VECTOR-VALUED OBSTACLE PROBLEMS FOR NON-LOCAL ENERGIES

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Abstract. We investigate the asymptotics of obstacle problems for non-local energies in a vector-valued setting. Motivations arise, in particular, in phase field models for ferroelectric materials and variational theories for dislocations.

1. Introduction

The homogenization of obstacle problems for non-local energies has been object of recent researches. Interesting applications can be found in several fields such as fractional diffusion, contact mechanics, theories of Markov processes, and stock options pricing (see [7] and [14] for an exhaustive list of references).

In a simplified setting, the problem consists in understanding the asymptotic behaviour of the (global) minimizers of the energies in the sequel supplemented by appropriate boundary conditions

$$\mathcal{F}_j(u) = \int_{U\times U} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \qquad \text{if } u\in W^{s,p}(U,\mathbf{R}^m),\ \tilde{u}\in E \ \mathrm{cap}_{s,p} \text{ q.e. on } T_j\cap U \tag{1.1}$$

$+\infty$ otherwise in $L^p(U,\mathbf{R}^m)$. Here, $U\subset\mathbf{R}^n$, $n\geq 1$, is a Lipschitz open set, $W^{s,p}(U,\mathbf{R}^m)$ is the Sobolev-Slobodeckij space for $s\in(0,1)$, $p\in(1,+\infty)$ and $sp\in[1,n]$, $\mathrm{cap}_{s,p}$ is the related variational (p,s)-capacity, and $\tilde{u}$ denotes the precise representative of $u\in W^{s,p}(U,\mathbf{R}^m)$ which is defined except on a $\mathrm{cap}_{s,p}$-negligible set (see subsections 2.3 and 2.4). In addition, with fixed a set E in $\mathbf{R}^m$, a bounded subset T of $\mathbf{R}^n$, and a discrete and homogeneous distribution of points $\Lambda=\{\mathrm{x}^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$ (see Definition 2.1), for all $j\in\mathbf{N}$ the *obstacle set* $T_j\subseteq\mathbf{R}^n$ is defined by $T_j=\cup_{\mathtt{i}\in\mathbf{Z}^n}\left(\varepsilon_j\,\mathrm{x}^{\mathtt{i}}+\lambda_j\,T\right)$, where $(\varepsilon_j)_{j\in\mathbf{N}}$ and $(\lambda_j)_{j\in\mathbf{N}}$ are positive infinitesimal sequences.

The scalar framework with bilateral or unilateral conditions on the obstacles, corresponding to the choices $m=1$ and $E=\{0\}$ or $E=(0,+\infty)$ respectively, has been analyzed by means of different approaches (cp. with [6], [7], [13], [14]). In particular, in the Hilbertian framework, i.e. choose $p=2$ in (1.1), the asymptotic analysis can be reduced to the case of energies defined on standard (weighted) Sobolev spaces building upon an extension result analogous to the classical harmonic extension of $W^{1/2,2}$ functions (see [8]). Very recently a direct proof working directly at the level of non-local energies has been proposed in [14] for values of the parameters s, p such that $sp\in(1,n)$. Several possible generalizations are also

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highlighted there, for instance obstacles with random sizes and shapes or centred on random distribution of points are also analyzed (see [14, Section 4]).

In this paper we push forward the approach introduced in [14] into another direction by extending it to vector-valued problems motivated by phase fields models for ferroelectric solids and for dislocations.

In a first model, the vector valued field $u : U \subseteq \mathbf{R}^2 \to \{z \in \mathbf{R}^2 : \|z\| \leq r\}$ represents the spontaneous polarization of a ferroelectric material, the obstacles are then to be considered as zones where an insulator is present, and the non-local energy represents the electrostatic energy related to the electric field created by the charges induced by the spontaneuos polarization field on ∂U (see [12] for more details). In Theorem 3.3 we determine the homogenization limit of the model. Let us remark that the additional constraint that admissible fields take values into a disk is actually not affecting the asymptotics below (see Remark 3.2 for more comments).

Instead, in the second occurrence we refer to the variational theory for dislocations in an elastic crystal under the action of an applied shear stress introduced in [19] (see also [9] for related analytical results). In this model, supposing that only one slip system is active, $u : \mathbf{R}^2 \to \mathbf{R}^2$ is the slip field induced by the presence of dislocations, the obstacle sets can be interpreted as pinning sites modeling impurities in the material restraining the motion of those line defects. The free energy is given by the sum of two competing terms: a nonlocal term analogous to the $W^{1/2,2}$-seminorm, representing the long-range elastic interaction energy related to dislocations, and a nonconvex multiwell potential favouring vector-valued phase fields taking integer values, in order to penalize slips not compatible with the underlying crystalline structure. In the analysis below the latter extra energy contribution shall not be included (for the study of the full model in the scalar case, with $p = 2$ and $sp = 1$, see [16] and [17]), in the spirit of the second order Γ-development performed in the one-dimensional case in [15].

The description of the asymptotics of the energies $(\mathcal{F}_j)_{j\in\mathbf{N}}$ is addressed in this paper via Γ-convergence. This variational theory is known to be particularly well-suited to study the behaviour of the sequence of (global) minimizers corresponding to $(\mathcal{F}_j)_{j\in\mathbf{N}}$ under appropriate boundary conditions or by adding appropriate forcing terms (see [11], [3] and [4]). In what follows, we shall consider only the leading part of the energy since neither forcing terms nor Dirichlet boundary conditions imposed on ∂U, if properly formulated (cp. with Remark 2.6), change the asymptotics of the problem.

We will show that $\Gamma(L^p)$-limits of the family $(\mathcal{F}_j)_{j\in\mathbf{N}}$ take the form

$$\mathcal{F}(u) = \int_{U\times U} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy + \vartheta \int_U \varphi(u(x))\beta(x)\, dx \tag{1.2}$$

if $u \in W^{s,p}(U, \mathbf{R}^m)$, $+\infty$ otherwise in $L^p(U, \mathbf{R}^m)$. In the previous formula the prefactor ϑ is related to the mutual relationship between the scalings of the problem (see (3.12) or (3.35) for a precise definition according to the different ranges of sp in $[1, n]$), β describes the limit distribution of the points in Λ (see (3.2)); eventually the function φ is defined in formulas (3.11) if $sp = n$, and in (3.34) if $sp \in [1, n)$. More precisely, the previous result holds true upon the extraction of subsequences in the scaling-invariant framework. Actually, we will analyze energies as in (1.1) comparable to the fractional seminorms, that is defined through singular kernels K that are anisotropic versions of those above. In such a generality, Γ-convergence to a functional analogous to that in (1.2) holds true only up to subsequences unless extra-assumptions are imposed on the kernel K (cp. with Theorems 3.3 and 3.6 and Section 4).

The energy density of the obstacle penalization term, the function φ in (1.2), describes the asymptotic behaviour of non-linear, vector-valued relative capacitary problems defined by the non-local energy under consideration.

We will focus our attention mainly on the scaling invariant case, corresponding to the choice $sp = n$ in (1.1), since the subcritical framework $sp \in [1, n)$ can be deduced from the results of the ensuing sections combined with those in [14] (see Subsection 3.2). Actually, we will slighlty improve upon [14] by including the case $sp = 1$ which was not covered there.

From a technical point of view the main novelties of the paper are contained in Proposition 2.8 (see also Proposition 2.9 for related results) where instrumental properties of the mentioned non-linear, vector-valued (relative) capacitary problems are established. The latter, combined with a joining lemma in varying domains for non-local energies established by the Author in [14, Lemma 3.9], are relevant in the analysis developed in the subsequent sections.

Finally, the homogenization of obstacle problems for gradient energies in the local framework, i.e. defined on the standard Sobolev spaces $W^{1,p}$, has been studied by several Authors. We mention [2], [4], [6], [7], [13], [14] and [21] for exhaustive references. Here, we only stress that such a problem was recently addressed by [2] in the subcritical case, i.e. $p \in (1, n)$, and by [21] in the scaling-invariant case, $p = n$. Vector-valued problems for quasiconvex energies with $E = \{0\}$ were analyzed in both papers. Extensions to more general constraint sets E using the methods of Proposition 2.8 are likely to be obtained in that setting, too.

A brief resume of the paper is as follows: Section 2 is devoted to introduce the notations adopted throughout the whole paper and the instrumental preliminary results. In particular, Subsection 2.4 deals with (relative) capacities for Sobolev-Slobodeckij spaces both in the scaling invariant and subcritical cases.

In Section 3 we prove the Γ-convergence statements contained in Theorems 3.3 and 3.6. Some possible generalizations are considered in Section 4.

2. Preliminaries and Notations

2.1. **Basic Notations.** The Euclidean norm in $\mathbf{R}^n$ shall be denoted by $|\cdot|$, the maximum one by $|\cdot|_\infty$. We will write $B_r(x)$ for the Euclidean ball in $\mathbf{R}^n$ with centre x and radius $r > 0$, and simply B_r in case $x = \underline{0}$. As usual, we will set $\omega_n := \mathcal{L}^n(B_1)$.

Given a set $E \subset \mathbf{R}^n$ its complement will be indifferently denoted by E^c or $\mathbf{R}^n \setminus E$; instead, its interior and closure will be denoted by int(E) and $\overline{E}$, respectively.

Given an open set $A \subseteq \mathbf{R}^n$ the collections of its open subsets will be indicated by $\mathcal{A}(A)$, the diagonal set in $\mathbf{R}^n \times \mathbf{R}^n$ by Δ, and for every $\delta > 0$ its open δ-neighborhood by $\Delta_\delta := \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n : |x - y| < \delta\}$. Accordingly, for any set $E \subseteq \mathbf{R}^n$ and for any $\delta > 0$

$$E_\delta := \{x \in \mathbf{R}^n : \operatorname{dist}(x, E) < \delta\}. \tag{2.1}$$

In the following, U will always be an open and connected subset of $\mathbf{R}^n$ whose boundary is Lipschitz regular.

We shall use standard notations for Lebesgue and Hausdorff measures, and for Lebesgue and Sobolev function spaces. In addition, u_O will denote the mean value of a summable function u on a $\mathcal{L}^n$-measurable set O with positive measure, i.e. $u_O = \fint_O u\,dx$.

In several computations below the letter c shall generically denote a positive constant. We assume this convention since it is not essential to distinguish from one specific constant to another, leaving understood that the constant may change from line to line. The parameters on which each constant c depends will be explicitely highlighted.

2.2. **Non-periodic tilings.** Aperiodic sets of points are considered in the ensuing sections. More precisely, the *Voronoï tessellation* related to a *Delone set of points* Λ will substitute the usual periodic lattice.

The book by M. Senechal [20] is the standard reference for all the results quoted below.

Definition 2.1. *A point set* $\Lambda \subset \mathbf{R}^n$ *is a* Delone (or Delaunay) set *if it satisfies*

(i) Discreteness*: there exists* $r > 0$ *such that for all* $x, y \in \Lambda$, $x \neq y$, $|x - y| \geq 2r$*;*
(ii) Homogeneity*: there exists* $R > 0$ *such that* $\Lambda \cap B_R(x) \neq \emptyset$ *for all* $x \in \mathbf{R}^n$*.*

It is then easy to show that Λ is countably infinite. Hence, from now on we use the notation $\Lambda = \{\mathrm{x}^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$. By the very definition the quantities

$$r_\Lambda := \frac{1}{2}\inf\{|x - y| : x, y \in \Lambda,\, x \neq y\}, \quad R_\Lambda := \inf\{R > 0 : \Lambda \cap B_R(x) \neq \emptyset \ \ \forall x \in \mathbf{R}^n\} \tag{2.2}$$

are finite and strictly positive; R_Λ is called the *covering radius* of Λ.

Definition 2.2. *Let* $\Lambda \subset \mathbf{R}^n$ *be a Delone set, the* Voronoï cell *of a point* $\mathrm{x}^{\mathtt{i}} \in \Lambda$ *is the set of points*

$$V^{\mathtt{i}} := \{y \in \mathbf{R}^n : |y - \mathrm{x}^{\mathtt{i}}| \leq |y - \mathrm{x}^{\mathtt{k}}|, \ \textit{for all } \mathtt{i} \neq \mathtt{k}\}.$$

The Voronoï tessellation *induced by* Λ *is the partition of* $\mathbf{R}^n$ *given by* $\{V^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$*.*

The sets $V^{\mathtt{i}}$'s are closed, convex polytopes intersecting only along their boundaries. Several other interesting properties of Voronoï tessellations are collected in [20, Propositions 2.7, 5.2] (see also [14, Propositions 2.4 and 2.5]). Here, we will only recall some results that will be used in the subsequent analysis. We omit their proofs since they are justified by elementary counting arguments. For any $A \in \mathcal{A}(\mathbf{R}^n)$ set

$$\mathcal{I}_\Lambda(A) := \{\mathtt{i} \in \mathbf{Z}^n : V^{\mathtt{i}} \subseteq A\}, \qquad \mathscr{I}_\Lambda(A) := \{\mathtt{i} \in \mathbf{Z}^n : V^{\mathtt{i}} \cap \partial A \neq \emptyset\}, \tag{2.3}$$

then the following results hold true.

Proposition 2.3. *Let $\Lambda \subset \mathbf{R}^n$ be a Delone set and $\{V^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$ its induced Voronoï tessellation. Then,*

$$\omega_n r_\Lambda^n \#(\mathcal{I}_\Lambda(A)) \leq \mathcal{L}^n(A), \tag{2.4}$$

$$\omega_n r_\Lambda^n \#(\mathscr{I}_\Lambda(A)) \leq \mathcal{L}^n((\partial A)_{R_\Lambda}), \tag{2.5}$$

There exists a constant $c = c(n) > 0$ such that if $R_\Lambda \leq 1$ for every $\mathtt{i} \in \mathbf{Z}^n$ and $h \in \mathbf{N}$ it holds

$$\#\{\mathtt{k} \in \mathcal{I}_\Lambda(A) : h r_\Lambda < |\mathrm{x}^{\mathtt{i}} - \mathrm{x}^{\mathtt{k}}|_\infty \leq (h+1) r_\Lambda\} \leq c\, h^{n-1}. \tag{2.6}$$

2.3. **Sobolev-Slobodeckij spaces.** Let $A \subseteq \mathbf{R}^n$ be any bounded open Lipschitz set and fix $s \in (0,1)$ and let $p \in (1,+\infty)$ be such that $sp \in [1,n]$. By $W^{s,p}(A,\mathbf{R}^m)$ we denote the usual Sobolev-Slobodeckij space, or Besov space $B^s_{p,p}(A,\mathbf{R}^m)$. The space is Banach if equipped with the norm $\|u\|_{W^{s,p}(A,\mathbf{R}^m)} = \|u\|_{L^p(A,\mathbf{R}^m)} + |u|_{W^{s,p}(A,\mathbf{R}^m)}$, where

$$|u|^p_{W^{s,p}(A,\mathbf{R}^m)} := \int_{A\times A} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\,.$$

We will use several properties of fractional Sobolev spaces, giving precise references for those employed in the sequel in the respective places mainly referring to [1] and [22].

In the indicated ranges for the parameters p, s the space $W^{s,p}$ turns out to be reflexive and separable (see [22, Thm 4.8.2]).

We recall Poincaré-Wirtinger and Poincaré inequalities in fractional Sobolev spaces.

Theorem 2.4. *Let $n \geq 1$, $s \in (0,1)$ and $p \in (1,+\infty)$ as above. Let $A \subset \mathbf{R}^n$ be a bounded, connected open set, and O any $\mathcal{L}^n$-measurable subset of A with $\mathcal{L}^n(O) > 0$. Then for any function $u \in W^{s,p}(A,\mathbf{R}^m)$,*

$$\|u - u_O\|^p_{L^p(A,\mathbf{R}^m)} \leq c_{PW} |u|^p_{W^{s,p}(A,\mathbf{R}^m)}, \tag{2.7}$$

for a constant $c_{PW} = c_{PW}(n,s,O,A)$.

Moreover, for any $u \in W^{s,p}_0(A,\mathbf{R}^m)$ we have

$$\|u\|^p_{L^p(A,\mathbf{R}^m)} \leq c_P |u|^p_{W^{s,p}(A,\mathbf{R}^m)}, \tag{2.8}$$

for a constant $c_P = c_P(n,s,A)$.

Remark 2.5. *A scaling argument and Hölder inequality yield for any $x \in \mathbf{R}^n$ and $r > 0$ and for some $c = c(n,s,O,A) > 0$*

$$\|u - u_{x+rO}\|^p_{L^p(x+rA,\mathbf{R}^m)} \leq c\, r^{sp} |u|^p_{W^{s,p}(x+rA,\mathbf{R}^m)}. \tag{2.9}$$

A similar conclusion holds for Poincaré inequality (2.8).

Remark 2.6. *In case $sp = 1$ traces are not well defined. Indeed, equality $W^{s,p}(A, \mathbf{R}^m) = W^{s,p}_0(A, \mathbf{R}^m)$ holds (see [22, Theorem 4.3.2/1]). Hence, Dirichlet boundary value data can be imposed by fixing the inner trace on $A \setminus A'$, A' a bounded open set in $\mathbf{R}^n$ with $A' \subset\subset A$.*

We end this subsection by recalling an useful elementary bound on standard singular kernels (cp. with [14, Lemma A.1]): there exists a positive constant $c = c(n, p, s)$ such that for any $\mathcal{L}^n$-measurable set O and for any point x with $\mathrm{dist}(x, O) > 0$ it holds

$$\int_O \frac{1}{|y-x|^{n+sp}}\, dy \le c\, (\mathrm{dist}(x, O))^{-sp}. \tag{2.10}$$

2.4. **Fractional capacities.** Capacitary type problems for fractional Sobolev spaces are introduced and analyzed in what follows. For any set $T \subseteq \mathbf{R}^n$ define

$$\mathrm{cap}_{s,p}(T) := \inf_{\{A \in \mathcal{A}(\mathbf{R}^n):\, A \supseteq T\}} \inf \left\{ |u|^p_{W^{s,p}(\mathbf{R}^n)} : u \in W^{s,p}(\mathbf{R}^n),\, u \ge 1\, \mathcal{L}^n \text{ a.e. on } A \right\}, \tag{2.11}$$

with the usual convention $\inf \emptyset = +\infty$. It is also worth introducing relative capacities. To this aim, first localize (2.11) for open sets $A \in \mathcal{A}(\mathbf{R}^n)$ setting

$$C_{s,p}(A, B_\rho) := \inf \left\{ |w|^p_{W^{s,p}(\mathbf{R}^n)} : w \in W^{s,p}(\mathbf{R}^n),\, w = 0 \text{ on } \mathbf{R}^n \setminus \overline{B}_\rho,\, w \ge 1\, \mathcal{L}^n \text{ a.e. on } A \right\}$$

and then extend it to all subsets of $\mathbf{R}^n$ by outer regularity as

$$C_{s,p}(T, B_\rho) := \inf_{\{A \in \mathcal{A}(B_\rho):\, A \supseteq T\}} C_{s,p}(A, B_\rho). \tag{2.12}$$

Arguing as in [10, Section 3], the set functions above turn out to be Choquet capacities (see also [1, Chapter V] and [18, Theorem 2.2]). Recall that a set T in $\mathbf{R}^n$ is said to be of (s,p)-*capacity zero* if

$$C_{s,p}(T \cap B_\rho, B_\rho) = 0$$

for all $\rho > 0$. We also say that a property holds (s,p)-*quasi everywhere*, in short $\mathrm{cap}_{s,p}$ q.e., if it holds up to a set of (s,p)-capacity zero. In particular, any function u in $W^{s,p}(A)$, $A \in \mathcal{A}(\mathbf{R}^n)$, has a *precise representative* $\tilde{u}$ defined $\mathrm{cap}_{s,p}$ q.e. and the following formula holds (see [10, Section 4]),

$$C_{s,p}(T, B_\rho) = \inf \left\{ |u|^p_{W^{s,p}(\mathbf{R}^n)} : u \in W^{s,p}(\mathbf{R}^n),\, u = 0 \text{ on } \mathbf{R}^n \setminus \overline{B}_\rho,\, \tilde{u} \ge 1 \text{ q.e. on } T \right\}. \tag{2.13}$$

The behaviour of relative capacities can be distinguished according to whether $sp = n$ or $sp \in [1, n)$. Before proceeding into this direction let us enlarge the framework of interest to a vector-valued setting and also to singular kernels different from those defining the fractional seminorms.

More generally, we shall be concerned in the sequel with non-linear, vector-valued capacitary problems related to translation-invariant singular kernels $K : \mathbf{R}^n \setminus \{0\} \to (0, +\infty)$ satisfying for some constant $\alpha \ge 1$ and for all $x \in \mathbf{R}^n \setminus \{0\}$

$$\alpha^{-1}|x|^{-(n+sp)} \le K(x) \le \alpha |x|^{-(n+sp)} \tag{2.14}$$

(see Section 4 for generalizations). For every $A \in \mathcal{A}(\mathbf{R}^n)$, the kernel K defines a functional $\mathcal{K} : L^p(A, \mathbf{R}^m) \to [0, +\infty]$ by

$$\mathcal{K}(u, A) = \int_{A\times A} K(x-y)|u(x) - u(y)|^p dxdy \tag{2.15}$$

if $u \in W^{s,p}(A, \mathbf{R}^m)$, $+\infty$ otherwise on $L^p(A, \mathbf{R}^m)$. We shall drop the dependence on A if $A = \mathbf{R}^n$. A relevant notion related to $\mathcal{K}$ is that of *locality defect*: for any $\mathcal{L}^n$-measurable function w and any $\mathcal{L}^{n\times n}$-measurable set $E \subseteq \mathbf{R}^n \times \mathbf{R}^n$

$$\mathcal{D}_{\mathcal{K}}(w, E) := \int_E K(x-y)|w(x) - w(y)|^p \, dxdy.$$

Clearly, $\mathcal{K}(w, A) = \mathcal{D}_{\mathcal{K}}(w, A \times A)$; the terminology is justified since given two disjoint subdomains $A, B \subseteq \mathbf{R}^n$, for $C = A \cup B$ we get

$$\mathcal{K}(w, C \times C) = \mathcal{K}(w, A \times A) + \mathcal{K}(B \times B) + \mathcal{D}_{\mathcal{K}}(w, A \times B) + \mathcal{D}_{\mathcal{K}}(w, B \times A). \tag{2.16}$$

2.4.1. *The scaling-invariant case.* Let us first focus our attention to parameters $p \in (1, +\infty)$ and $s \in (0, 1)$ satisfying $sp = n$ (for related results and references in the local case see [21]).

The scaling invariance of the kernel yields that for any subset T of $\mathbf{R}^n$ and any $\lambda > 0$

$$\liminf_{\rho\to+\infty} C_{s,p}(T, B_\rho) = \liminf_{\rho\to+\infty} C_{s,p}(\lambda T, B_\rho), \quad \limsup_{\rho\to+\infty} C_{s,p}(T, B_\rho) = \limsup_{\rho\to+\infty} C_{s,p}(\lambda T, B_\rho). \tag{2.17}$$

In addition, the following estimates can be obtained as in [10, Theorem 3.11]: for some constant $c = c(n, p) > 0$

$$c^{-1} \left(\ln \frac{\rho}{t}\right)^{1-p} \leq C_{s,p}(B_t(x), B_\rho(x)) \leq c \left(\ln \frac{\rho}{t}\right)^{1-p} \tag{2.18}$$

for every $x \in \mathbf{R}^n$ and for every pairs of positive numbers t, ρ such that $t < \rho/2$. Hence, one can show that $\mathrm{cap}_{s,p}(T) = 0$ for all sets T as in the standard (local) Sobolev setting.

By formula (2.18) a logarithmic rescaling is then needed. We have not been able to prove that the function $\rho \to (\ln \rho)^{p-1} C_{s,p}(B_t, B_\rho)$ has actually a limit as ρ diverges. Such a property is well-known for (standard) Sobolev relative n-capacities (for a homogenization-like proof see [21, Proposition 5.1]).

Despite this, (2.17) and (2.18) imply that for all bounded subsets T of $\mathbf{R}^n$ with non-empty interior part we have

$$\begin{aligned} c^{-1} \leq \liminf_{\rho\to+\infty} (\ln \rho)^{p-1} C_{s,p}(B_1, B_\rho) &= \liminf_{\rho\to+\infty} (\ln \rho)^{p-1} C_{s,p}(T, B_\rho) \\ &\leq \limsup_{\rho\to+\infty} (\ln \rho)^{p-1} C_{s,p}(T, B_\rho) \leq \limsup_{\rho\to+\infty} (\ln \rho)^{p-1} C_{s,p}(B_1, B_\rho) \leq c \end{aligned} \tag{2.19}$$

for some constant $c = c(n, p) \geq 1$.

With fixed subsets $T \subseteq \mathbf{R}^n$ and $E \subseteq \mathbf{R}^m$, radii ρ and R such that $T \subset\subset B_\rho \subset\subset B_R$, and a point $z \in \mathbf{R}^m$ define

$$\varphi^{K,T}_{\rho,R}(z) := (\ln \rho)^{p-1} \inf_{w\in AD_z(T,B_\rho)} \mathcal{K}(w, B_R), \tag{2.20}$$

where the set of admissible test functions is given by

$$AD_z(T,B_\rho) := \Big\{ w \in L^p_{loc}(\mathbf{R}^n,\mathbf{R}^m) : w - z \in W^{s,p}(\mathbf{R}^n,\mathbf{R}^m),$$
$$w = z \text{ on } \mathbf{R}^n \setminus \overline{B}_\rho,\ \tilde{w}(x) \in E \ \ \mathrm{cap}_{s,p} \text{ q.e. on } T \Big\}. \quad (2.21)$$

For the sake of simplicity we shall drop the dependence on T and K in the quantities defined in (2.20) and (2.21) when there will be no risk of confusion, and write only φ_ρ if in addition $R = +\infty$.

Remark 2.7. *Note that if $m = 1$, $E = \{0\}$ and $K(x-y) = |x-y|^{-2n}$, then $\varphi^T_\rho(z)$ reduces to $|z|^p |\ln\rho|^{p-1} C_{s,p}(T,B_\rho)$ for all $\rho > 0$. With the same choices of E and K a similar characterization can be given in the vectorial setting, too.*

In general, the properties enjoyed by $(\varphi_{\rho,R})_{\rho>0}$, as described in what follows, are not obtained by explicit characterizations.

In the next proposition we shall establish several properties of the families $(\varphi_{\rho,R})_{\rho>0}$. We remark that the estimates below will always be uniform for those kernels satisfying the growth conditions in (2.14). In addition, in what follows the letter α will always denote the constant introduced there.

Proposition 2.8. *Suppose T bounded. Then, for every $0 < \rho < R$ it holds*

(i) $(\varphi_{\rho,R})_{\rho>0}$ is pointwise equi-bounded: there exist a non-negative constant c_1 and a positive constant c_2 depending on T, m and α such that for every $z \in \mathbf{R}^m$

$$0 \le \varphi_{\rho,R}(z) \le c_2 \,\mathrm{dist}^p(z,E), \quad (2.22)$$

and in addition,

$$c_1 \,\mathrm{dist}^p(z,E) \le \varphi_\rho(z). \quad (2.23)$$

(ii) $(\varphi_{\rho,R})_{\rho>0}$ is locally equi-Lipschitz continuous: there exists a positive constant c depending only on E, T, m, p and α such that for all z_1, $z_2 \in \mathbf{R}^m$

$$|\varphi_{\rho,R}(z_2) - \varphi_{\rho,R}(z_1)| \le c\left(1 + |z_1|^{p-1} + |z_2|^{p-1}\right)|z_1 - z_2|; \quad (2.24)$$

(iii) there exists a positive constant c depending on m, n, p and α such that

$$\varphi_{\rho,R}(z) \le \varphi_\rho(z) \le \left(1 + c\left(\frac{\rho}{R-\rho}\right)^n\right)\varphi_{\rho,R}(z). \quad (2.25)$$

Proof. We start off with item (i). Given $\varepsilon > 0$ and take a test function ψ for the capacitary problem of T in B_ρ such that

$$|\psi|^p_{W^{s,p}(\mathbf{R}^n)} \le (1+\varepsilon) C_{s,p}(T,B_\rho).$$

For any point $\zeta \in E$ consider $w := \psi\zeta + (1-\psi)z$, then $w \in AD_z(T, B_\rho)$ and satisfies

$$\mathcal{K}(w, B_R) \leq \alpha |z-\zeta|^p |\psi|^p_{W^{s,p}(\mathbf{R}^n)}.$$

The latter inequality implies

$$\varphi_{\rho,R}(z) \leq \alpha(1+\varepsilon)(\ln\rho)^{p-1}|z-\zeta|^p C_{s,p}(T, B_\rho).$$

By letting first $\varepsilon \downarrow 0^+$, and then passing to the infimum on $\zeta \in E$ we deduce the upper bound in (2.22) by (2.18) with $c_2 := \alpha \limsup_\rho (\ln\rho)^{p-1} C_{s,p}(T, B_\rho)$.

To show (2.23), choose $z \in \mathbf{R}^n$ with $\mathrm{dist}(z, E) > 0$ and let $\delta := \frac{1}{m}\mathrm{dist}(z, E)$. Given a map w in $AD_z(T, B_\rho)$ denote by $T_i := \{x \in T : \delta \leq |\tilde{w}_i(x) - z_i|\}$. Then, the set $T \setminus \cup_{i=1}^m T_i$ has (s,p)-capacity zero, and for every $\rho > 0$ there exists $\ell \in \{1, \ldots, m\}$ such that

$$\frac{1}{m} C_{s,p}(T, B_\rho) \leq C_{s,p}(T_\ell, B_\rho).$$

Note that w_ℓ satisfies $w_\ell - z_\ell \in W^{s,p}(\mathbf{R}^n)$ and $\tilde{w}_\ell \in (-\infty, z_\ell - \delta] \cup [z_\ell + \delta, +\infty)$ $\mathrm{cap}_{s,p}$ q.e. on T_ℓ. The embedding of $W^{s,p}$ into VMO (see, for instance [5, Section I.2]) implies that $\tilde{w}_\ell$ takes values either in $(-\infty, z_\ell - \delta]$ or in $[z_\ell + \delta, +\infty)$. In the first case, we infer

$$\begin{aligned} \varphi_\rho(z) \geq \alpha^{-1}(\ln\rho)^{p-1} \inf \Big\{ &|u|^p_{W^{s,p}(\mathbf{R}^n)} : u - z_\ell \in W^{s,p}(\mathbf{R}^n), \\ &u = z_\ell \text{ on } \mathbf{R}^n \setminus \overline{B}_\rho, \ \tilde{u}(x) \leq z_\ell - \delta \ \mathrm{cap}_{s,p} \text{ q.e. on } T_\ell \Big\} \\ &= \alpha^{-1}\delta^p (\ln\rho)^{p-1} C_{s,p}(T_\ell, B_\rho) \geq \frac{\mathrm{dist}^p(z,E)}{m^{p+1}\alpha} (\ln\rho)^{p-1} C_{s,p}(T, B_\rho). \end{aligned}$$

The second instance is completely analogous. The conclusion follows at once with $c_1 := m^{-(p+1)}\alpha^{-1} \liminf_\rho (\ln\rho)^{p-1} C_{s,p}(T, B_\rho)$.

We now turn to the proof of item (ii). To this aim we use special external variations.

Clearly, by (2.22) it is not restrictive to assume $\mathbf{R}^m \setminus \overline{E} \neq \emptyset$. Fix z_1 and $z_2 \in \mathbf{R}^m$, we shall estimate the oscillation of $\varphi_{\rho,R}$ in those points. Fix $\delta > 0$ and first suppose $\mathrm{dist}(z_1, E) \geq 2\delta$. Note then that

$$\delta \leq \mathrm{dist}(z_1, E_\delta) \leq \mathrm{dist}(z_1, E) \leq \mathrm{dist}(z_1, E_\delta) + \delta,$$

from which we infer

$$1 \leq \frac{\mathrm{dist}(z_1, E)}{\mathrm{dist}(z_1, E_\delta)} \leq 2. \tag{2.26}$$

In addition, Kirszbraun's extension theorem supplies the existence of a map $\Psi \in \mathrm{Lip}(\mathbf{R}^m, \mathbf{R}^m)$ such that $\Psi|_{E_\delta} = \mathrm{Id}$ and $\Psi(z_1) = z_2$, with

$$\mathrm{Lip}(\Psi) = \mathrm{Lip}(\Psi|_{E_\delta \cup \{z_1\}}) = 1 \vee \sup_{\zeta \in E_\delta} \frac{|\zeta - z_2|}{|\zeta - z_1|} \leq 1 + \frac{|z_1 - z_2|}{\mathrm{dist}(z_1, E_\delta)}.$$

Now take any $w \in AD_{z_1}(T, B_\rho)$, and note that $\Psi \circ w \in AD_{z_2}(T, B_\rho)$ since E_δ is open and $\Psi|_{E_\delta} = \mathrm{Id}$. It is also immediate to check that

$$\varphi_{\rho,R}(z_2) \leq \mathrm{Lip}^p(\Psi)\varphi_{\rho,R}(z_1).$$

In turn, the latter estimate, (2.22), (2.26), and the convexity of $\mathbf{R} \ni t \to |t|^p$ imply

$$\varphi_{\rho,R}(z_2) - \varphi_{\rho,R}(z_1) \leq p \left(1 + \frac{|z_1 - z_2|^{p-1}}{\mathrm{dist}^{p-1}(z_1, E_\delta)}\right) \frac{|z_1 - z_2|}{\mathrm{dist}(z_1, E_\delta)}\, \varphi_{\rho,R}(z_1)$$
$$\stackrel{(2.22),\,(2.26)}{\leq} 2^p p\, c_2 \left(\mathrm{dist}^{p-1}(z_1, E) + |z_1 - z_2|^{p-1}\right) |z_1 - z_2|.$$

Hence, if in addition $\mathrm{dist}(z_2, E) \geq 2\delta$, exchanging the roles of z_1 and z_2 we infer

$$|\varphi_{\rho,R}(z_2) - \varphi_{\rho,R}(z_1)| \leq c_3 \left(1 + |z_1|^{p-1} + |z_2|^{p-1}\right) |z_1 - z_2| \tag{2.27}$$

for a positive constant c_3 depending on E, T, p, m and α, and above all independent from δ.

Hence, the arbitrariness of $\delta > 0$ yields that (2.27) holds for all points $z_1, z_2 \notin \overline{E}$. Actually, (2.27) is still valid if $z_1, z_2 \in \overline{E}$ by (2.22). Finally, if $z_1 \in \overline{E}$ and $z_2 \notin \overline{E}$ we have

$$|\varphi_{\rho,R}(z_2) - \varphi_{\rho,R}(z_1)| = |\varphi_{\rho,R}(z_2)| \stackrel{(2.22)}{\leq} c_2\, \mathrm{dist}^p(z_2, E)$$
$$\leq c_2 |z_2 - z_1|^p \leq c_3 \left(1 + |z_2|^{p-1} + |z_1|^{p-1}\right) |z_2 - z_1|.$$

In conclusion, we have established (2.27) for all couple of points z_1, z_2 in $\mathbf{R}^m$.

To prove item (iii), fix any test function w in $AD_z(T, B_\rho)$, for $R > \rho$ we infer from the scaled Poincaré-Wirtinger inequality in (2.9)

$$0 \leq \mathcal{K}(w) - \mathcal{K}(w, B_R) \stackrel{w|_{\mathbf{R}^n \setminus B_\rho} = z}{=} \mathcal{D}_{\mathcal{K}}(w, B_\rho \times (\mathbf{R}^n \setminus B_R)) + \mathcal{D}_{\mathcal{K}}(w, (\mathbf{R}^n \setminus B_R) \times B_\rho)$$
$$\stackrel{(2.14)}{\leq} 2\alpha \int_{B_\rho} |w(x) - z|^p \int_{\mathbf{R}^n \setminus B_R} \frac{1}{|x - y|^{2n}} dy\, dx \stackrel{(2.10)}{\leq} \frac{c}{(R - \rho)^n} \int_{B_\rho} |w(x) - z|^p\, dx$$
$$\stackrel{(2.9)}{\leq} c \left(\frac{\rho}{R - \rho}\right)^n |w|^p_{W^{s,p}(B_\rho, \mathbf{R}^m)} \stackrel{(2.14)}{\leq} c\, \alpha \left(\frac{\rho}{R - \rho}\right)^n \mathcal{K}(w, B_\rho).$$

Multiplying by $(\ln \rho)^{p-1}$ the previous inequality and taking into account the arbitrariness of w imply the conclusion. □

Furthermore, let us highlight some additional properties enjoyed by the families $(\varphi_{\rho,R})_\rho$ due to symmetries of E. Their proof is omitted since it is a straightforward consequence of the definition of $\varphi_{\rho,R}$ and Proposition 2.8.

Proposition 2.9. *Suppose T bounded, then*

(i) E is convex if and only if φ_ρ is convex;

(ii) if E is a cone, then $\varphi_{\rho,R}$ is p-homogeneous;

(iii) if E is invariant under an isometry Φ, then $\varphi_{\rho,R} \circ \Phi = \varphi_{\rho,R}$.

Remark 2.10. *The proofs of (2.22) and (2.23) show that the constants c_1, c_2 depend on T through the inferior, superior limit of $(\ln \rho)^{p-1} C_{s,p}(T, B_\rho)$, respectively. Hence, if T is bounded and has non-empty interior, those constants do not actually depend on T itself by (2.19).*

Hence, if in addition the kernel K is $(-2n)$-homogeneous, under the same hypotheses on T, one can argue as in (2.19) and show that for all $z \in \mathbf{R}^n$ the inferior and superior limits as $\rho \to +\infty$ of $(\ln \rho)^{p-1}\varphi_\rho(z)$ do not depend on T either.

On the other hand, the functions φ_ρ do depend on the set E as it follows, for instance, from item (i) in Proposition 2.9. Note also that $\overline{E} = \mathbf{R}^m$ if and only if $\varphi_\rho = 0$ by (2.22) and (2.23), in case $\liminf_\rho (\ln \rho)^{p-1} C_{s,p}(T, B_\rho) > 0$, that is $c_1 > 0$.

2.4.2. *The subcritical case.* In case $sp \in [1, n)$ the behaviour of (relative) capacities is rather different. Indeed, an elementary scaling argument shows that for all $t > 0$ and $x \in \mathbf{R}^n$

$$\mathrm{cap}_{s,p}(B_t(x)) = t^{n-sp}\mathrm{cap}_{s,p}(B_1). \tag{2.28}$$

One can also show that the latter expression is strictly positive.

Consider a kernel K satisfying (2.14) and let

$$\varphi^{K,T}_{\rho,R}(z) := \inf_{w \in AD_z(T,B_\rho)} \mathcal{K}(w, B_R), \tag{2.29}$$

$AD_z(T, B_\rho)$ given by (2.21). Following step by step the arguments of Propositions 2.8 and 2.9 one can establish results completely analogous to those contained there. The proofs being even easier since in the current range no logarithmic rescaling is needed in the definition of $\varphi^{K,T}_{\rho,R}$ (see also [14, Lemma 2.12]). Details will not be worked-out and left to the interested reader.

In addition, for all $z \in \mathbf{R}^m$ the function $\rho \to \varphi^{K,T}_\rho(z)$ turns out to be non-decreasing, thus monotonicity implies the convergence

$$\varphi^{K,T}(z) := \lim_{\rho \to +\infty} \varphi^{K,T}_\rho(z). \tag{2.30}$$

Actually, the pointwise convergence above is uniform on compact subsets of $\mathbf{R}^m$ thanks to (2.27) in this setting. Furthermore, by (2.25) we have that similar convergences to $\varphi^{K,T}$ hold for families $(\varphi^{K,T}_{\rho,R(\rho)})$ provided $R(\rho)/\rho \to +\infty$ as $\rho \to +\infty$ (cp. with [14, Lemma 2.12]).

3. Γ-convergence statement

Consider Delone sets $\Lambda_j = \{\mathrm{x}^{\mathtt{i}}_j\}_{\mathtt{i} \in \mathbf{Z}^n}$, and let $r_j := r_{\Lambda_j}$ $R_j := R_{\Lambda_j}$, $\mathcal{I}_j(A) := \mathcal{I}_{\Lambda_j}(A)$, for all $A \in \mathcal{A}(U)$, dropping the dependence on the set if $A = U$, that is $\mathcal{I}_j := \mathcal{I}_j(U)$ (see (2.3) for the definition of $\mathcal{I}_{\Lambda_j}(\cdot)$). Assume that the Λ_j's are such that

$$\lim_j r_j = 0, \quad (1 \leq) \limsup_j (R_j/r_j) < +\infty, \tag{3.1}$$

$$\mu_j := \frac{1}{\#\mathcal{I}_j(U)} \sum_{\mathtt{i} \in \mathcal{I}_j(U)} \delta_{\mathrm{x}^{\mathtt{i}}_j} \to \mu := \beta\, \mathcal{L}^n \llcorner U \qquad w^*\text{-}(C_b(U))^*, \tag{3.2}$$

for some $\beta \in L^1(U, (0, +\infty))$ with $\|\beta\|_{L^1(U)} = 1$.

It has been shown in [14, Remark 3.2] that condition (3.2) above holds true upon the extraction of subsequences. Some non-trivial, i.e. non-periodic, examples of Delone sets are discussed there, too (cp. with [14, Examples 3.5-3.7]).

Note that by (2.4), (2.5) and (3.1) we infer

$$0 < \liminf_j r_j^n \#\mathcal{I}_j \leq \limsup_j r_j^n \#\mathcal{I}_j < +\infty. \tag{3.3}$$

With fixed subsets $E \subset \mathbf{R}^m$, and $T \subset \mathbf{R}^n$ bounded, for all $j \in \mathbf{N}$ define the *obstacle set* $T_j := \cup_{\mathtt{i} \in \mathbf{Z}^n} T_j^{\mathtt{i}}$ where

$$T_j^{\mathtt{i}} := \mathrm{x}_j^{\mathtt{i}} + \lambda_j T, \qquad \text{and } \lambda_j \in (0, r_j). \tag{3.4}$$

Note that $T_j^{\mathtt{i}} \subseteq V_j^{\mathtt{i}}$ for all $\mathtt{i} \in \mathbf{Z}^n$, $j \in \mathbf{N}$. Then define functionals $\mathcal{K}_j : L^p(U, \mathbf{R}^m) \to [0, +\infty]$ by

$$\mathcal{K}_j(u) = \begin{cases} \mathcal{K}(u, U) & \text{if } u \in W^{s,p}(U, \mathbf{R}^m),\ \tilde{u} \in E \ \mathrm{cap}_{s,p} \text{ q.e. on } T_j \cap U \\ +\infty & \text{otherwise.} \end{cases} \tag{3.5}$$

Subsections 3.1 and 3.2 contains the asymptotic analysis of the energies above in the scaling-invariant and subcritical cases, respectively. The proofs are strongly linked though some details are different. In particular, in the former case the energy does not concentrate at the same scale as the radii of the perforations.

The Γ-convergence statement established in Theorems 3.3 and 3.6 relies upon a technical result proved in [14, Lemma 3.9] in the scalar case. There is no difficulty in extending that result in the vector-valued setting currently under investigation, thus we limit ourselves to present its statement only. On a technical side, it reduces the verification of liminf and limsup inequalities on sequences of functions almost matching the values of their limit on suitable annuli surrounding the obstacle sets. Following an early idea by De Giorgi, a clever slicing and averaging argument is exploited to change boundary values increasing the energy in a controlled and infinitesimal way (for more comments see [14, Subection 3.2]).

To recall the statement of [14, Lemma 3.9] we fix some more notation: for all $\mathtt{i} \in \mathcal{I}_j$, N and $h \in \mathbf{N}$ let

$$B_j^{\mathtt{i},h} := \{x \in \mathbf{R}^n : |x - \mathrm{x}_j^{\mathtt{i}}| < N^{-3h} r_j\}, \quad C_j^{\mathtt{i},h} := \{x \in \mathbf{R}^n : N^{-3h-2} r_j < |x - \mathrm{x}_j^{\mathtt{i}}| < N^{-3h-1} r_j\}.$$

Clearly, the inclusions $C_j^{\mathtt{i},h} \subset B_j^{\mathtt{i},h} \setminus \overline{B}_j^{i,h+1} \subset V_j^{\mathtt{i}}$ hold true.

Lemma 3.1. *Let $(u_j)_{j \in \mathbf{N}}$ be converging to u in $L^p(U, \mathbf{R}^m)$ with $\sup_j |u_j|_{W^{s,p}(U, \mathbf{R}^m)} < +\infty$. With fixed $N \in \mathbf{N}$, for every $j \in \mathbf{N}$ there exists $h_j \in \{1, \ldots, N\}$ and $w_j \in W^{s,p}(U, \mathbf{R}^m)$ such that*

$$w_j = u_j \ \text{ on } U \setminus \cup_{\mathtt{i} \in \mathcal{I}_j} (\overline{B}_j^{\mathtt{i},h_j} \setminus B_j^{\mathtt{i},h_j+1}), \tag{3.6}$$

$$w_j = (u_j)_{C_j^{\mathtt{i},h_j}} \ \text{ on } C_j^{\mathtt{i},h_j}, \tag{3.7}$$

for some $c = c(m, n, p, s, \alpha) > 0$ it holds for every measurable set E in $U \times U$

$$|\mathcal{D}_{\mathcal{K}}(u_j, E) - \mathcal{D}_{\mathcal{K}}(w_j, E)| \leq \frac{c}{N} \mathcal{K}(u_j, U), \tag{3.8}$$

and the sequence $(w_j)_{j \in \mathbf{N}}$ converge to u in $L^p(U, \mathbf{R}^m)$. If, in addition $u_j \in L^\infty(U, \mathbf{R}^m)$, then

$$\|w_j\|_{L^\infty(U, \mathbf{R}^m)} \leq \|u_j\|_{L^\infty(U, \mathbf{R}^m)}. \tag{3.9}$$

Eventually, if $\zeta_j := \sum_{\mathbf{i}\in\mathcal{I}_j}(u_j)_{C_j^{\mathbf{i},h_j}}\chi_{V_j^{\mathbf{i}}}$, *then* $(\zeta_j)_{j\in\mathbf{N}}$ *converges to* u *in* $L^p(U,\mathbf{R}^m)$.

Remark 3.2. *A similar statement can be proved in case sequences* $(u_j)_{j\in\mathbf{N}}$ *taking values into a convex set are considered. Indeed, the method to change boundary data performed in* [14, Lemmata 3.8, 3.9], *is realized through a convexity argument, which is then compatible with the constraint on the target codomain.*

This clarification is necessary to fit the analysis of the phase-field model for ferroelectric solids mentioned in the Introduction in our setting.

3.1. **The scaling-invariant case.** In the sequel we will consider parameters $s\in(0,1)$ and $p\in(1,+\infty)$ fixed and such that $sp=n$. Notice then that $p>n$.

To describe the asymptotic behaviour of the sequence $(\mathcal{K}_j)_{j\in\mathbf{N}}$ we shall use the auxiliary functions in (2.20) given by

$$\varphi_j^q(z) := \varphi_{\rho_j^q}^{\lambda_j^{2n}K(\lambda_j\cdot)}(z), \quad \text{where } \rho_j^q := q\lambda_j^{-1}r_j, \tag{3.10}$$

for any $q\in\mathbf{Q}^+$. In what follows we shall assume that the sequences $(\varphi_j^q)_{j\in\mathbf{N}}$ converge uniformly on compact subsets of $\mathbf{R}^m$ to functions φ^q for every $q\in\mathbf{Q}^+$. Note that since $\varphi^{q_2}\le\varphi^{q_1}$ if $q_1\le q_2$ there exists the limit

$$\varphi(z) := \lim_{q\to0^+}\varphi^q(z) = \lim_{q\to0^+}\lim_{j\to+\infty}\varphi_j^q(z) \tag{3.11}$$

and the convergence is uniform on compact sets of $\mathbf{R}^m$.

The convergence assumptions in (3.11) are not restrictive, they are always satisfied upon the extraction of subsequences thanks to (i) and (ii) in Proposition 2.8.

Theorem 3.3. *Let* $U\in\mathcal{A}(\mathbf{R}^n)$ *be bounded and connected with Lipschitz regular boundary.*

Given sets of points Λ_j *satisfying* (3.1)-(3.2), *and functions* φ_j^q *satisfying* (3.11), *suppose that the following limit exists*

$$\vartheta := \lim_j \#\mathcal{I}_j\,|\ln\lambda_j|^{1-p}\in[0,+\infty]. \tag{3.12}$$

Then, the sequence $(\mathcal{K}_j)_{j\in\mathbf{N}}$ Γ*-converges in the* $L^p(U,\mathbf{R}^m)$ *topology to the functional* $\mathscr{K}$: $L^p(U,\mathbf{R}^m)\to[0,+\infty]$ *defined by*

$$\mathscr{K}(u) = \mathcal{K}(u,U) + \vartheta\int_U\varphi(u(x))\beta(x)\,dx \tag{3.13}$$

if $u\in W^{s,p}(U,\mathbf{R}^m)$, $+\infty$ *otherwise in* $L^p(U,\mathbf{R}^m)$, *where* φ *is defined in* (3.11).

3.1.1. *Proof of the* Γ*-convergence.* In Propositions 3.4 below we show the lower bound inequality. By Lemma 3.1 we may consider only sequences assuming constant values around the obstacles, which are then approximately mean values of the target function close to the $T_j^{\mathbf{i}}$'s (cp. with (ζ_j) in Lemma 3.1). Then, a separation of scale argument shows that the capacitary contribution is concentrated along any neighborhood of the diagonal set Δ. Instead, the remaining part of the energy provides the long range interaction term since the kernel is no longer singular far from Δ.

Proposition 3.4. *For every $u_j \to u$ in $L^p(U, \mathbf{R}^m)$ we have*

$$\liminf_j \mathcal{K}_j(u_j) \geq \mathscr{K}(u).$$

Proof. Without loss of generality we shall assume $\vartheta > 0$ in what follows, the lower bound inequality being trivial otherwise.

Fix $\delta > 0$, $N \in \mathbf{N}$, and consider the sequence $(w_j)_{j\in\mathbf{N}}$ provided by Lemma 3.1. Then, $(w_j)_{j\in\mathbf{N}}$ converges to u in $L^p(U, \mathbf{R}^m)$ and for some $c = c(m, n, p, s, \alpha) > 0$ it holds

$$\left(1 + \frac{c}{N}\right) \liminf_j \mathcal{K}_j(u_j) \geq \liminf_j \mathcal{K}_j(w_j). \tag{3.14}$$

Upon extracting a subsequence, not relabeled for convenience, we may assume that the right hand side above is actually a limit, and in addition the index $h_j \in \{1, \ldots, N\}$ in Lemma 3.1 to be independent of j. Hence, from now on we shall denote it simply by h.

Note that for j sufficiently big $\cup_{\mathtt{i}\in\mathcal{I}_j}(V_j^{\mathtt{i}} \times V_j^{\mathtt{i}}) \subseteq \Delta_\delta$, and thus

$$\begin{aligned}\liminf_j \mathcal{K}_j(w_j) &\geq \liminf_j \left(\mathcal{D}_{\mathcal{K}}(w_j, U \times U \setminus \Delta_\delta) + \sum_{\mathtt{i}\in\mathcal{I}_j} \mathcal{K}(w_j, V_j^{\mathtt{i}})\right) \\ &\geq \mathcal{D}_{\mathcal{K}}(u, U \times U \setminus \Delta_\delta) + \liminf_j \sum_{\mathtt{i}\in\mathcal{I}_j} \mathcal{K}(w_j, V_j^{\mathtt{i}}),\end{aligned} \tag{3.15}$$

thanks to Fatou's lemma. We claim that for $q = N^{-(3h+2)}$

$$\liminf_j \sum_{\mathtt{i}\in\mathcal{I}_j} \mathcal{K}(w_j, V_j^{\mathtt{i}}) \geq (1 - \epsilon_N)\, \vartheta \int_U \varphi^q(u(x))\beta(x)dx, \tag{3.16}$$

with $\epsilon_N > 0$ infinitesimal as $N \to +\infty$. Given this for granted, by (3.14) inequality (3.15) rewrites as

$$\left(1 + \frac{c}{N}\right) \liminf_j \mathcal{K}_j(u_j) \geq \mathcal{D}_{\mathcal{K}}(u, U \times U \setminus \Delta_\delta) + (1 - \epsilon_N)\, \vartheta \int_U \varphi^q(u(x))\beta(x)dx.$$

The thesis then follows by passing to the limit first as $N \to +\infty$ and then as $\delta \to 0^+$ in the last inequality.

To conclude we are left with proving (3.16). We keep the notations of Lemma 3.1 and formula (3.10) with $q = N^{-(3h+2)}$; note that $B_{\lambda_j N \rho_j^q}(\mathrm{x}_j^{\mathtt{i}}) \subseteq V_j^{\mathtt{i}}$ for all $\mathtt{i} \in \mathcal{I}_j$, and that $\lambda_j \rho_j^q = q r_j$.

A change of variables and item (iii) in Proposition 2.8 give

$$\begin{aligned}&\mathcal{K}(w_j, V_j^{\mathtt{i}}) \\ &\quad\geq \inf\Big\{\mathcal{K}(w, B_{\lambda_j N \rho_j^q}(\mathrm{x}_j^{\mathtt{i}})) : w \in W^{s,p}(\mathbf{R}^n, \mathbf{R}^m),\, w = (u_j)_{C_j^{\mathtt{i},h}} \text{ on } C_j^{\mathtt{i},h},\, \tilde{w} \in E \text{ q.e. on } T_j^{\mathtt{i}}\Big\} \\ &\quad\geq \inf\Big\{\int_{B_{N\rho_j^q} \times B_{N\rho_j^q}} \lambda_j^{2n} K(\lambda_j(x - y))|w(x) - w(y)|^p\, dx\, dy : w \in W^{s,p}(\mathbf{R}^n, \mathbf{R}^m), \\ &\qquad\quad w = (u_j)_{C_j^{\mathtt{i},h}} \text{ on } \mathbf{R}^n \setminus \overline{B}_{\rho_j^q},\, \tilde{w} \in E \text{ q.e. on } T\Big\} = (\ln \rho_j^q)^{1-p}\, \varphi_{\rho_j^q, N\rho_j^q}^{\lambda_j^{2n} K(\lambda_j \cdot)}\left((u_j)_{C_j^{\mathtt{i},h}}\right)\end{aligned}$$

$$\stackrel{(2.25)}{\geq} \frac{(\ln \rho_j^q)^{1-p}}{1+c(N-1)^{-n}}\,\varphi_j^q\left((u_j)_{C_j^{\mathtt{i},h}}\right) = (1-\epsilon_N)(\ln \rho_j^q)^{1-p}\,\varphi_j^q\left((u_j)_{C_j^{\mathtt{i},h}}\right). \tag{3.17}$$

Hence, if $A \in \mathcal{A}(U)$ is such that $A \subset\subset U$, for j sufficiently big we infer

$$\sum_{\mathtt{i}\in\mathcal{I}_j} \mathcal{K}(w_j, V_j^{\mathtt{i}}) \geq (1-\epsilon_N)(\ln \rho_j^q)^{1-p}\,\#\mathcal{I}_j \int_A \varphi_j^q\,(\zeta_j(x))\,\Psi_j(x)\,dx,$$

where ζ_j is defined in Lemma 3.1 and $\Psi_j(x) := (\#\mathcal{I}_j)^{-1}\sum_{\mathtt{i}\in\mathcal{I}_j}(\mathcal{L}^n(V_j^{\mathtt{i}}))^{-1}\chi_{V_j^{\mathtt{i}}}(x)$. Note that $\|\Psi_j\|_{L^\infty(U)} \leq (\omega_n\, r_j^n\,\#\mathcal{I}_j)^{-1}$, so that $(\Psi_j)_{j\in\mathbf{N}}$ is equi-bounded in $L^\infty(U)$ by (2.4), (2.5) and (3.1). Furthermore, it is easy to show that $\Psi_j \to \beta$ weak* $L^\infty(U)$ (cp. with [14, Proposition 3.10]). Then, recalling that $\varphi_j^q \to \varphi^q$ uniformly on compact subsets of $\mathbf{R}^m$, and $\zeta_j \to u$ in $L^p(U,\mathbf{R}^m)$ (see Lemma 3.1), it follows

$$\liminf_j \sum_{\mathtt{i}\in\mathcal{I}_j} \mathcal{K}(w_j, V_j^{\mathtt{i}}) \geq (1-\epsilon_N)\,\vartheta \int_A \varphi^q(u(x))\,\beta(x)\,dx.$$

Eventually, estimate (3.16) follows at once by letting A increase to U. □

In the next proposition we prove that the lower bound established in Proposition 3.4 is sharp. Thanks to the insight provided by Proposition 3.4, we are able to construct a sequence for which there's no loss of energy asymptotically in all the estimates there.

Proposition 3.5. *For every $u \in L^p(U,\mathbf{R}^m)$ there exists a sequence $(u_j)_{j\in\mathbf{N}}$ such that $u_j \to u$ in $L^p(U,\mathbf{R}^m)$ and*

$$\limsup_j \mathcal{K}_j(u_j) \leq \mathscr{K}(u).$$

Proof. To begin with we note that it is not restrictive to take $\vartheta < +\infty$.

Furthermore, we may assume $u \in W^{1,\infty}(U,\mathbf{R}^m)$ by a standard density argument, the lower semicontinuity of Γ- $\limsup \mathcal{K}_j$, and the continuity of $\mathscr{K}(\cdot)$ with respect to strong convergence in $W^{s,p}$ as follows from item (ii) in Proposition 2.8.

In addition, we may also take $u \in W^{1,\infty}(U',\mathbf{R}^m)$ on an open and bounded smooth set U' such that $U \subset\subset U'$.

Fix $N \in \mathbf{N}$, and let $(w_j)_{j\in\mathbf{N}}$ be the sequence obtained from u by applying Lemma 3.1 on U'. To simplify the notation introduced there set

$$\varphi_j(z) := \varphi_j^{N^{-(3h_j+2)}}(z), \quad \phi_j(z) := \varphi_j^{N^{-5}}(z), \quad \varrho_j := \lambda_j^{-1} r_j\, N^{-(3h_j+2)}.$$

Note that since $h_j \in \{1,\ldots,N\}$, then $\varphi_j \leq \phi_j$. In addition, if $\mathtt{i} \in \mathcal{I}_j' := \mathcal{I}_j \cup \mathscr{I}_j(U)$ let

$$B_j^{\mathtt{i}} := B_{\lambda_j N \varrho_j}(\mathrm{x}_j^{\mathtt{i}}),\;\; u_j^{\mathtt{i}} := u_{C_j^{\mathtt{i},h_j}},\;\; U_j := U \setminus \left(\cup_{\mathtt{i}\in\mathcal{I}_j'} B_j^{\mathtt{i}}\right),$$

and notice then that $C_j^{\mathtt{i},h_j} \subset B_j^{\mathtt{i}}$.

For every $j \in \mathbf{N}$ consider $\xi_j^{\mathtt{i}} \in AD_{u_j^{\mathtt{i}}}(T, B_{\varrho_j})$ such that

$$\mathcal{K}(\xi_j^{\mathtt{i}}, B_{\varrho_j}) \leq |\ln \varrho_j|^{1-p}\left(\varphi_j\left(u_j^{\mathtt{i}}\right) + \frac{1}{N}\right).$$

Then, define

$$u_j(x) := \begin{cases} w_j(x) & U_j \\ \xi_j^{\mathtt{i}}\left(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})\right) & B_j^{\mathtt{i}} \cap U, \text{ if } \mathtt{i} \in \mathcal{I}'_j. \end{cases} \tag{3.18}$$

For the sake of notational simplicity we have not highlighted the dependence of the sequence $(u_j)_{j\in\mathbf{N}}$ on the parameter $N \in \mathbf{N}$. Clearly, $(u_j)_{j\in\mathbf{N}}$ converges strongly to u in $L^p(U, \mathbf{R}^m)$, and moreover it satisfies the obstacle condition by construction. The rest of the proof is devoted to show that $u_j \in W^{s,p}(U, \mathbf{R}^m)$ with

$$\limsup_j \mathcal{K}_j(u_j) \le \mathscr{K}(u) + \epsilon_\delta + \epsilon_N, \tag{3.19}$$

where $\epsilon_\delta \to 0^+$ as $\delta \to 0^+$ and $\epsilon_N \to 0^+$ as $N \to +\infty$. A recovery sequence as in the statement of Proposition 3.5 can be then constructed via a diagonal arument.

We first reduce ourselves to compute the energy of u_j on a neighborhood of the diagonal Δ. Indeed, let $\delta > 0$, Lebesgue dominated convergence and the stated convergence of $(u_j)_{j\in\mathbf{N}}$ to u in $L^p(U, \mathbf{R}^m)$ imply

$$\lim_j \mathcal{D}_{\mathcal{K}}(u_j, (U \times U) \setminus \Delta_\delta) = \mathcal{D}_{\mathcal{K}}(u, (U \times U) \setminus \Delta_\delta).$$

In addition, since $u_j = w_j$ on U_j by (3.8) in Lemma 3.1 we have for some positive constant $c = c(m, n, p, s, \alpha)$

$$\begin{aligned} &\limsup_j \mathcal{D}_{\mathcal{K}}(u_j, (U_j \times U_j) \cap \Delta_\delta) \\ &\qquad \le \limsup_j \mathcal{D}_{\mathcal{K}}(w_j, (U \times U) \cap \Delta_\delta) \le \left(1 + \frac{c}{N}\right) \mathcal{D}_{\mathcal{K}}(u, (U \times U) \cap \Delta_\delta) = \epsilon_\delta. \end{aligned} \tag{3.20}$$

The conclusion then follows provided we show that

$$\begin{aligned} &\limsup_j \big(\mathcal{D}_{\mathcal{K}}(u_j, (U \times (U \setminus \overline{U}_j)) \cap \Delta_\delta) + \mathcal{D}_{\mathcal{K}}(u_j, ((U \setminus \overline{U}_j) \times U_j) \cap \Delta_\delta)\big) \\ &\qquad\qquad \le \vartheta \int_U \varphi(u(x))\beta(x)\,dx + \epsilon_N + \epsilon_\delta. \end{aligned} \tag{3.21}$$

In order to prove this we introduce the following splitting of the left hand side above:

$$\begin{aligned} &\mathcal{D}_{\mathcal{K}}(u_j, (U \times (U \setminus \overline{U}_j)) \cap \Delta_\delta) + \mathcal{D}_{\mathcal{K}}(u_j, ((U \setminus \overline{U}_j) \times U_j) \cap \Delta_\delta) \\ &\qquad \le \sum_{\mathtt{i}\in\mathcal{I}'_j} \mathcal{K}(u_j, B_j^{\mathtt{i}}) + \sum_{\{(\mathtt{i},\mathtt{k})\in\mathcal{I}'_j\times\mathcal{I}'_j:\, 0<|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|<\delta\}} \mathcal{D}_{\mathcal{K}}(u_j, B_j^{\mathtt{i}} \times B_j^{\mathtt{k}}) \\ &\qquad\qquad + 2\sum_{\mathtt{i}\in\mathcal{I}'_j} \mathcal{D}_{\mathcal{K}}(u_j, (B_j^{\mathtt{i}} \times U_j) \cap \Delta_\delta) =: I_j^1 + I_j^2 + I_j^3. \end{aligned}$$

Next we estimate separately each term I_j^r, $r \in \{1, 2, 3\}$. All the constants c appearing in the rest of the proof will depend only on m, n, p, s, α and $\|u\|_{W^{1,\infty}(U',\mathbf{R}^m)}$. Hence, this dependence will no longer be indicated.

Step 1. Estimate of I_j^1:

$$\limsup_j I_j^1 \leq \vartheta \int_U \varphi(u(x))\beta(x)\,dx + \epsilon_N. \tag{3.22}$$

A straightforward change of variables, the very definition of u_j and (2.22) yield

$$\begin{aligned} I_j^1 &\leq |\ln \varrho_j|^{1-p} \sum_{\mathtt{i}\in\mathcal{I}_j'} \left(\varphi_j(u_j^{\mathtt{i}}) + \frac{1}{N}\right) \\ &\leq \#\mathcal{I}_j |\ln \varrho_j|^{1-p} \int_U \left(\phi_j(\zeta_j(x)) + \frac{1}{N}\right) \Psi_j(x)\,dx + \left(c + \frac{1}{N}\right) \#(\mathscr{I}_j(U))\,|\ln \varrho_j|^{1-p} \end{aligned} \tag{3.23}$$

where $\Psi_j(x) = (\#\mathcal{I}_j)^{-1} \sum_{\mathtt{i}\in\mathcal{I}_j} (\mathcal{L}^n(V_j^{\mathtt{i}}))^{-1} \chi_{V_j^{\mathtt{i}}}(x)$ and ζ_j is defined in Lemma 3.1 and it is related to the sequence with constant terms equal to u. Arguing as in Proposition 3.4, the convergences of $(\phi_j)_{j\in\mathbf{N}}$ to $\varphi^{N^{-5}}$ uniform on compact subsets of $\mathbf{R}^m$, and of $(\Psi_j)_{j\in\mathbf{N}}$ to β weak*-$L^\infty(U)$, give (3.22) by passing to the limit as $j \to +\infty$ on the right hand side above, once one notices that

$$\lim_j \#(\mathscr{I}_j(U))\,|\ln \varrho_j|^{1-p} = 0. \tag{3.24}$$

The last equality follows from (2.5), (3.1), (3.3) and (3.12), $\vartheta < +\infty$, and the fact that U is Lipschitz regular.

Step 2. Estimate of I_j^2:

$$\limsup_j I_j^2 \leq \epsilon_N + \epsilon_\delta. \tag{3.25}$$

The very definition of u_j in (3.18) implies that for any $(x,y) \in B_j^{\mathtt{i}} \times B_j^{\mathtt{k}}$, $\mathtt{i} \neq \mathtt{k}$ and $\mathtt{i},\mathtt{k} \in \mathcal{I}_j'$ it holds

$$u_j(x) - u_j(y) = \left(u_j^{\mathtt{i}} - u_j^{\mathtt{k}}\right) + \left(\xi_j^{\mathtt{i}}\left(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})\right) - u_j^{\mathtt{i}}\right) - \left(\xi_j^{\mathtt{k}}\left(\lambda_j^{-1}(y - \mathrm{x}_j^{\mathtt{k}})\right) - u_j^{\mathtt{k}}\right).$$

Hence, we can bound I_j^2 as follows

$$\begin{aligned} I_j^2 \leq\ & 3^{p-1}\alpha \sum_{\{(\mathtt{i},\mathtt{k})\in\mathcal{I}_j'\times\mathcal{I}_j' :\, 0<|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|<\delta\}} \int_{B_j^{\mathtt{i}}\times B_j^{\mathtt{k}}} \frac{|u_j^{\mathtt{i}} - u_j^{\mathtt{k}}|^p}{|x-y|^{2n}} dxdy \\ &+ 3^p\alpha \sum_{\{(\mathtt{i},\mathtt{k})\in\mathcal{I}_j'\times\mathcal{I}_j' :\, 0<|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|<\delta\}} \int_{B_j^{\mathtt{i}}\times B_j^{\mathtt{k}}} \frac{|\xi_j^{\mathtt{i}}(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})) - u_j^{\mathtt{i}}|^p}{|x-y|^{2n}} dxdy =: I_j^{2,1} + I_j^{2,2}. \end{aligned}$$

To estimate the right hand side above we note that $|\mathrm{x}_j^{\mathtt{i}} - \mathrm{x}_j^{\mathtt{k}}|/2 \leq |x-y| \leq 2|\mathrm{x}_j^{\mathtt{i}} - \mathrm{x}_j^{\mathtt{k}}|$ for any $(x,y) \in B_j^{\mathtt{i}} \times B_j^{\mathtt{k}}$, $\mathtt{i},\mathtt{k} \in \mathcal{I}_j'$ with $\mathtt{i} \neq \mathtt{k}$, and that $|u_j^{\mathtt{i}} - u_j^{\mathtt{k}}| \leq 2\,\|u\|_{W^{1,\infty}(U',\mathbf{R}^m)} |\mathrm{x}_j^{\mathtt{i}} - \mathrm{x}_j^{\mathtt{k}}|$. Hence, we infer

$$\int_{B_j^{\mathtt{i}}\times B_j^{\mathtt{k}}} \frac{|u_j^{\mathtt{i}} - u_j^{\mathtt{k}}|^p}{|x-y|^{2n}} dxdy \leq c\, \frac{r_j^{2n}}{|\mathrm{x}_j^{\mathtt{i}} - \mathrm{x}_j^{\mathtt{k}}|^{2n-p}}. \tag{3.26}$$

In addition, for every fixed $\mathtt{i} \in \mathcal{I}_j'$ we have

$$\{\mathtt{k} \in \mathcal{I}_j' :\, 0 < |\mathrm{x}_j^{\mathtt{i}} - \mathrm{x}_j^{\mathtt{k}}|_\infty < \delta\} \subseteq \cup_{h=2}^{\lfloor \delta/r_j \rfloor} \{\mathtt{k} \in \mathcal{I}_j' :\, h\,r_j \leq |\mathrm{x}_j^{\mathtt{i}} - \mathrm{x}_j^{\mathtt{k}}|_\infty < (h+1)r_j\},$$

where $\lfloor t \rfloor$ denotes the integer part of t. The latter inclusion together with (2.4), (2.6) and (3.26) entails

$$I_j^{2,1} \le c \sum_{\mathtt{i}\in\mathcal{I}_j'} \sum_{h=2}^{\lfloor \delta/r_j \rfloor} \sum_{\{\mathtt{k}\in\mathcal{I}_j':\, h\, r_j \le |\mathtt{x}_j^{\mathtt{i}} - \mathtt{x}_j^{\mathtt{k}}|_\infty < (h+1) r_j\}} \frac{r_j^p}{h^{2n-p}} \overset{(2.4),\,(2.6)}{\le} c \sum_{h=2}^{\lfloor \delta/r_j \rfloor} \frac{r_j^{p-n}}{h^{1+n-p}} \le c\, \delta^{p-n}, \tag{3.27}$$

since $\sum_{h=2}^{M} h^{-(1+\gamma)} \le (M^{-\gamma})/(-\gamma)$, for any $\gamma < 0$ and $M \in \mathbf{N}$ (recall that $p > n$ if $sp = n$).

To bound $I_j^{2,2}$ we argue as follows: for every $\mathtt{i} \in \mathcal{I}_j'$ we have

$$\sum_{\{\mathtt{k}\in\mathcal{I}_j':\, \mathtt{k}\neq\mathtt{i}\}} \int_{B_j^{\mathtt{k}}} \frac{1}{|x-y|^{2n}} dy \le c \sum_{\{\mathtt{k}\in\mathcal{I}_j':\, \mathtt{k}\neq\mathtt{i}\}} \frac{r_j^n}{|\mathtt{x}_j^{\mathtt{i}} - \mathtt{x}_j^{\mathtt{k}}|^{2n}} \overset{(2.6)}{\le} \frac{c}{r_j^n} \sum_{h\ge1} \frac{1}{h^{1+n}}.$$

Therefore, a change of variables, Poincaré-Wirtinger inequality, and the very definition of ϱ_j yield

$$\begin{aligned} I_j^{2,2} &\le c \frac{\lambda_j^n}{r_j^n} \sum_{\mathtt{i}\in\mathcal{I}_j'} \int_{B_{\varrho_j}} |\xi_j^{\mathtt{i}}(x) - u_j^{\mathtt{i}}|^p dx \le \frac{c}{N^{n(3h_j+2)}} \sum_{\mathtt{i}\in\mathcal{I}_j'} |\xi_j^{\mathtt{i}}|^p_{W^{s,p}(B_{\varrho_j},\mathbf{R}^m)} \\ &\overset{(2.14)}{\le} \frac{c}{N^{5n}} |\ln \varrho_j|^{1-p} \sum_{\mathtt{i}\in\mathcal{I}_j'} \left(\varphi_j(u_j^{\mathtt{i}}) + \frac{1}{N} \right) \\ &\le \frac{c}{N^{5n}} \#\mathcal{I}_j' \, |\ln \varrho_j|^{1-p} \left(\int_U \phi_j(\zeta_j(x)) \Psi_j(x)\, dx + c \right) \overset{(3.23),\,(3.24)}{=} \epsilon_N. \end{aligned} \tag{3.28}$$

Clearly, (3.27) and (3.28) imply (3.25).

Step 3. Estimate of I_j^3:

$$\limsup_j I_j^3 \le \epsilon_\delta + \epsilon_N. \tag{3.29}$$

Being $u_j = w_j$ on U_j, we find

$$\begin{aligned} I_j^3 &\le c \sum_{\mathtt{i}\in\mathcal{I}_j'} \int_{(B_j^{\mathtt{i}}\times U_j)\cap\Delta_\delta} K(x-y) \left| \xi_j^{\mathtt{i}} \left(\lambda_j^{-1}(x - \mathtt{x}_j^{\mathtt{i}}) \right) - u_j^{\mathtt{i}} \right|^p dxdy \\ &\quad + c \sum_{\mathtt{i}\in\mathcal{I}_j'} \int_{(B_j^{\mathtt{i}}\times U_j)\cap\Delta_\delta} K(x-y) |u_j^{\mathtt{i}} - w_j(x)|^p \, dxdy \\ &\quad + c \sum_{\mathtt{i}\in\mathcal{I}_j'} \int_{(B_j^{\mathtt{i}}\times U_j)\cap\Delta_\delta} K(x-y) |w_j(x) - w_j(y)|^p dxdy =: I_j^{3,1} + I_j^{3,2} + I_j^{3,3}. \end{aligned}$$

Since $\xi_j^{\mathtt{i}} = u_j^{\mathtt{i}}$ out of $B_{\varrho_j}(\mathtt{x}_j^{\mathtt{i}})$, by a change of variables the first integral above can be bounded by

$$I_j^{3,1} \le c \sum_{\mathtt{i}\in\mathcal{I}_j'} \int_{B_{N\varrho_j}} \int_{B^c_{N\varrho_j}} \lambda_j^{2n} K(\lambda_j(x-y)) |\xi_j^{\mathtt{i}}(x) - \xi_j^{\mathtt{i}}(y)|^p \, dxdy$$

$$\leq c\,|\ln \varrho_j|^{1-p}\left(\sum_{\mathtt{i}\in\mathcal{I}'_j}\left|\varphi_j(u_j^{\mathtt{i}})-\varphi^{\lambda_j^{2n}K(\lambda_j\cdot)}_{\varrho_j,N\varrho_j}(u_j^{\mathtt{i}})\right|+\frac{\#\mathcal{I}'_j}{N}\right)$$

$$\stackrel{(2.25)}{\leq} c\,|\ln \varrho_j|^{1-p}\left(\frac{1}{(N-1)^n}\sum_{\mathtt{i}\in\mathcal{I}'_j}\varphi_j(u_j^{\mathtt{i}})+\frac{\#\mathcal{I}'_j}{N}\right)$$

$$\leq c\,\#\mathcal{I}'_j\,|\ln \varrho_j|^{1-p}\left(\frac{1}{(N-1)^n}\int_U \phi_j(\zeta_j(x))\Psi_j(x)\,dx+\frac{1}{N}\right)\stackrel{(3.23),(3.24)}{=}\epsilon_N. \quad (3.30)$$

To deal with the term $I_j^{3,2}$, we use the growth conditions on K in (2.14), integrate out y thanks to (2.10), and observe that $w_j|_{C_j^{\mathtt{i},h_j}}=u_j^{\mathtt{i}}$ to apply the scaled Poincaré-Wirtinger inequality in (2.9) and infer:

$$I_j^{3,2}\stackrel{(2.14),(2.10)}{\leq} c\sum_{\mathtt{i}\in\mathcal{I}'_j}\int_{B_j^{\mathtt{i}}}\frac{|w_j(x)-u_j^{\mathtt{i}}|^p}{\mathrm{dist}^n(x,\partial B_j^{\mathtt{i}})}\,dx = c\sum_{\mathtt{i}\in\mathcal{I}'_j}\int_{B_{\lambda_j\varrho_j}(\mathtt{x}_j^{\mathtt{i}})}\frac{|w_j(x)-u_j^{\mathtt{i}}|^p}{\mathrm{dist}^n(x,\partial B_j^{\mathtt{i}})}\,dx$$

$$\leq c\left(\frac{N^{3h_j+1}}{r_j}\right)^n\sum_{\mathtt{i}\in\mathcal{I}'_j}\int_{B_{\lambda_j\varrho_j}(\mathtt{x}_j^{\mathtt{i}})}|w_j(x)-u_j^{\mathtt{i}}|^p dx\stackrel{(2.9)}{\leq}\frac{c}{N^n}\sum_{\mathtt{i}\in\mathcal{I}'_j}|w_j|^p_{W^{s,p}(B_j^{\mathtt{i}},\mathbf{R}^m)}$$

$$\stackrel{(2.14)}{\leq}\frac{c}{N^n}\mathcal{D}_{\mathcal{K}}(w_j,(U'\times U')\cap\Delta_\delta)=\epsilon_N\,\epsilon_\delta. \quad (3.31)$$

Finally, for what $I_j^{3,3}$ is concerned we have

$$I_j^{3,3}\leq c\,\mathcal{D}_{\mathcal{K}}(w_j,(U\times U)\cap\Delta_\delta)\stackrel{(3.20)}{=}\epsilon_\delta. \quad (3.32)$$

By collecting (3.30)-(3.32) we infer (3.29).

Step 4: Conclusion. The conclusion follows at once from Step 1 - Step 3. □

3.2. **The subcritical case.** We now establish a result analogous to Theorem 3.3 in case the singular kernel K satisfies (2.14) with $sp\in[1,n)$. We limit ourselves to state the result and comment on it since the scalar setting for homogeneous kernels has been investigated in details in [14] and few changes are needed to deal with the vector-valued one considered in this paper once Propositions 2.8 and 2.9 are at disposal.

In doing that we slightly extend the conclusions in [14, Theorem 3] by including the case $sp=1$ that was originally excluded in that statement, the reason for that being the use of Hardy inequality in some estimates. Indeed, such an inequality does not hold true if $sp=1$ (see [22, Theorem 4.3.2/1, Remark 2 pp. 319-320]).

Despite this, in view of the argument leading to (3.31) in Theorem 3.3, an inspection of the proof of [14, Proposition 3.11] shows that the use of Hardy inequality is actually pointless in [14, Theorem 3] (cp. with formulas (3.42) and (3.46) there). Hence, in what follows, we state an asymptotic result for the full subcritical range of values of p and s.

To describe the limit behaviour of $(\mathcal{K}_j)_{j\in\mathbf{N}}$ we shall consider auxiliary functions as in (2.29) given by

$$\varphi_j^q(z) := \varphi_{\rho_j^q}^{\lambda_j^{n+sp}K(\lambda_j\cdot)}(z), \qquad \text{where } \rho_j^q := q\lambda_j^{-1}r_j, \tag{3.33}$$

for any $q \in \mathbf{Q}^+$. In addition, we shall assume that the sequences $(\varphi_j^q)_{j\in\mathbf{N}}$ converge uniformly on compact subsets of $\mathbf{R}^m$ to functions φ^q for every $q \in \mathbf{Q}^+$. Note that since $\varphi^{q_2} \leq \varphi^{q_1}$ if $q_1 \leq q_2$ there exists the limit

$$\varphi(z) := \lim_{q\to 0^+} \varphi^q(z) = \lim_{q\to 0^+} \lim_{j\to+\infty} \varphi_j^q(z) \tag{3.34}$$

and the convergence is uniform on compact sets of $\mathbf{R}^m$.

Let us point out that in case the kernel K is $(-n-sp)$-homogeneous φ_j^q reduces to $\varphi_{\rho_j^q}^K$ so that the convergence in (3.34) holds true as noticed in (2.30). Otherwise, it is guaranteed only up to subsequences.

Theorem 3.6. *Let $U \in \mathcal{A}(\mathbf{R}^n)$ be bounded and connected with Lipschitz regular boundary.*

Given sets of points Λ_j satisfiying (3.1)-(3.2), and functions φ_j^q as in (3.33), suppose that (3.34) holds and that the following limit exists

$$\vartheta := \lim_j \#\mathcal{I}_j\, \lambda_j^{n-sp} \in [0, +\infty]. \tag{3.35}$$

Then, the sequence $(\mathcal{K}_j)_{j\in\mathbf{N}}$ Γ-converges in the $L^p(U,\mathbf{R}^m)$ topology to the functional $\mathscr{K} : L^p(U,\mathbf{R}^m) \to [0,+\infty]$ defined by

$$\mathscr{K}(u) = \mathcal{K}(u,U) + \vartheta \int_U \varphi(u(x))\beta(x)\,dx \tag{3.36}$$

if $u \in W^{s,p}(U,\mathbf{R}^m)$, $+\infty$ otherwise in $L^p(U,\mathbf{R}^m)$.

4. Generalizations

Several generalizations are possible following the path in [14]. In this section we limit ourselves to consider energies defined by non-translation invariant kernels in the scaling invariant case $sp = n$. Clearly, analogous results can be obtained for $sp \in [1,n)$. Given a $\mathcal{L}^n \times \mathcal{L}^n$-measurable function $K : \mathbf{R}^n \times \mathbf{R}^n \setminus \Delta \to [0,+\infty)$, the growth conditions in (2.14) rewrite as

$$\alpha^{-1}|x-y|^{-2n} \leq K(x,y) \leq \alpha|x-y|^{-2n} \tag{4.1}$$

for some $\alpha \geq 1$, and for all $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n \setminus \Delta$.

We shall also assume that K satisfies the continuity condition

$$|K(x,y) - K(x+p,y+p)| \leq \omega(|p|)\, K(x,y) \tag{4.2}$$

for all points $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n \setminus \Delta$, vectors $p \in \mathbf{R}^n \setminus \{0\}$, and for some modulus of continuity ω, i.e. $\omega : [0,+\infty) \to [0,+\infty]$ with $\omega(t) \to 0$ as $t \to 0^+$.

For instance, it is straightforward to check that estimate (4.2) holds true in case the function $\mathbf{R}^n \times \mathbf{R}^n \setminus \Delta \ni (x,y) \to |x-y|^{2n}K(x,y)$ is 0-homogeneous, i.e. K is $(-2n)$-homogeneous, and belongs to $C^0(\mathbf{S}^{2n-1})$. Consequently, a functional $\mathcal{K}: L^p(A,\mathbf{R}^m)\times\mathcal{A}(\mathbf{R}^n)\to[0,+\infty]$ can be defined as in (2.15).

Assuming (4.1) and (4.2), let $\varphi_{\rho,R}:\mathbf{R}^m\to[0,+\infty)$ be the function defined in (2.20) and consider the following capacitary problems

$$\varphi_{\rho,R}(x,z):=\varphi_{\rho,R}^{K(x+\cdot,x+\cdot),T}(z). \tag{4.3}$$

In turn, the latter identity, Proposition 2.8, (4.1) and (4.2) yield that $\varphi_{\rho,R}$ belongs to $C^0(\mathbf{R}^n\times\mathbf{R}^m)$ with

$$\begin{aligned}|\varphi_{\rho,R}(x_1,z_1)&-\varphi_{\rho,R}(x_2,z_2)|\\ &\leq c\,(1+|z_1|^p+|z_2|^p)\,\omega(|x_1-x_2|)+c\,(1+|z_1|^{p-1}+|z_2|^{p-1})|z_1-z_2|,\end{aligned} \tag{4.4}$$

for some constant $c=c(m,n,p,s,\alpha)>0$ and for all $(x_i,z_i)\in\mathbf{R}^n\times\mathbf{R}^m$, $i\in\{1,2\}$. Actually, estimate (4.4) is uniform for families of kernels satisfying (4.1) and (4.2). Hence, up to subsequences, if

$$\varphi_j^q(x,z):=\varphi_{\rho_j^q}^{\lambda_j^{2n}K(x+\lambda_j\cdot,\,x+\lambda_j\cdot),T}(z),\quad\text{where } \rho_j^q:=q\lambda_j^{-1}r_j, \tag{4.5}$$

the sequences $(\varphi_j^q)_{j\in\mathbf{N}}$ converge uniformly on compact subsets of $\mathbf{R}^n\times\mathbf{R}^m$ to functions φ^q for any $q\in\mathbf{Q}^+$. Then, since $\varphi^{q_2}\leq\varphi^{q_1}$ if $q_1\leq q_2$ there exists the limit

$$\varphi(x,z):=\lim_{q\to 0^+}\varphi^q(x,z)=\lim_{q\to 0^+}\lim_{j\to+\infty}\varphi_j^q(x,z). \tag{4.6}$$

Notice that $\varphi\in C^0(\mathbf{R}^n\times\mathbf{R}^m)$ since the convergence is uniform on compact subsets.

In view of this convergence and of (4.4), it is then easy to adapt the proof and infer a statement similar to Theorem 3.3, the Γ-limit being given as in (3.36) with the non-local energy defined by the kernel K above, and with the function φ in (4.6) substituting that in (3.11) in the density of the obstacle penalization term.

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