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Asymptotic analysis and relaxation of Mumford-Shah type energies with obstacles

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We discuss in a model case some results concerning the homogenization and relaxation of free-discontinuity energies subjected to (unilateral) obstacle conditions.

1 Introduction

Homogenization of particle-reinforced composites and porous bodies may involve minimum problems for free-discontinuity energies with obstacle conditions. In the framework of Griffith theory of brittle fracture, we consider bodies with a periodic distribution of holes and study in the antiplane setting the behaviour of the total energy as the diameter of the holes tends to 0. Thus, by selecting the Mumford-Shah energy as a prototype, we analyze the asymptotics as $\varepsilon \to 0^+$ of the minimum problems

$$\inf\left\{\int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S_u):\quad u\in SBV(\Omega),\ u^+=0\ \ \mathcal{H}^{n-1}\text{ a.e. on }\mathbf{E}_\varepsilon,\ u=\varphi\,\mathcal{H}^{n-1}\text{ a.e. on }\partial\Omega\right\}. \qquad (1)$$

In formula (1) above $\Omega\subset\mathbf{R}^n$ is a given regular bounded open set, $\mathbf{E}_\varepsilon$ is the set of perforations obtained by repeating periodically a copy of the reference perforation set $E\subseteq(-1/2,1/2)^n$ rescaled by a factor $r_\varepsilon\in(0,\varepsilon)$, namely $\mathbf{E}_\varepsilon = \Omega\cap\cup_{i\in\mathbf{Z}^n} r_\varepsilon(i+E)$, and $\varphi\in W^{1,2}_{\rm loc}(\mathbf{R}^n)$ is a given boundary datum. The admissible displacements u belong to the space of *Special functions with Bounded Variation* introduced by De Giorgi and Ambrosio. Such a functional setting has been widely used in the last years to give weak formulations of variational problems in fracture mechanics (see [1]). In particular, ∇u and S_u in (1) are respectively the (approximate) gradient and the set of (approximate) discontinuities of u, u^+ is a suitable $\mathcal{L}^n$ representant of u, and the equality $u=\varphi$ on $\partial\Omega$ has to be intended in the sense of traces (see [1]). The total energy is the sum of a bulk and surface term, the former representing the elastic energy stored in the uncraked part of the material $\Omega\setminus S_u$ (which is assumed to be hyperlastic), the latter the energy dissipated to make the crack S_u grow.

Periodically perforated domains in Sobolev space have been the object of many researches after the pioneering works of Marchenko and Khruslov [5], Rauch and Taylor [6], [7] and Cioranescu and Murat [2]. It turns out that if one restricts the competitors in (1) to $W^{1,2}$ functions the minimum problems converge to a limit problem where the energy to be minimized contains an extra term which is a finite penalization keeping track of the local capacity density of the homogenizing holes. In order to handle this *relaxation phenomenon* De Giorgi, Dal Maso and Longo [4] took up a variational viewpoint by using Γ-convergence[1] analysis for the associated Dirichlet energies. Along this line of thought we study the asymptotic behaviour of the problems (1) *via* the Γ-convergence of the energies

$$\mathcal{F}_\varepsilon(u)=\int_\Omega |\nabla u|^2\,dx+\mathcal{H}^{n-1}(S_u)\quad \text{if } u\in SBV(\Omega),\ u^+=0\ \ \mathcal{H}^{n-1}\text{ a.e. on }\mathbf{E}_\varepsilon,\ u=\varphi\,\mathcal{H}^{n-1}\text{ a.e. on }\partial\Omega \qquad (2)$$

$+\infty$ otherwise in $L^1(\Omega)$ (see Theorem 2.1 below). With this notation (1) rewrites as $\inf_{L^1(\Omega)}\mathcal{F}_\varepsilon$, and the Γ-convergence of the family $(\mathcal{F}_\varepsilon)$ enforces the convergence of (1) as $\varepsilon\to 0^+$ to the corresponding minimum problem for the limit functional.

In addition, Γ-convergence implies the L^1 convergence of minimizers of (1) to mimimum points of the limit problem. To ensure the existence of minimizers for (1) the Direct Methods of the Calculus of Variations call for coercivity and lower semicontinuity of $\mathcal{F}_\varepsilon$ in $L^1(\Omega)$. While the former is ensured by a well known result of Ambrosio provided a control on the L^∞ norm is added (see [1]), in principle the energies in (2) are not L^1 lower semicontinuous since we allow also *thin* obstacles, i.e. $\mathcal{L}^n(E)=0$. More generally, in [9] we characterize the lower semicontinuous envelope (the *relaxation*) of free-discontinuity energies subjected to unilateral constraints. Hence, in our model setting we consider the functionals

$$F_\psi(u)=\int_\Omega |\nabla u|^2dx+\mathcal{H}^{n-1}(S_u)\quad\text{if }\ u\in SBV(\Omega),\ u^+\geq\psi\,\mathcal{H}^{n-1}\text{ a.e. on }\Omega,\ u=\varphi\,\mathcal{H}^{n-1}\text{ a.e. on }\partial\Omega \qquad (3)$$

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[1] Given a metric space (X,d), a family $G_\varepsilon: X\to\mathbf{R}\cup\{\pm\infty\}$ Γ-converges w.r.t. the topology induced by d to $\mathcal{G}: X\to\mathbf{R}\cup\{\pm\infty\}$ if: (i) for every $x\in X$ and $x_\varepsilon\xrightarrow{d}x$ it holds $\liminf_\varepsilon G_\varepsilon(x_\varepsilon)\geq\mathcal{G}(x)$, (ii) for every $x\in X$ there exists $z_\varepsilon\xrightarrow{d}x$ such that $\limsup_\varepsilon G_\varepsilon(z_\varepsilon)\leq\mathcal{G}(x)$. In particular, if $G_\varepsilon\equiv G$ the Γ-limit always exists and coincides with the d-lower semicontinuous envelope of G.

and $+\infty$ otherwise in $L^1(\Omega)$, where $\psi : \Omega \to \mathbf{R} \cup \{\pm\infty\}$ is a given a Borel function. In order to deal with this relaxation problem we consider a variational measure σ introduced by De Giorgi [3] for minimal surfaces with obstacles and prove that the L^1 lower semicontinuous envelope of F_ψ can be written in terms of the measure σ (see Theorem 2.2 below).

2 Overview of the results

Let us first deal with the homogenization problem. In the following statement we summarize in the model case presented above the results contained in Theorems 3.1 and 4.1, Propositions 3.3 and 3.4 in [8] (see also Theorem 5.1 there for generalizations). To avoid technicalities we exemplify the Γ-limit only for functions in $SBV(\Omega)$.

Theorem 2.1 (Γ-convergence) *Let $E \subseteq \Omega$ be $\mathcal{H}^{n-1}$ measurable, and suppose that the limit $\beta = \lim_{\varepsilon\to 0^+} r_\varepsilon/\varepsilon^{\frac{n}{n-1}} \in [0, +\infty]$ exists. Then the family $(\mathcal{F}_\varepsilon)$ defined in* (2) *Γ-converges in the L^1 topology to a functional $\mathcal{F} : L^1(\Omega) \to [0, +\infty]$ given for every $u \in SBV(\Omega)$ by*

$$\begin{aligned}\mathcal{F}(u) = &\int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{n-1}(S_u) \\ &+C_1(E)\beta^{n-1}\mathcal{L}^n\left(\{x \in \Omega : u(x) \neq 0\}\right) + \mathcal{H}^{n-1}(\{x \in \partial\Omega\, :\, u(x) \neq \varphi(x)\}),\end{aligned} \tag{4}$$

where $C_1(\cdot)$ is the functional capacity of degree 1.

The last term in (4) is due to a well known relaxation phenomenon in BV for the non-attainment of the Dirichlet boundary conditions imposed on $\partial\Omega$. Instead, the third term comes from the presence of the obstacles. In particular, if $\beta = 0$ the holes are too small and their effect disappears in the limit, while if $\beta = +\infty$ they are too big and only the trivial displacement $u \equiv 0$ is allowed. At the critical scale $r_\varepsilon \sim \varepsilon^{\frac{n}{n-1}}$ a non-trivial finite penalization appears (see [8] for further comments and details). Notice that the relevant scale depends only on the dimension n contrary to the Sobolev setting (see [2], [4]).

Let us now pass to the relaxation topic. In this setting we follow the approach of De Giorgi to study parametric Plateau problems with an obstacle [3] and introduce the Borel measure σ on Ω defined by

$$\sigma(E) = \sup_{\varepsilon>0} \left(\inf\left\{\mathrm{Per}(D) + \varepsilon^{-1}\mathcal{L}^n(D) :\, D \subseteq \Omega \text{ open, } D \supseteq E\right\}\right).$$

In the formula above $\mathrm{Per}(\cdot)$ is the perimeter according to De Giorgi (see [1]). Among the properties of σ collected in [3] we recall the following ones: there exist two constants $c_i = c_i(n) > 0$ such that $c_1\mathcal{H}^{n-1}(E) \leq \sigma(E) \leq c_2\mathcal{H}^{n-1}(E)$ for every set $E \subseteq \Omega$, and $\sigma(E) = 2\mathcal{H}^{n-1}(E)$ if E is a $\mathcal{H}^{n-1}$-rectifiable set.

In the next theorem we summarize in the model case mentioned in the introduction the results contained in Theorems 4.1 and 5.1 in [9] (see also Theorem 6.1 there for generalizations). To avoid technicalities we exemplify the relaxed functional only on $SBV(\Omega)$.

Theorem 2.2 (Relaxation) *Suppose that there exists $w \in SBV(\Omega)$ such that $F_\psi(w) < +\infty$ with F_ψ defined in* (3)*. Then the lower semicontinuous envelope $\mathcal{F}_\psi : L^1(\Omega) \to [0, +\infty]$ of the functional F_ψ is given for every $u \in SBV(\Omega)$ by*

$$\begin{aligned}\mathcal{F}_\psi(u) = &\int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{n-1}(S_u) + \frac{1}{2}\sigma\left(\{x \in S_u : u^+(x) < \psi(x)\}\right) \\ &+\sigma\left(\{x \in \Omega \setminus S_u : u^+(x) < \psi(x)\}\right) + \mathcal{H}^{n-1}(\{x \in \partial\Omega\, :\, u(x) \neq \varphi(x)\}).\end{aligned}$$

In view of the properties of σ recalled before, one can approximate with finite energy functions u which violate the constraint if and only if $\mathcal{H}^{n-1}(\{x \in \Omega : u^+(x) < \psi(x)\}) < +\infty$. Moreover, such a set appears in the relaxed functional *via* the measure σ (see [9] for further comments and details).

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