = u_j^{\mathtt{i}}$. Hence, we apply the scaled Poincaré-Wirtinger inequality (5.4) on the ball $B_j^{\mathtt{i}}\setminus C_j^{\mathtt{i},h_j} = B_{2^{-3h_j-2}\varepsilon_j}(\varepsilon_j\mathtt{i})$ to infer

$$I_j^{3,2} \leq c\sum_{\mathtt{i}\in\mathcal{I}_j'}\int_{B_j^{\mathtt{i}}}\frac{|w_j(x)-u_j^{\mathtt{i}}|^2}{\mathrm{dist}^{2s}(x,\partial B_j^{\mathtt{i}})}dx = c\sum_{\mathtt{i}\in\mathcal{I}_j'}\int_{B_j^{\mathtt{i}}\setminus C_j^{\mathtt{i},h_j}}\frac{|w_j(x)-u_j^{\mathtt{i}}|^2}{\mathrm{dist}^{2s}(x,\partial B_j^{\mathtt{i}})}dx$$
$$\leq c\left(\frac{2^{3h_j+1}}{\varepsilon_j}\right)^{2s}\sum_{\mathtt{i}\in\mathcal{I}_j'}\int_{B_j^{\mathtt{i}}\setminus C_j^{\mathtt{i},h_j}}|w_j(x)-u_j^{\mathtt{i}}|^2dx$$
$$\leq c\sum_{\mathtt{i}\in\mathcal{I}_j'}|w_j|^2_{W^{s,2}(B_j^{\mathtt{i}})} \leq c\,\mathcal{D}_s(w_j,(\Omega\times\Omega)\cap\Delta_\delta) \stackrel{(5.25)}{=} \epsilon_\delta. \qquad (5.35)$$

Finally, for what $I_j^{3,3}$ is concerned we have

$$I_j^{3,3} \leq c\,\mathcal{D}_s(w_j,(\Omega\times\Omega)\cap\Delta_\delta) \stackrel{(5.25)}{=} \epsilon_\delta. \qquad (5.36)$$

By collecting (5.34)-(5.36) we infer (5.33).

Step 4: Conclusion. By collecting Step 1 - Step 3 we infer

$$\limsup_j \mathscr{F}_j(u_j) \leq \mathscr{F}(u) + \epsilon_\delta + \epsilon_N,$$

with the two terms on the right hand side above infinitesimal as $\delta\downarrow 0^+$ and as $N\uparrow+\infty$, respectively. □

We end this section by mentioning several possible generalizations of Theorem 5.2 which are obtained slightly refining the arguments outlined here. The analysis performed above is sufficiently robust to deal with suitable aperiodic distributions of points (Delone set of points) on which the perforations are centered, with holes with random sizes and shapes, and with holes centered on stochastic lattices (see [39]). Vector-valued obstacles can also be taken into account (see [40]).

Eventually, all the quoted results are proved also for broad classes of anisotropic and non-homogeneous kernels defined on Sobolev-Slobodeckij spaces $W^{s,p}$, with $s \in (0,1)$, $p \in (1,+\infty)$ and $sp \in (0,n]$ (the case $sp = n$ deserves a slightly different analysis due to the scaling invariance property of the related energy, see [40]).

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Università di Firenze

E-mail address: focardi@math.unifi.it