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# FRACTURE MECHANICS IN PERFORATED DOMAINS: A VARIATIONAL MODEL FOR BRITTLE POROUS MEDIA

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This paper deals with fracture mechanics in periodically perforated domains. Our aim is to provide a variational model for *brittle porous media* in the case of anti-planar elasticity.

Given the perforated domain $\Omega_\varepsilon \subset \mathbb{R}^N$ ($\varepsilon$ being an internal scale representing the size of the periodically distributed perforations), we will consider a total energy of the type

$$\mathcal{F}_\varepsilon(u) := \int_{\Omega_\varepsilon} |\nabla u(x)|^2\, dx + \mathcal{H}^{N-1}(S_u).$$

Here $u$ is in $SBV(\Omega_\varepsilon)$ (the space of *special functions of bounded variation*), $S_u$ is the set of discontinuities of $u$, which is identified with a macroscopic crack in the porous medium $\Omega_\varepsilon$, and $\mathcal{H}^{N-1}(S_u)$ stands for the $(N-1)$-Hausdorff measure of the crack $S_u$.

We study the asymptotic behavior of the functionals $\mathcal{F}_\varepsilon$ in terms of $\Gamma$-convergence as $\varepsilon \to 0$. As a first (nontrivial) step we show that the domain of any limit functional is $SBV(\Omega)$ despite the degeneracies introduced by the perforations. Then we provide explicit formula for the bulk and surface energy densities of the $\Gamma$-limit, representing in our model the effective elastic and brittle properties of the porous medium, respectively.

*Keywords*: Brittle fracture; homogenization; perforated domains; composite and mixture properties; cavities; variational methods; special functions of bounded variation.

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## 1. Introduction

There is a huge mathematical literature concerning the dependence of solutions of partial differential equations, as well as minimum problems, on their domain of definition. In particular, the asymptotic behavior for minimizers $u_n$ defined in varying domains $\Omega_n$ with homogenizing small holes, usually referred to as *perforated domains* (we refer to the books Refs. 11, 13, 21 and 37) has been extensively studied. Typically, the integral functionals to be minimized depend on $u$ and on its gradient, and on the perforations it is imposed either a Dirichlet type boundary condition (see Refs. 9, 18, 22, 27, 40, 43, 44 and references therein) or a Neumann type boundary condition (see Refs. 1–3, 15, 16, 20, 24, 42 and references therein). Under standard growth assumptions this kind of minimization problems can be settled in the framework of Sobolev spaces.

The aim of this paper is to study the problem of periodic homogenization of small holes in the framework of fracture mechanics, i.e. for total energies involving not only a bulk term, but also a surface term, obtaining in the homogenized limit a variational model for *brittle porous media* (see Refs. 19 and 39 for related topics). The homogenizing holes represent traction free micro-cracks in the body, so that our analysis will focus on natural Neumann boundary conditions on the perforations. The case of Dirichlet conditions has been considered in Ref. 32 in connection with the study of the asymptotic limit of obstacle problems for Mumford–Shah type functionals (see Ref. 41) in perforated domains.

From a mathematical point of view, the minimization of total energies involving both bulk and surface terms can be settled within the theory of $SBV$-deformations. The functional space $SBV$ of *special functions of bounded variation* has been introduced by De Giorgi and Ambrosio[29] to deal with free discontinuity problems arising in image segmentation (see Ref. 41), and was proposed by Ambrosio and Braides[4] as a suitable framework for fracture mechanics.

Variational models to describe equilibria of brittle hyperelastic bodies have been largely developed in recent years. Inspired by Griffith's theory of crack propagation, these models in fracture mechanics are based on the assumption that the cracked deformed configuration of the body is reached through a minimization process driven by the competition of surface and bulk energies. The surface energy represents the energy dissipated to break atomic bonds, and hence spent to enlarge the crack, while the bulk energy represents the elastic energy stored in the body, and partially released during the crack growth.

Let us consider for a while a non-porous brittle body (i.e. without perforations). We will consider only the case of *generalized anti-planar elasticity*, in which $\Omega \subset \mathbb{R}^N$ represents a section of a cylindrical body (in the relevant physical case we have $N = 2$), the displacement function $u \in SBV(\Omega)$ is assumed to be scalar, and the crack is implicitly identified with the set $S_u$ of discontinuities of $u$. Concerning the total energy, we will consider for simplicity the following model case:

$$E(u) := \int_\Omega |\nabla u(x)|^2 \, dx + \mathcal{H}^{N-1}(S_u). \tag{1.1}$$

Here $\mathcal{H}^{N-1}$ stands for the $(N-1)$-Hausdorff measure, so that if $N = 2$ and $S_u$ is a smooth curve, $\mathcal{H}^{N-1}(S_u)$ is just the usual length of the crack. More general energies could be considered, as for instance surface energies eventually depending also on the normal $\nu$ of the crack, due to anisotropy of the body, while the dependence on the opening of the crack for cohesive models would require a specific analysis. Critical points (and in particular minimum points) of the total energy (1.1) represent stable configurations of the cracked domain according to Griffith's theory.

To study the effect that the perforations have on the variational problem, let us begin by discussing the case of a single crack $K$ present in the body. Assume that $K$ is a closed subset of $\Omega$ and that in $\Omega\setminus K$ the elastic behavior of the body is unperturbed, so that the density of the elastic energy remains the same in $\Omega\setminus K$, while the surface energy will be dissipated only to enlarge the pre-existing crack $K$. We have that the total energy is now given by

$$E(K, u) := \int_{\Omega\setminus K} |\nabla u(x)|^2\, dx + \mathcal{H}^{N-1}(S_u\setminus K). \tag{1.2}$$

This kind of energy plays an essential role in the variational approaches to quasi-static crack growth as proposed by Francfort and Marigo[35] and developed in many subsequent papers in the framework of *SBV*-functions (we refer to Refs. 34, 26, 36 and the references therein).

We model the presence of homogenizing cracks considering a sequence $K_\varepsilon := \varepsilon(K + \mathbb{Z}^N)$, with $\varepsilon \to 0$ and $K$ closed, and study the asymptotic behavior in terms of $\Gamma$-convergence of the corresponding energy functionals $\mathcal{F}_\varepsilon(u) := E(K_\varepsilon, u)$.[a] The bulk and surface energy densities of the $\Gamma$-limit $\mathcal{F}_{\text{hom}}$ will represent the *effective* elastic and brittle properties of the porous brittle medium. Notice that we do not prescribe the shape of the holes, assuming only that they are closed sets, each component well contained in its periodic cell. They may have positive Lebesgue measure, being for instance small holes, as well as finite length, being one-dimensional cracks. In any case, we will refer to them as *micro-cracks*.

A natural question is whether a quasi-static crack growth corresponding to the energy $\mathcal{F}_\varepsilon$ (i.e. in the perforated domain) converges (as $\varepsilon \to 0$ and with respect to a suitable topology) to a quasi-static growth in the porous brittle material represented by $\mathcal{F}_{\text{hom}}$. We believe that this happens in the framework of the variational approach to crack evolution proposed by Francfort and Marigo.[35] The reason is that the key condition that the displacement $u$ has to satisfy in their definition of crack growth is a suitable minimality property, called *unilateral minimality*, that enjoys good stability properties with respect to $\Gamma$-convergence (see Ref. 36). In our opinion this is the main justification to identify the homogenized brittle porous material with $\mathcal{F}_{\text{hom}}$. On the other hand, the asymptotic behavior of stable configurations in the sense of Griffith

[a] In our anti-planar setting both the crack $S_u$ and the perforations $K_\varepsilon$ are defined in a horizontal section $\Omega$ of the cylindrical body and they are assumed to be invariant with respect to the vertical direction of the body. This assumption has to be understood as a mere mathematical simplification of the problem.

theory that are not unilateral minimizers of the corresponding total energy (being for instance local minima or just critical points) is not catched by our analysis based on $\Gamma$-convergence, and requires a specific challenging analysis. Summarizing, we expect that in a material with fine microstructure, the quasi-static crack growth by Francfort and Marigo is governed by the macroscopic properties of the homogenized material, since it is based on global minimization and it admits *brutal* growth of the crack. Therefore, we consider our total energy $\mathcal{F}_{\rm hom}$ to be the natural candidate to represent a porous brittle material in this approach to crack evolution.

A similar mathematical problem has been considered in Ref. 36 in connection to stability properties of equilibria in fracture mechanics for sequences of $(N-1)$-rectifiable sets satisfying $\mathcal{H}^{N-1}(K_n) \leq c$. In that case they prove that the $\Gamma$-limit of the functionals $E(K_n, \cdot)$ still has the form (1.2), where $K$ is a suitable $(N-1)$-rectifiable set which represents the limit fracture, in a suitable sense, corresponding to the sequence $K_n$. In that model the fractures $K_n$ represent the cracks created in the body during a load process. Therefore the assumption $\mathcal{H}^{N-1}(K_n) \leq c$ is very natural in their setting, meaning that $K_n$ have finite energy according to Griffith's theory. Our situation of periodically distributed cracks $K_\varepsilon = \varepsilon(K + \mathbb{Z}^N)$ is very different, having $K_\varepsilon$ by definition diverging area as $\varepsilon \to 0$. Indeed, in our case the homogenizing micro-cracks will affect both bulk and surface energies in the $\Gamma$-convergence process.

Our main result is twofold: in the first part of the paper we deal with the natural lack of coercivity of the problem, establishing a compactness property for sequences with equi-bounded energies, under some natural assumptions ensuring that $K_\varepsilon$ does not disconnect the body. Furthermore, we prove that the energy functionals $\mathcal{F}_\varepsilon$ $\Gamma$-converge (with respect to a suitable topology) to the functional $\mathcal{F}_{\rm hom}$ given by

$$\mathcal{F}_{\rm hom}(u) := \int_\Omega f_{\rm hom}(\nabla u)\, dx + \int_{S_u} g_{\rm hom}(\nu_u)\, d\mathcal{H}^{N-1},$$

where $f_{\rm hom}$ and $g_{\rm hom}$ are defined through cell type formulas below, and represent the material properties of the porous medium.

Concerning $f_{\rm hom}$ we have for every $\xi \in \mathbb{R}^N$

$$f_{\rm hom}(\xi) := \inf\left\{\int_{Q\setminus K} |\nabla w + \xi|^2 dx :\, w \in W^{1,2}_\sharp(Q\setminus K)\right\}, \tag{1.3}$$

where $Q$ is the unit cube and $W^{1,2}_\sharp(Q\setminus K)$ denotes the class of $Q$-periodic functions in $W^{1,2}(Q\setminus K)$, i.e. Sobolev functions on $Q\setminus K$ whose traces on opposite faces of $Q$ coincide.

This homogenization formula is well known in the context of periodic homogenization in Sobolev spaces and represents the effective energy density in perforated domains subject to Neumann conditions (see for instance Ref. 1). It turns out that the same formula represents the effective bulk energy density also for brittle materials. In this respect we conclude that there is no interaction between macroscopic cracks and micro-cracks for the elastic properties of a brittle porous medium.

Passing to the density of the homogenized surface energy $g_{\text{hom}}$, for all $(a, b, \nu) \in \mathbb{R} \times \mathbb{R} \times \mathbf{S}^{N-1}$ let $u_{a,b,\nu} : \mathbb{R}^N \to \mathbb{R}$ be given by

$$u_{a,b,\nu}(y) := \begin{cases} b & \text{if } y \cdot \nu \geq 0, \\ a & \text{if } y \cdot \nu < 0. \end{cases} \tag{1.4}$$

The surface energy density $g_{\text{hom}} : \mathbf{S}^{N-1} \to [0, +\infty)$ is given by

$$\begin{aligned} g_{\text{hom}}(\nu) := \lim_{\varepsilon \to 0^+} \{ & \mathcal{H}^{N-1}(S_w \backslash K_\varepsilon) : w \in P(Q^\nu \backslash K_\varepsilon), \\ & w = u_{0,1,\nu} \text{ on a neighborhood of } \partial Q^\nu \}. \end{aligned} \tag{1.5}$$

Above $Q^\nu$ is any unit cube centered at the origin with one face orthogonal to $\nu$, and $P(Q^\nu \backslash K_\varepsilon)$ is the family of characteristic functions (see (2.2)). We show the existence of the limit in (1.5) in Lemma 5.1. Note that formula (1.5) involves only locally constant functions. We deduce that the toughness of the porous medium does not depend on the elastic properties of the corresponding non-porous material.

Let us finally discuss our result under a slightly different perspective. The porous brittle material in our model has been obtained by homogenizing a constituent material with holes. The problem can be settled in the framework of homogenization of composite materials, in which one of the constituent materials is the void. From a mathematical point of view, we deal with energy densities fast oscillating with respect to $x$, taking values in 0 and 1, and the presence of the coefficient 0 (that is of the void) brings high degeneracy into the problem. Homogenization problems in $SBV$ spaces have been largely studied in the last years, as for instance in Refs. 5, 6 and 14. Our homogenization formulas extend those given in the mentioned papers to our context, in which the homogenizing coefficients do not satisfy standard ellipticity conditions. The lack of ellipticity produces many specific difficulties in our analysis, the most remarkable being in the proof of suitable compactness properties for minimizers. In this respect, our approach has been to provide a localized Poincaré type inequality in $SBV$ in dimension two, which allows us to truncate the minimizers at suitable levels around each perforation (see Sec. 4.2 for a comparison with De Giorgi's Poincaré inequality in $SBV$). In view of this, we can extend the minimizers by means of standard cutoff techniques inside the perforations, filling the holes with good control of the total energy. Finally we are in a position to use Ambrosio's compactness results for sequences in $SBV$ with bounded energy. The general $N$-dimensional case is then recovered by a slicing argument, using the results obtained in dimension two. A different approach to the problem, based on excision techniques introduced by De Giorgi, Carriero and Leaci[30] (see also Ref. 8, Chap. 7), has been developed in the recent paper.[17] Their approach provides, as for the Sobolev setting, a family of uniformly bounded extension operators to fill the holes.

Finally, we mention that a related problem concerns fiber reinforced brittle materials, in which, instead of homogenizing holes, the composite material is made by homogenizing reinforced (e.g. unbreakable) fibers. In this direction we refer to the recent papers Refs. 10, 28 and 45.

The paper is organized as follows. In Sec. 2, we provide some preliminary results used in the rest of the paper. In Sec. 3, we set the mathematical framework to study the asymptotic behavior of energy functionals $\mathcal{F}_\varepsilon$. In Sec. 4, we provide a Poincaré type inequality in $SBV$ in dimension two, and we prove suitable compactness properties for sequences with bounded energy. In Sec. 5, we prove the Γ-convergence result of the functionals $\mathcal{F}_\varepsilon$, and in Sec. 6, we give the analogous Γ-convergence result for energy functionals taking into account Dirichlet boundary conditions on $\partial\Omega$. Finally, in Sec. 7, we will discuss the validity of our results for more general energy functionals.

## 2. Preliminaries

In this section we will fix some notation and introduce some notions of geometric measure theory we will need in the sequel.

For every $r, s$ with $0 < r < s$ we set

$$Q_r := \{x \in \mathbb{R}^N : \|x\|_\infty < r/2\}, \quad Q_{r,s} := \{x \in \mathbb{R}^N : r/2 < \|x\|_\infty < s/2\}, \tag{2.1}$$

and, for simplicity we denote the unitary cube $Q_1$ by $Q$.

Throughout the paper $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $\mathcal{A}(\Omega)$ denotes the family of all open subsets of $\Omega$. Let $A \in \mathcal{A}(\Omega)$. We denote by $SBV(A)$ the space of *special functions of bounded variation*, and by $SBV^2(A)$ the subspace

$$SBV^2(A) := \{u \in SBV(A) \,:\, \nabla u \in L^2(A, \mathbb{R}^N), \mathcal{H}^{N-1}(S_u) < +\infty\}.$$

Here $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure, and $S_u$ denotes the jump set of $u$. For the notations and the general theory concerning the function space $SBV(A)$ we refer the reader to Ref. 8. We indicate by $SBV_0(A)$ the subset of piecewise constant functions in $SBV(A)$ defined by

$$SBV_0(A) := \{u \in SBV(A) : \nabla u = 0 \text{ for } \mathcal{L}^N \text{ a.e. } x \in A\}.$$

Moreover, let us consider the family of sets with finite perimeter in $A$, which will be identified by the functional space $P(A)$ defined by

$$P(A) = \{u \in SBV_0(A) : u(x) \in \{0, 1\} \text{ for } \mathcal{L}^N \text{ a.e. } x \in A\}. \tag{2.2}$$

### 2.1. *Rectifiable sets and Coarea formula*

In this subsection we recall the definition of rectifiable sets and several notions dealing with the tangential calculus which can be developed on them (see Ref. 8, Definition 2.57 and Proposition 2.76).

**Definition 2.1.** Let $E \subset \mathbb{R}^N$ be an $\mathcal{H}^m$-measurable set. We say that $E$ is countably $\mathcal{H}^m$-rectifiable if $E \subseteq N \cup \bigcup_{i\geq 1} \Gamma_i$ where $\mathcal{H}^m(N) = 0$ and each $\Gamma_i$ is the graph of a function $f_i \in C^1(\mathbb{R}^m, \mathbb{R}^N)$.

Countably $\mathcal{H}^m$-rectifiable sets $E$ have nice tangential properties. In particular, they can be endowed with a tangent space $\mathrm{Tan}^m(E, x)$, called *approximate tangent space*, for $\mathcal{H}^m$ a.e. $x \in E$. Essentially, this follows from the locality property of the tangent space of $C^1$ graphs and the decomposition of $E$ into such sets (see Ref. 8, Proposition 2.85 and Definition 2.86).

Furthermore, any Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}^k$ exhibits good differentiability properties on $E$. Indeed, it turns out that for $\mathcal{H}^m$ a.e. $x \in E$ the restriction of $f$ to the affine space $x + \mathrm{Tan}^m(E, x)$ is differentiable at $x$. The *tangential differential* of $f$ on $E$ at $x$, $d^E f_x$, is then defined as the differential of the restriction of $f$ to the affine space $x + \mathrm{Tan}^m(E, x)$ for $\mathcal{H}^m$ a.e. $x \in E$ (see Ref. 8, Definition 2.89 and Theorem 2.90).

Given this, we can state a version of the Coarea formula valid on countably rectifiable sets (see Ref. 8, Theorem 2.93).

**Theorem 2.1.** *Let $f : \mathbb{R}^N \to \mathbb{R}^k$ be a Lipschitz function and let $E \subset \mathbb{R}^N$ be a countably $\mathcal{H}^m$-rectifiable set, with $m \geq k$. Then the function $t \to \mathcal{H}^{m-k}(E \cap f^{-1}(t))$ is $\mathcal{L}^k$ measurable in $\mathbb{R}^k$, $E \cap f^{-1}(t)$ is countably $\mathcal{H}^{m-k}$-rectifiable for $\mathcal{L}^k$ a.e. $t \in \mathbb{R}^k$ and*

$$\int_E \mathbf{C}_k(d^E f_x) d\mathcal{H}^m(x) = \int_{\mathbb{R}^k} \mathcal{H}^{m-k}(E \cap f^{-1}(t))\, dt. \tag{2.3}$$

*In the formula above $\mathbf{C}_k(d^E f_x)$ is the k-dimensional Coarea factor of $d^E f_x$ defined by*

$$\mathbf{C}_k(d^E f_x) = \sqrt{\det((d^E f_x) \circ (d^E f_x)^*)},$$

*where $(d^E f_x)^* : \mathbb{R}^k \to (\mathrm{Tan}^m(E, x))^*$ is the transpose operator.*

## 3. Formulation of the Problem

In this section we will introduce the perforated domains $\Omega_\varepsilon$ and the energy functionals $\mathcal{F}_\varepsilon$.

### 3.1. *The perforated domain*

We consider a compact subset $K$ of the open unitary cube $Q$ such that $Q \backslash K$ is connected. We stress the fact that neither regularity nor dimensional assumptions are imposed on the reference perforation $K$ (see Fig. 1). For every $\varepsilon > 0$ set

$$K_\varepsilon := \varepsilon(K + \mathbb{Z}^N)$$

and

$$\Omega_\varepsilon := \Omega \backslash K_\varepsilon.$$

The sets $K_\varepsilon$ represent the $\varepsilon$-perforations, while $\Omega_\varepsilon$ the perforated domains.

### 3.2. *The energy functionals*

Let us fix a boundary datum $\psi$ (which is the trace of a function) in $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and introduce the functionals $\mathcal{F}_\varepsilon^\psi : L^1(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}_\varepsilon^\psi(u) := \begin{cases} \displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2\, dx + \mathcal{H}^{N-1}(S_u^{\psi,\varepsilon}) & \text{if } u \in SBV^2(\Omega_\varepsilon), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{3.1}$$

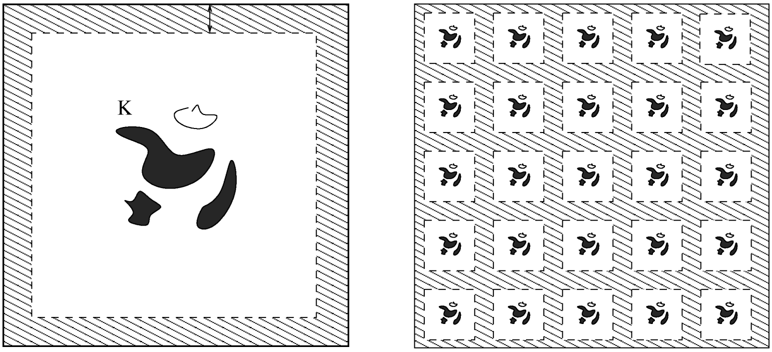


Fig. 1. The "larger" obstacle $Q_{1-2\delta}$ and the related perforated domain $R_n$.

where

$$S_u^{\psi,\varepsilon} := (S_u \cap \Omega_\varepsilon) \cup \{x \in \partial\Omega \cap \partial\Omega_\varepsilon : \psi(x) \neq u(x)\},$$

and the inequality is intended in the sense of traces. The set $S_u^{\psi,\varepsilon}$ takes into account the crack formed inside $\Omega_\varepsilon$, and the part of $\partial\Omega_\varepsilon \cap \partial\Omega$ where $u$ does not agree with the imposed deformation $\psi$ (which is thus considered as part of the crack which has reached the boundary).

Our aim is to study the asymptotic behavior of the energy functionals $\mathcal{F}_\varepsilon^\psi$ defined in (3.1) as $\varepsilon \to 0$ in terms of $\Gamma$-convergence with respect to a suitable topology and to prove compactness properties for sequences of corresponding of minimizers.

**Remark 3.1.** The choice of the $L^1$ setting is rather natural since it provides suitable compactness properties for minimizers (see Sec. 4). In this respect we notice that compactness for sequences of functions with bounded energy cannot hold in general since the energy functionals are not affected by the values of the functions inside the holes $K_\varepsilon$. Nevertheless we will see that we can assign the values inside $K_\varepsilon$ for sequences with bounded energy in order to gain compactness. Furthermore any limit point obtained with this procedure is uniquely determined by the values outside the holes $K_\varepsilon$. Indeed, it is easy to prove that if $(u_\varepsilon), (v_\varepsilon) \subset L^1(\Omega)$ are such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $v_\varepsilon \to v$ in $L^1(\Omega)$ and $u_\varepsilon \equiv v_\varepsilon$ in $\Omega_\varepsilon$, then $u = v$ $\mathcal{L}^N$ a.e. in $\Omega$.

## 4. Compactness

The main aim of this section is to prove a compactness result in $SBV^2$ for suitable extensions of sequences of functions in $L^1(\Omega_{\varepsilon_n})$ with bounded energy, where $(\varepsilon_n)$ is a fixed vanishing sequence. This result will allow us to identify the domain of any $\Gamma$-limit of the functionals $\mathcal{F}_\varepsilon^\psi$ defined in (3.1) and to take advantage of integral representation techniques.

We will consider only sequences uniformly bounded in $L^\infty$. This framework is not restrictive in our setting of the problem, since the boundary datum $\psi$ is in $L^\infty$, and the energy functionals decrease by truncation. Therefore we can assume the minimizing sequences to have $L^\infty$ norm bounded by that of $\psi$.

First we focus on sequences of functions defined in more regular perforated domains obtained substituting the original reference set $K$ with the larger one $Q_{1-2\delta}$, defined according to (2.1) where $0 < \delta < \operatorname{dist}(K, \partial Q)$ is a fixed parameter (see Fig. 1). In addition, let us set

$$R := \{x \in \overline{Q} : \operatorname{dist}(x, \partial Q) < \delta\}, \quad R_n := \varepsilon_n(R + \mathbb{Z}^N) \cap \Omega.$$

Notice that $R = Q_{1-2\delta,1}$, or more explicitly $R = Q \backslash Q_{1-2\delta}$. Throughout the section $(v_n)$ will be a sequence in $L^1(R_n)$ bounded in energy and in $L^\infty$, i.e. satisfying

$$\int_{R_n} |\nabla v_n|^2 \, dx + \mathcal{H}^{N-1}(S_{v_n}) \leq c, \quad \|v_n\|_{L^\infty(R_n)} \leq \|\psi\|_{L^\infty(R_n)}, \tag{4.1}$$

where $c$ is a constant independent of $n$.

In our applications the functions $v_n$ will be given by the restriction to $R_n$ of functions $u_n \in L^1(\Omega)$ with uniformly bounded energy. In view of Remark 3.1 the cluster points of $(u_n)$ in $L^1(\Omega)$ (suitably modified on $K_{\varepsilon_n}$) are determined by those of $(v_n)$ (suitably extended on $\Omega$).

For these sequences $(v_n)$ we provide the existence of suitable $BV$ and $SBV^2$ extensions (these last ones only in the two-dimensional case) preserving a uniform bound of the corresponding energy. By a slicing argument and taking advantage of Remark 3.1 we will then prove that, up to subsequences, we have convergence in $L^1(\Omega)$ to a function belonging to $SBV^2(\Omega)$ (see Sec. 4.3).

The desired compactness result for sequences defined on general perforated sets $\Omega_{\varepsilon_n}$ will then be achieved by an approximation argument (see Sec. 4.4).

Under assumption (4.1) we establish the following results (see Secs. 4.1–4.3, respectively):

(i) *$BV$-compactness*: there exist functions $\tilde{v}_n, v \in BV(\Omega)$ such that $\tilde{v}_n \equiv v_n$ in $R_n$ and, up to a subsequence, $\tilde{v}_n \to v$ in $L^1(\Omega)$;

(ii) *Two-dimensional $SBV^2$-compactness*: if $N = 2$ there exist functions $\widehat{v}_n, v \in SBV^2(\Omega)$ such that $\widehat{v}_n \equiv v_n$ in $R_n$ and, up to a subsequence, $\widehat{v}_n \to v$ in $L^1(\Omega)$;

(iii) *$N$-dimensional $SBV^2$-closure*: if $\tilde{v}_n \to v$ in $L^1(\Omega)$ and $\tilde{v}_n \equiv v_n$ in $R_n$ then $v$ is in $SBV^2(\Omega)$.

The most difficult part of this program is to prove the two-dimensional $SBV^2$-extension result in (ii). The hypothesis on the dimension comes into play only into a technical result, Lemma 4.1, where the *smallness* of a set in terms of area and perimeter implies some estimate on the diameter of its “connected components”. In view of this estimate we are able to prove a Poincaré type inequality in $SBV$ (see Theorem 4.1), which allows us to perform the construction of the functions $\widehat{v}_n$ in (ii) without creating new jumps. Moreover, if the original sequence $(v_n)$ belongs to $W^{1,2}(R_n)$ or to $SBV_0(R_n)$, then $(\widehat{v}_n)$ belongs to $W^{1,2}(\Omega)$, $SBV_0(\Omega)$, respectively.

### 4.1. *BV-compactness*

Here we prove a compactness result in $BV(\Omega)$.

**Proposition 4.1.** (Compactness in $BV(\Omega)$) *Let $(v_n) \subset L^1(R_n)$ be a sequence such that*

$$\sup_n (|Dv_n|(R_n) + \|v_n\|_{L^\infty(R_n)}) < +\infty,$$

*then there exist functions $\tilde{v}_n \in BV(\Omega)$ such that*

$$\tilde{v}_n \equiv v_n \text{ on } R_n \quad \text{and} \quad |D\tilde{v}_n|(\Omega) + \|\tilde{v}_n\|_{L^\infty(\Omega)} \leq c(|Dv_n|(R_n) + \|v_n\|_{L^\infty(R_n)})$$

*for a constant c independent of n. In particular, up to a subsequence, $(\tilde{v}_n)$ converges to $v$ in $L^1(\Omega)$ for some $v \in BV(\Omega)$.*

**Proof.** Let us fix some notation: for $i \in \mathbb{Z}^N$ set $Q_n^i := \varepsilon_n(i + Q)$, $R_n^i := \varepsilon_n(i + R) \cap \Omega$. Let also $\mathcal{I}_n = \{i \in \mathbb{Z}^N : Q_n^i \cap \partial\Omega \neq \emptyset\}$, and for every $Q_n^i \subset \Omega$ set

$$m_n^i := \frac{1}{|R_n^i|} \int_{R_n^i} v_n(x)\, dx,$$

and

$$\tilde{v}_n(x) := \begin{cases} v_n(x) & \text{if } x \in R_n^i, \\ m_n^i & \text{if } x \in Q_n^i \backslash R_n^i,\ i \notin \mathcal{I}_n, \\ 0 & \text{elsewhere in } \Omega. \end{cases}$$

We claim that $(\tilde{v}_n)$ defined above satisfies the thesis. Indeed, by construction $\tilde{v}_n \equiv v_n$ on $R_n$ and $\|\tilde{v}_n\|_{L^\infty(\Omega)} \leq \|v_n\|_{L^\infty(R_n)}$. Standard trace results in $BV$ (see Ref. 8, Theorems 3.84 and 3.87), yield that the function $\tilde{v}_n$ belongs to $BV(\Omega)$ with distributional derivative

$$D\tilde{v}_n = Dv_n \llcorner R_n + \sum_{i \in \mathbb{Z}^N} D\tilde{v}_n \llcorner (\partial R_n^i \cap Q_n^i \cap \Omega),$$

and for $i \notin \mathcal{I}_n$

$$D\tilde{v}_n \llcorner (\partial R_n^i \cap Q_n^i \cap \Omega) = ((\mathrm{tr}(v_n) - m_n^i)\nu_{\partial R_n^i})\mathcal{H}^{N-1} \llcorner (\partial R_n^i \cap Q_n^i),$$

being $\mathrm{tr}(v_n)$ the trace left by $v_n$ on the boundary $\partial R_n^i$. Since by hypothesis $\sup_n |Dv_n|(R_n) < +\infty$, to conclude it suffices to give a uniform estimate of the total variation of $D\tilde{v}_n$ concentrated on the union of $\partial R_n^i \cap Q_n^i \cap \Omega$.

To this aim notice that $\#\mathcal{I}_n \leq c/\varepsilon_n^{N-1}$, with $c$ depending only on $\mathcal{H}^{N-1}(\partial\Omega)$ since $\partial\Omega$ is Lipschitz. Here $\#$ denotes the cardinality of the relevant set. Thus, by taking into account that $\mathcal{H}^{N-1}(\partial R_n^i) \leq c\varepsilon_n^{N-1}$ and $\sup_n \|\tilde{v}_n\|_{L^\infty(R_n)} < +\infty$, we deduce $\sup_n \sum_{i \in \mathcal{I}_n} |D\tilde{v}_n|(\partial R_n^i \cap \Omega) < +\infty$. Furthermore, to control $|D\tilde{v}_n|(\partial R_n^i \cap Q_n^i)$ for $i \notin \mathcal{I}_n$ we use a scaling argument and the continuity of the Trace Operator on $R$ (see Ref. 8, Theorem 3.87). For $i \notin \mathcal{I}_n$ let $w_n^i : R \to \mathbb{R}$ be defined as $w_n^i(y) =$

$v_n(\varepsilon_n(i+y))$. It is easy to check that $w_n^i \in BV(R)$, the mean value of $w_n^i$ on $R$ equals $m_n^i$, and $|Dw_n^i|(R) = \varepsilon_n^{1-N}|Dv_n|(R_n^i)$. Moreover, there exists a positive constant $c(R)$ independent of $n$ such that

$$\int_{\partial R \cap Q} |\mathrm{tr}(w_n^i) - m_n^i|\, d\mathcal{H}^{N-1} \le c(R)|Dw_n^i|(R).$$

A scaling argument gives

$$\int_{\partial R_n^i \cap Q_n^i} |\mathrm{tr}(v_n) - m_n^i|\, d\mathcal{H}^{N-1} = \varepsilon_n^{N-1} \int_{\partial R \cap Q} |\mathrm{tr}(w_n^i) - m_n^i|\, d\mathcal{H}^{N-1},$$

from which we infer that for every $i \notin \mathcal{I}_n$

$$|D\tilde{v}_n|(\partial R_n^i \cap Q_n^i) = \int_{\partial R_n^i \cap Q_n^i} |\mathrm{tr}(v_n) - m_n^i|\, d\mathcal{H}^{N-1} \le c(R)|Dv_n|(R_n^i)$$

and this gives the desired estimate. The rest of the statement is a direct consequence of the $BV$ compactness theorem (see Ref. 8, Theorem 3.23). □

**Remark 4.1.** The $BV$ compactness result still holds if we replace the $\delta$-neighborhood $R_n$ with any connected neighborhood $C$ of $\partial Q$ with Lipschitz continuous boundary. It is also possible to consider varying domains $C_n$, provided they ensure the continuity of the trace operator together with a uniform estimate on the relative constants.

### 4.2. *Compactness in $SBV^2(\Omega)$: The case $N = 2$*

This subsection is focused on $SBV$ compactness properties in dimension two. In this setting given a sequence $(v_n) \subset L^1(R_n)$ with bounded energy (see (4.1)) we construct an $SBV^2(\Omega)$ extension with uniform control on the increase of the energy. In this respect we remark that the $BV$ extensions of Sec. 4.1 have diverging jump energy and so cannot be exploited to infer $SBV$ regularity of their limit. Further arguments are then needed.

We first extend any function $v \in SBV^2(R)$ with quantified small jump set (see Proposition 4.2) to a function $\widehat{v} \in SBV^2(Q)$ such that $v \equiv \widehat{v}$ in $R$ and

$$\int_Q |\nabla \widehat{v}|^2 + \mathcal{H}^{N-1}(S_{\widehat{v}}) \le c\left(\int_R |\nabla v|^2 + \mathcal{H}^{N-1}(S_v)\right),$$

with $c$ independent of $v$ and depending only on the geometry of $R$. Then, the extension for $v_n$ is obtained by exploiting the periodicity of the problem by repeating the construction in each $\varepsilon_n$-square contained in $\Omega$ in which $v_n$ has small jump set (up to the usual scaling argument) and using the $BV$-extension $\tilde{v}_n$ in the holes of the remaining squares (see Proposition 4.3 for more details).

To describe briefly the idea to accomplish the extension in the case of fixed geometry consider a function $v \in SBV^2(R)$, and by a standard argument based on composition with bi-Lipschitz functions assume that $v \in SBV^2(Q_{r_0,1})$, with $1 - 2\,\mathrm{dist}(K, \partial Q) < r_0 < 1 - 2\delta$. Set now $r_2 = 1 - 2\delta$ and let $r_1 \in (r_0, r_2)$ be arbitrarily chosen. In Theorem 4.1 and Proposition 4.2 we will show that if the jump set of $v$ is sufficiently small, we are able to modify $v$ in a region containing $Q_{r_0,r_1}$ and contained in $Q_{r_0,r_2}$. The construction acts by truncating $v$ at suitable levels, in such a way that this truncated function has oscillation on $Q_{r_0,r_1}$ controlled in terms of $|D^a v|(Q_{r_0,r_2})$, and above all without creating any new jump. In view of this Poincaré type inequality, the extension of $v$ to the whole $Q$ is obtained by joining it smoothly to a suitable constant through a cutoff function (see Fig. 2 for a sketch of the construction).

A Poincaré type inequality for $SBV$ functions in any space dimension has been first established in Ref. 30 (see Ref. 8, Theorem 4.14). There the truncation levels are selected via the $BV$ Coarea Formula to control the $L^p$ oscillation on the whole set of the modified function from its median only by its gradient energy.

In this respect the main difference of our approach is the selection procedure of the truncation levels which preserves the boundary values of the original function and which does not introduce new jump set. Such a requirement is essential for the compactness issue in the homogenization problem (see Proposition 4.3) and can be developed in dimension $N = 2$ thanks to a capacitary argument (see Lemmas 4.1, 4.2 and Remark 4.3).

Let us begin with the truncation procedure that we set primarily for functions in $SBV_0$. Let us fix some more preliminary notation. As already mentioned we fix positive radii $r_0, r_1, r_2$ as follows:

$$r_0 \in (1 - 2\,\mathrm{dist}(K, \partial Q), 1 - 2\delta), \quad r_2 = 1 - 2\delta, \quad r_1 \in (r_0, r_2).$$

Moreover, for every $s \in \mathbb{R}$ we denote by $E_s$ the $s$ sub-level of $v$ in $Q_{r_1,r_2}$, i.e.

$$E_s := \{x \in Q_{r_1,r_2} : v(x) \le s\}, \tag{4.2}$$

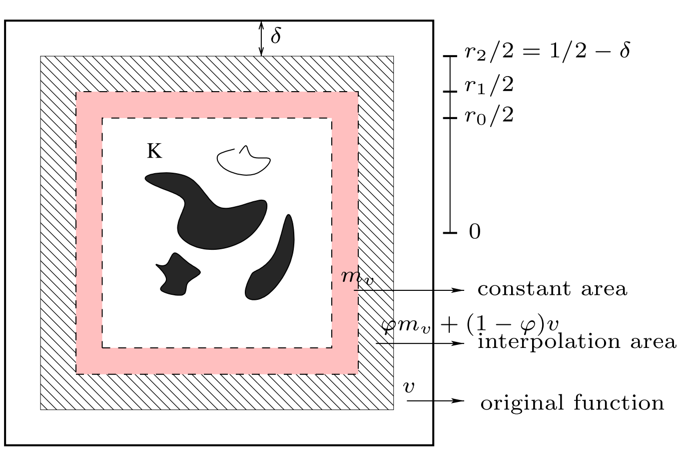


Fig. 2. Definition of $\widehat{v}$ in different areas.

and by med$(v)$ a *median* of $v$ in $Q_{r_1,r_2}$, namely

$$\mathrm{med}(v) := \sup\{s \in \mathbb{R} : |E_s| \le |Q_{r_1,r_2}|/2\}. \tag{4.3}$$

In formula above, the two-dimensional Lebesgue measure $\mathcal{L}^2$ has been indicated with $|\cdot|$, a notation that we will use for the rest of the subsection.

**Lemma 4.1.** (Truncation lemma in $SBV_0(Q_{r_1,r_2})$) *For every $v \in SBV_0(Q_{r_1,r_2})$ with $\mathcal{H}^1(S_v) < (r_2 - r_1)/2$, the set $I = \{r \in (r_1, r_2) : \mathcal{H}^0(\partial Q_r \cap S_v) = 0\}$ is such that $\mathcal{L}^1(I) > 0$. In particular, for $\mathcal{L}^1$ a.e. $r \in I$ the trace of $v$ on $\partial Q_r$ is constant.*

**Proof.** Set

$$J := \{r \in (r_1, r_2) : \mathcal{H}^0(\partial Q_r \cap S_v) \ge 1\},$$

then the thesis is equivalent to proving that $\mathcal{L}^1(J) < r_2 - r_1$ (see Ref. 8, Theorem 3.87).

In order to estimate $\mathcal{L}^1(J)$ we use the Coarea Formula 2.1 applied with $k = 1$, $m = 1$, $f(y) = \|y\|_\infty$ and $E = S_v$. A simple computation shows that $d^{S_v} f_x = \langle \nabla f(x), \nu_v^\perp(x)\rangle \nu_v^\perp(x)$ for $\mathcal{H}^1$ a.e. $x \in S_u$, so that

$$\mathbf{C}_1(d^{S_v} f_x) = |\langle \nabla f, \nu_v^\perp(x)\rangle|.$$

Since $f^{-1}(r/2) = Q_r$ and $|\nabla f| = 1$ $\mathcal{L}^2$ a.e. in $\mathbb{R}^2$, by (2.3) we infer

$$\mathcal{L}^1(J) \le \int_J \mathcal{H}^0(\partial Q_r \cap S_v)\, dr \le 2\mathcal{H}^1(S_v) < r_2 - r_1,$$

from which the result follows. □

**Remark 4.2.** The same result can also be obtained by using classical slicing results (separately in suitable sectors of $Q_{r_1,r_2}$), instead of the Coarea Formula.

In the following we will deal with one-dimensional sections of a set of finite perimeter $E$. To make the framework rigorous we could fix a $\mathcal{L}^2$-representantive of $E$, e.g. $E^+ = \{x \in \mathbb{R}^2 : \limsup_{r\to 0+} r^{-2}|B_r(x) \cap E| > 0\}$. A careful reading shows anyway that all the statements below are independent of the $\mathcal{L}^2$ representantive of $E$.

Given a set of finite perimeter $E$, we denote by $\partial^* E$ the *essential boundary* of $E$ (see Ref. 8, Definition 3.60). By applying Lemma 4.1 to the characteristic function of a set with finite perimeter we immediately deduce the following corollary.

**Corollary 4.1.** *For any set of finite perimeter $E \subseteq Q_{r_1,r_2}$ with $\mathcal{H}^1(\partial^* E) < (r_2 - r_1)/2$ there exists a set of positive $\mathcal{L}^1$ measure in $(r_1, r_2)$ such that for any $r$ in this set either $\mathcal{H}^1(E \cap \partial Q_r) = 0$ or $\mathcal{H}^1(E \cap \partial Q_r) = \mathcal{H}^1(\partial Q_r)$.*

Under some additional conditions on the smallness of both $|E|$ and $\mathcal{H}^1(\partial^* E)$ the previous result can be refined.

**Lemma 4.2.** *There exists a constant $C_1 \in (1, +\infty)$ depending only on $r_1$ and $r_2$ such that the following holds true. For any set of finite perimeter $E \subseteq Q_{r_1,r_2}$, with*

*$|E| \leq |Q_{r_1,r_2}|/2$ and $\mathcal{H}^1(\partial^* E) \leq (r_2 - r_1)/C_1$, there exists a set $\mathcal{I} \subseteq (r_1, r_2)$ of positive $\mathcal{L}^1$ measure such that $\mathcal{H}^1(E \cap \partial Q_r) = 0$ for $\mathcal{L}^1$ a.e. $r \in \mathcal{I}$.*

**Proof.** Let us set

$$\begin{aligned} I &:= \{r \in (r_1, r_2) : \mathcal{H}^1(\partial Q_r \cap E) = \mathcal{H}^1(\partial Q_r)\}, \\ J &:= \{r \in (r_1, r_2) : 0 < \mathcal{H}^1(\partial Q_r \cap E) < \mathcal{H}^1(\partial Q_r)\}. \end{aligned}$$

We have to prove that if $C_1$ is large enough, then $\mathcal{L}^1(I \cup J) < r_2 - r_1$.

By the Coarea Formula we have

$$\int_{r_1}^{r_2} \mathcal{H}^1(\partial Q_r \cap E)\, dr = 2|E|.$$

If $\tilde{c}$ is the constant of the Relative Isoperimetric Inequality in $Q_{r_1,r_2}$ (see Ref. 8, formula (3.43)), a rearrangement argument[b] gives

$$\begin{aligned} \int_{r_1}^{r_1+\mathcal{L}^1(I)} 4r\, dr &\leq \int_I \mathcal{H}^1(\partial Q_r)\, dr = \int_I \mathcal{H}^1(\partial Q_r \cap E)\, dr \\ &\leq 2|E| \leq 2\tilde{c}(\mathcal{H}^1(\partial^* E))^2 \leq 2\frac{\tilde{c}}{C_1^2}(r_2 - r_1)^2, \end{aligned}$$

from which we immediately obtain

$$\mathcal{L}^1(I) \leq \frac{\sqrt{\tilde{c}}}{C_1}(r_2 - r_1). \tag{4.4}$$

In order to estimate $\mathcal{L}^1(J)$ we use a slicing argument and notice that for $\mathcal{L}^1$ a.e. $r \in J$ $\partial Q_r \cap E$ is a set of finite perimeter on each side of $\partial Q_r$. Since by assumption $\partial Q_r \cap E$ is not of full measure in $\partial Q_r$ nor it is the empty set, hence $\mathcal{H}^0(\partial Q_r \cap \partial^* E) \geq 1$ for $\mathcal{L}^1$ a.e. $r \in J$. As a consequence by the Coarea Formula we infer

$$\mathcal{L}^1(J) \leq \int_J \mathcal{H}^0(\partial Q_r \cap \partial^* E)\, dr \leq 2\mathcal{H}^1(\partial^* E) \leq 2\frac{r_2 - r_1}{C_1}. \tag{4.5}$$

From (4.4) and (4.5) we easily conclude. □

**Remark 4.3.** In dimension greater than two the result of Lemma 4.2 is no longer true. Indeed, one can exhibit sets with small perimeter intersecting the boundary of each $Q_r$ in a set of positive $\mathcal{H}^{N-1}$ measure. In this case an analogous of Lemma 4.2 should deal with a suitable quantification of the measure of the subset intersecting the boundary of each cube. We did not investigate further this kind of result since our techniques allow us to prove the closure and compactness result in any dimension arguing by sections, taking advantage of the two-dimensional case.

[b] To justify the first inequality it is enough to consider the function mapping $[r_1, \sup I]$ in $[r_1, r_1 + \mathcal{L}^1(I)]$ whose derivative is equal to the characteristic function of the set $I$. By taking into account that this map is 1-Lipschitz, monotone increasing and surjective, changing variables by means of the Coarea Formula 2.1 yields the desired inequality.

From Lemmas 4.1 and 4.2 we can deduce a localized Poincaré type inequality for functions in $SBV(Q_{r_0,1})$.

**Theorem 4.1.** (A Poincaré type inequality in $SBV(Q_{r_0,1})$) *Let $C_1$ be as in Lemma 4.2 and let $v \in SBV(Q_{r_0,1})$ with $\mathcal{H}^1(S_v) \le (r_2 - r_1)/2C_1$. Then there exist a function denoted by $T(v)$ in $SBV(Q_{r_0,1})$ and a constant $m_v \in \mathbb{R}$ satisfying*

(i) $T(v) = v$ *in* $R$;
(ii) $|DT(v)| \le |Dv|$ *in* $Q_{r_0,1}$ *in the sense of measures*;
(iii) $\|T(v) - m_v\|_{L^\infty(Q_{r_0,r_1})} \le 2C_1 |D^a v|(Q_{r_1,r_2})/(r_2 - r_1)$.

**Proof.** If $|D^a v|(Q_{r_1,r_2}) = 0$, we apply Lemma 4.1 and select $\bar{r} \in (r_1, r_2)$ such that the trace of $v$ on $\partial Q_{\bar{r}}$ is constant. In particular, choosing $m_v$ equal to such a constant and setting $T(v) := m_v$ in $Q_{\bar{r}}$, all the conditions of the theorem are satisfied.

Otherwise we have $|D^a v|(Q_{r_1,r_2}) > 0$, then the $BV$ Coarea Formula (see Ref. 8, Theorem 3.40) implies

$$\int_{\mathrm{med}(v) - 2C_1 |D^a v|(Q_{r_1,r_2})/(r_2 - r_1)}^{\mathrm{med}(v)} \mathcal{H}^1(\partial^* E_s \backslash S_v)\, ds \le \int_{\mathbb{R}} \mathcal{H}^1(\partial^* E_s \backslash S_v)\, ds = |D^a v|(Q_{r_1,r_2}),$$

where $E_s$ is the sub-level of $v$ in $Q_{r_1,r_2}$ defined in (4.2) and $\mathrm{med}(v)$ is defined in (4.3). By the Mean Value Theorem, there exists $s' \in (\mathrm{med}(v) - 2C_1|D^a v|(Q_{r_1,r_2})/(r_2 - r_1), \mathrm{med}(v))$ such that

$$\mathcal{H}^1(\partial^* E_{s'} \backslash S_v) \le (r_2 - r_1)/2C_1$$

and so

$$\mathcal{H}^1(\partial^* E_{s'}) \le \mathcal{H}^1(\partial^* E_{s'} \backslash S_v) + \mathcal{H}^1(S_v) \le (r_2 - r_1)/C_1. \tag{4.6}$$

Analogously, we can find $s'' \in (\mathrm{med}(v), \mathrm{med}(v) + 2C_1|D^a v|(Q_{r_1,r_2})/(r_2 - r_1))$ such that

$$\mathcal{H}^1(\partial^* E_{s''}) \le (r_2 - r_1)/C_1. \tag{4.7}$$

The definition of median (4.3) and the choice $s' < \mathrm{med}(v)$ yield $|E_{s'}| \le |Q_{r_1,r_2}|/2$, and by arguing similarly, the same inequality holds for the set $Q_{r_1,r_2} \backslash E_{s''}$ as well. By taking into account (4.6) and (4.7) we may apply Lemma 4.2 separately to the two sets $E_{s'}$, $Q_{r_1,r_2} \backslash E_{s''}$ and find radii $r_1 < r' < r'' < r_2$ with $\mathcal{H}^1(E_{s'} \cap \partial Q_{r'}) = 0$ and $\mathcal{H}^1((Q_{r_1,r_2} \backslash E_{s''}) \cap \partial Q_{r''}) = 0$. Set $m_v := \mathrm{med}(v)$ and define $T(v) := s' \vee v \wedge s''$ in $Q_{r_0,r'}$, $T(v) := v \wedge s''$ in $Q_{r',r''}$ and $T(v) = v$ in $Q \backslash Q_{r''}$. The thesis follows easily by construction. □

**Proposition 4.2.** (An extension result) *There exists a constant $C_2 > 0$ depending only on $r_0, r_1, r_2$ such that for any $v \in SBV^2(R)$ with $\mathcal{H}^1(S_v) \le (r_2 - r_1)/2C_1$ there exists $\widehat{v} \in SBV^2(Q)$ such that $\widehat{v} = v$ in $R$, $\|\nabla \widehat{v}\|_{L^2(Q)} \le C_2 \|\nabla v\|_{L^2(R)}$ and $\mathcal{H}^{N-1}(S_{\hat{v}}) \le C_2 \mathcal{H}^{N-1}(S_v)$. Moreover, if $v \in W^{1,2}(R)$ $(v \in SBV_0(R))$, then $\widehat{v} \in W^{1,2}(Q)$ $(v \in SBV_0(Q))$.*

**Proof.** Thanks to the regularity of the sets $Q_{r,s}$, by a standard technique that relies on inner composition with bi-Lipschitz functions (see Ref. 8, Theorem 3.16 and also Ref. 7 for a deeper insight) we may assume the function $v$ to be extended in $Q_{r_0,1}$ in such a way that

$$\int_{Q_{r_0,1}} |\nabla v|^2\, dx \le c \int_R |\nabla v|^2\, dx, \quad \mathcal{H}^{N-1}(S_v) \le c\mathcal{H}^{N-1}(S_v \cap R),$$

for a universal constant $c > 0$ depending only on the geometry of $R$.

Let now $T(v)$ and $m_v$ be as in Theorem 4.1. If $v \in SBV_0(Q_{r_0,1})$ define $\widehat{v}$ simply by extending $T(v)$ to the whole of $Q$ with constant value $m_v$. Otherwise, we consider a cutoff function $\varphi \in C^1(Q, [0,1])$ such that $\varphi \equiv 0$ on $Q_{r_0}$, $\varphi \equiv 1$ on $Q_{r_1,1}$. Define the function $\widehat{v}$ on $Q$ as $\widehat{v} := \varphi T(v) + (1-\varphi)m_v$. A straightforward computation shows that

$$\int_Q |\nabla \widehat{v}|^2 dx \le \int_{Q_{r_0,1}} |\nabla v|^2 dx + c\int_{Q_{r_0,r_1}} |T(v) - m_v|^2\, dx. \tag{4.8}$$

Taking into account (iii) of Theorem 4.1 and using Jensen inequality we obtain

$$\int_{Q_{r_0,r_1}} |T(v) - m_v|^2\, dx \le c\int_{Q_{r_1,r_2}} |\nabla v|^2 dx. \tag{4.9}$$

From (4.8) and (4.9), noticing that $S_{\hat{v}} \subseteq S_v$, the thesis follows. □

**Remark 4.4.** We notice that with the same techniques used in the proof of Proposition 4.2 one can prove an extension result for functions $v$ in $SBV(R)$ with $\mathcal{H}^1(S_v) \le (r_2 - r_1)/2C_1$ to functions which are in $SBV(Q)$.

We are now in a position to prove the compactness of sequences $(v_n)$ satisfying (4.1).

**Proposition 4.3.** (Compactness in $SBV^2(\Omega), \Omega \subset \mathbb{R}^2$) *Let $(v_n) \subseteq L^1(R_n)$ be satisfying* (4.1). *Then there exist functions $\widehat{v}_n \in SBV^2(\Omega)$ satisfying $\widehat{v}_n \equiv v_n$ in $R_n$ such that (up to a subsequence) $(\widehat{v}_n)$ converges in $L^1(\Omega)$ to some $v \in SBV^2(\Omega)$.*

**Proof.** Set $\mathcal{J}_n = \{i \in \mathbb{Z}^N :$ either $Q_n^i \not\subset \Omega$ or $\mathcal{H}^1(S_{v_n} \cap Q_n^i) > \varepsilon_n(r_2 - r_1)/2C_1\}$ with $C_1$ as in Lemma 4.2. For every $i \in \mathcal{J}_n$ we define $\widehat{v}_n$ on $Q_n^i$ to be equal to the $BV$ extension $\tilde{v}_n$ defined in Proposition 4.1. By the Lipschitz regularity of $\partial\Omega$ and the fact that $\sup_n \mathcal{H}^1(S_{v_n}) < +\infty$ we deduce that $\#(\mathcal{J}_n) \le c/\varepsilon_n$. This together with the assumption $\sup \|v_n\|_{L^\infty(R_n)} < +\infty$ provides the estimate

$$\int_{\cup_{i\in\mathcal{J}_n} Q_n^i} |\nabla \hat{v}_n|^2\, dx + \mathcal{H}^1(S_{\hat{v}_n} \cap \cup_{i\in\mathcal{J}_n} Q_n^i) \le c$$

for some $c$ independent of $n$.

Let us now consider a square $Q_n^i$ contained in $\Omega$ and satisfying $\mathcal{H}^1(S_{v_n} \cap Q_n^i) \le \varepsilon_n(r_2 - r_1)/2C_1$. Let $v_n^i : R \to \mathbb{R}$ be defined as $v_n^i(y) = v_n(\varepsilon_n(i + y))$. It can be checked that $v_n^i$ satisfies the hypotheses of Proposition 4.2. Let $\widehat{v}_n^i \in SBV^2(Q)$ be its extension provided by Proposition 4.2 and define $\widehat{v}_n$ as $\widehat{v}_n^i$ scaled back to $Q_n^i$. Using a standard scaling argument, we obtain $\|\nabla \widehat{v}_n\|_{L^2(Q_n^i)} \leq C_2 \|\nabla v_n\|_{L^2(R_n^i)}$, $\mathcal{H}^1(S_{\hat{v}_n} \cap Q_n^i) \leq C_2 \mathcal{H}^1(S_{v_n} \cap R_n^i)$, and $\|\widehat{v}_n\|_{L^\infty(\Omega)} \leq \|v_n\|_{L^\infty(R_n)}$. The compactness then follows by Ambrosio's *SBV* Theorem (see Ref. 8, Theorem 4.8). □

### 4.3. *Compactness in $SBV^2(\Omega)$: The general case*

Let us turn our attention to prove that in dimension greater than 2 the $L^1$ limit of any (extension of) $(v_n)$ as in (4.1) is actually in $SBV^2(\Omega)$. We argue by a slicing procedure that allows us to infer the result from Proposition 4.3.

**Proposition 4.4.** ($SBV^2$ closure) *Let $(v_n) \subset L^1(R_n)$ be a sequence as in (4.1) and let $v$ be the $L^1$ limit of some sequence $(\tilde{v}_n) \subset L^1(\Omega)$, with $\tilde{v}_n \equiv v_n$ in $R_n$. Then $v \in SBV^2(\Omega)$.*

**Proof.** First note that by Remark 3.1 $v$ is also the $L^1$ limit of the sequence of extensions constructed in Proposition 4.1, thus we deduce that $v \in BV(\Omega)$.

We argue by a slicing procedure that allows us to use the result in Proposition 4.3. Let $V_{i,j}$ be the two-dimensional subspace in $\mathbb{R}^N$ generated by the vectors $e_i$, $e_j$ of the canonical base. We use the standard notation $V_{i,j}^\perp$ to denote the space orthogonal to $V_{i,j}$.

Given $z \in V_{i,j}^\perp$ we denote by $v^{i,j,z}$ the restriction of the function $v$ to the planar set $\Omega^{i,j,z} := (V_{i,j} + z) \cap \Omega$. We claim that for $\mathcal{H}^{N-2}$ a.e. $z \in V_{i,j}^\perp$ the function $v^{i,j,z}$ belongs to $SBV^2(\Omega^{i,j,z})$, and

$$\int_{V_{i,j}^\perp} \left( \int_{\Omega^{i,j,z}} |\nabla v^{i,j,z}|^2 d\mathcal{H}^2 + \mathcal{H}^1(S_{v^{i,j,z}}) \right) d\mathcal{H}^{N-2}(z) < +\infty. \tag{4.10}$$

Once claim (4.10) is proved we conclude the proof of the proposition as follows. Fix $1 \leq i,\, j \leq N$, and let $z \in V_{i,j}^\perp$ be such that

$$\int_{\Omega^{i,j,z}} |\nabla v^{i,j,z}|^2 d\mathcal{H}^2 + \mathcal{H}^1(S_{v^{i,j,z}}) := M(z) < +\infty. \tag{4.11}$$

For every fixed $t \in \mathbb{R}$ let us set $L^{i,j,z,t} := \Omega \cap \{te_j + se_i + z,\ s \in \mathbb{R}\}$, and let $v^{i,j,z,t}$ be the restriction of $v$ to $L^{i,j,z,t}$. By (4.11) and standard one-dimensional slicing theory (see Ref. 8, Theorem 3.108), we have that for almost every $t \in \mathbb{R}$ the function $v^{i,j,z,t}$ belongs to $SBV^2(L^{i,j,z,t})$, and moreover

$$\int_{\mathbb{R}} \left( \int_{L^{i,j,z,t}} |\nabla v^{i,j,z,t}|^2 \, d\mathcal{H}^1 + \mathcal{H}^0(S_{v^{i,j,z,t}}) \right) dt \leq M(z). \tag{4.12}$$

Integrating (4.12) with respect to $z$ and taking into account (4.10), (4.11) we conclude that for almost all $\xi \in e_i^\perp$, setting $L^{i,\xi} := \Omega \cap \{se_i + \xi, s \in \mathbb{R}\}$, the restriction

$v^{i,\xi}$ of $v$ to $L^{i,\xi}$ belongs to $SBV^2(L^{i,\xi})$, and again by one-dimensional slicing theory

$$\int_{e_i^\perp}\left(\int_{L^{i,\xi}} |\nabla v^{i,\xi}|^2\, d\mathcal{H}^1 + \mathcal{H}^0(S_{v^{i,j,z,t}})\right) d\mathcal{H}^{N-1}(\xi) < +\infty. \tag{4.13}$$

Since the choice of the direction $e_i$ is arbitrary, we have that (4.13) holds true for all $1 \le i \le N$. By standard slicing argument we deduce that $v \in SBV^2(\Omega)$, that concludes the proof of the proposition using the claim.

It remains to prove the claim (4.10). Let us set $R_n^z := R_n \cap \Omega^{i,j,z}$ and

$$M_n(z) := \int_{R_n^z} |\nabla v_n^{i,j,z}|^2 d\mathcal{H}^2 + \mathcal{H}^1(S_{v_n^{i,j,z}}).$$

In view of (4.1), by Fubini Theorem and standard slicing arguments we have that

$$\int_{V_{i,j}^\perp} M_n(z)\, d\mathcal{H}^{N-2}(z) \le c. \tag{4.14}$$

Hence for $\mathcal{H}^{N-2}$-a.e. $z \in V_{i,j}^\perp$ the values $\liminf_n M_n(z)$ are finite and the restriction $v_n^{i,j,z}$ of $v_n$ to $R_n^z$ belongs to $SBV^2(R_n^z)$.

Let us fix $z \in V_{i,j}^\perp$ such that, up to a subsequence not relabeled, $M_n(z)$ is bounded uniformly in $n$. We observe that for given $n$ the set $R_n^z$ either coincides with $\Omega^{i,j,z}$, or with the two-dimensional $\delta$-neighborhood of the grid

$$\varepsilon_n(([-1/2,1/2]^2 \backslash [-1/2+\delta, 1/2-\delta]^2) + \mathbb{Z}^2) \cap \Omega^{i,j,z}$$

which we label as $R_n(\Omega^{i,j,z})$. In both cases we can apply Proposition 4.3 to the sequence $(v_n^{i,j,z})$ on $R_n(\Omega^{i,j,z})$ and get functions $w_n^{i,j,z}$ with $w_n^{i,j,z} \equiv v_n^{i,j,z}$ on $R_n(\Omega^{i,j,z})$ satisfying

$$\begin{aligned} &\int_{\Omega^{i,j,z}} |\nabla w_n^{i,j,z}|^2 d\mathcal{H}^2 + \mathcal{H}^1(S_{w_n^{i,j,z}}) \\ &\qquad \le c' \int_{R_n(\Omega^{i,j,z})} |\nabla v_n^{i,j,z}|^2 d\mathcal{H}^2 + c'\mathcal{H}^1(S_{v_n^{i,j,z}} \cap R_n(\Omega^{i,j,z})) \le c' M_n(z), \end{aligned} \tag{4.15}$$

where $c'$ is a constant depending only on $\delta$ and the fixed geometry of the perforations. In particular, a two-dimensional argument analogous to Remark 3.1 implies that $w_n^{i,j,z}$ converge to $v^{i,j,z}$ for $\mathcal{H}^{N-2}$ a.e. $z \in V^\perp$. Finally, (4.15) and Ambrosio's *SBV* Theorem yield

$$\int_{\Omega^{i,j,z}} |\nabla v^{i,j,z}|^2 d\mathcal{H}^2 + \mathcal{H}^1(S_{v^{i,j,z}}) \le c' \liminf_n M_n(z).$$

Integrating with respect to $z$, in view of (4.14) and using Fatou lemma we conclude

$$\begin{aligned} &\int_{V_{i,j}^\perp}\left(\int_{\Omega^{i,j,z}} |\nabla v^{i,j,z}|^2\, d\mathcal{H}^2 + \mathcal{H}^1(S_{v^{i,j,z}})\right) d\mathcal{H}^{N-2}(z) \\ &\qquad \le c' \int_{V_{i,j}^\perp} \liminf_n M_n(z) d\mathcal{H}^{N-2}(z) \le c' \liminf_n \int_{V_{i,j}^\perp} M_n(z) d\mathcal{H}^{N-2}(z) < +\infty. \end{aligned}$$

This concludes the proof of the claim (4.10) and of the proposition. □

### 4.4. $L^1$-compactness

In this section we will state the compactness result for sequences of functions on the perforated domains bounded in energy. In the sequel we will need the following lemma.

**Lemma 4.3.** *Let $K$ be a closed set in $Q$. Then there exists a sequence of sets $(C^m)$ in $Q$ that are closures of open sets with Lipschitz boundary such that $C^{m+1} \subset\subset C^m$, and $\cap_{m\geq 1} C^m = K$.*

*In particular, the sequence $(C^m)$ converges to $K$ in the Hausdorff metric on $\bar{Q}$, and $(\chi_{C^m})$ converges to $\chi_K$ in $L^1(Q)$.*

*Moreover, if $Q\backslash K$ is connected we can choose the sets $C^m$ such that $Q\backslash C^m$ is connected.*

**Proof.** For every $m \in \mathbb{N}$ consider an open set $A^m$ with Lipschitz boundary such that

$$\{x \in Q : \text{dist}(x, K) > 1/m\} \subset\subset A^m \subset\subset \{x \in Q : \text{dist}(x, K) > 1/(m+1)\}. \quad (4.16)$$

The existence of such a set can be justified by taking a finite covering of the set $\{x \in Q : \text{dist}(x, K) > 1/m\}$ made of cubes compactly contained in $\{x \in Q : \text{dist}(x, K) > 1/(m+1)\}$. Then the first part of the statement is proved choosing $C^m$ as the closure of the complementary of $A^m$ by (4.16).

Assume now that $Q\backslash K$ is connected. Let $B^m$ denote the connected component of $A^m$ whose closure contains $\partial Q$, in this case we set $C^m = Q\backslash B^m$. Clearly $C^m$ is the closure of an open set with Lipschitz boundary and $Q\backslash C^m = B^m$ is connected. Moreover, since $B^m, B^{m+1}$ are two connected components intersecting in a "neighborhood" of $\partial Q$, by construction we have $B^m \subset\subset B^{m+1}$, and thus $C^{m+1} \subset\subset C^m \subset\subset Q$.

To conclude we prove that $\cap_{m\geq 1} C^m = K$. As $C^m \supseteq \{x \in Q : \text{dist}(x, K) \leq 1/(m+1)\}$, therefore $\cap_{m\geq 1} C^m \supseteq K$. On the other hand, with fixed $x_0 \in Q\backslash K$ and $x_1 \in B^1$, there exists a continuous curve $\gamma : [0,1] \to Q\backslash K$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$ by the connectedness of $Q\backslash K$. Let $\eta = \text{dist}(K, \gamma([0,1]))$, then $\eta > 0$ and $x_0 \in B^m$ for any $m > [1/\eta]$. Hence, $x_0 \notin \cap_{m\geq 1} C^m$, and this yields the claim. □

**Theorem 4.2.** ($L^1$-Compactness for $(u_n)$) *Let $(u_n) \subset L^1(\Omega_{\varepsilon_n})$ be a sequence satisfying*

$$\int_{\Omega_{\varepsilon_n}} |\nabla u_n|^2 \, dx + \mathcal{H}^{N-1}(S_{u_n}) + \|u_n\|_{L^\infty(\Omega_{\varepsilon_n})} \leq c \quad (4.17)$$

*for some constant $c$ independent of $n$. Then there exist $u \in SBV^2(\Omega)$ and a sequence $(w_n) \subset L^1(\Omega)$, with $w_n \equiv u_n$ in $\Omega_{\varepsilon_n}$, such that (up to a subsequence) $(w_n)$ converges to $u$ in $L^1(\Omega)$.*

**Proof.** For any $m \in \mathbb{N}$ let $C^m$ be as in Lemma 4.3, that is a closed set with Lipschitz continuous boundary containing $K$ such that $Q\backslash C^m$ is connected. Set

$$C_n^m := \bigcup_{z\in\mathbb{Z}} \varepsilon_n(z + C^m), \quad \Omega_n^m := \Omega\backslash C_n^m.$$

By applying Remark 4.1 to the perforated domain $\Omega_n^m$ and by a standard induction argument we deduce that there exists a subsequence (not relabeled for convenience) $(\tilde{u}_n^m)$ converging in $L^1(\Omega)$ to some $u^m \in BV(\Omega)$, with $\tilde{u}_n^m \equiv u_n$ on $\Omega_n^m$ for every $m \in \mathbb{N}$. Actually, by Remark 3.1 we infer that the limit function $u^m$ does not depend on $m$; and thus we drop the superscript $m$ and denote it only by $u$.

A diagonalization argument allows us to find a sequence $(\tilde{u}_{n(m)}^m)$ which converges to $u$ in $L^1(\Omega)$. Finally set

$$w_m(x) := \begin{cases} \tilde{u}_{n(m)}^m(x) & \text{if } x \in K_{\varepsilon_{n(m)}}; \\ u_{n(m)}(x) & \text{if } x \in \Omega_{\varepsilon_{n(m)}}. \end{cases}$$

To conclude notice that the set $\{x \in \Omega : w_m(x) \neq \tilde{u}_{n(m)}^m(x)\}$ is contained in $C_{n(m)}^m \backslash K_{\varepsilon_{n(m)}}$, and $\mathcal{L}^N(C_{n(m)}^m \backslash K_{\varepsilon_{n(m)}}) \leq c\mathcal{L}^N(C^m \backslash K)$: being this last term infinitesimal as $m \to +\infty$ we get that $(w_m)$ converges to $u$ in measure. Hence, since $w_m$ are uniformly bounded in $L^\infty$, $w_m \to u$ in $L^1(\Omega)$. Finally, in view of Proposition 4.4 we conclude that $u \in SBV^2(\Omega)$. □

**Remark 4.5.** It is clear that if we remove the assumption that $Q\backslash K$ is connected the compactness result does not hold true anymore. For instance, it suffices to consider $K = Q_{1/4,1/2}$ and $u_n$ to be equal to 1 in all the inner squares (rescaled and translated copies of $Q_{1/4}$) and 0 otherwise.

Nevertheless the compactness still stands in a weaker form. Indeed, let $\tilde{\Omega}_{\varepsilon_n}$ be the connected component of $\Omega_{\varepsilon_n}$ containing $\varepsilon_n \mathbb{Z}^N$, then it is possible to prove that for any $(u_n) \subset L^1(\Omega_{\varepsilon_n})$ as in (4.17) there exists a subsequence $(w_n)$ with $w_n \equiv u_n$ on $\tilde{\Omega}_{\varepsilon_n}$, and locally constant in $\Omega_{\varepsilon_n} \backslash \tilde{\Omega}_{\varepsilon_n}$, such that (up to a subsequence) $(w_n)$ converges to $u$ in $L^1(\Omega)$ for some $u \in SBV^2(\Omega)$.

## 5. The Γ-Convergence Result

In the sequel we study the asymptotics as $\varepsilon \to 0$ of the family of functionals $\mathcal{F}_\varepsilon^\psi$ defined in (3.1) by using the direct methods of Γ-convergence. This abstract approach generalizes straightforwardly to the homogenization of more complex energies (see Sec. 7). In order to apply the direct methods of Γ-convergence we localize the energy functionals and for simplicity we first neglect the boundary conditions. We will set the problem in the ambient space $L^2(\Omega)$ and we represent the Γ-limit of $\mathcal{F}_\varepsilon$ with respect to the $L^2$-topology only on $SBV^2(\Omega) \cap L^2(\Omega)$. This formulation fits with the study of asymptotic behavior of minimizers of the functionals $\mathcal{F}_\varepsilon^\psi$ taking into account a $L^\infty$ boundary datum $\psi$ (see Sec. 6 and the related discussion therein).

For every $A \in \mathcal{A}(\Omega)$ and $\varepsilon > 0$ we set $A_\varepsilon = A \backslash K_\varepsilon$ and we introduce the functionals $\mathcal{F}_\varepsilon : L^2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ defined for every $A \in \mathcal{A}(\Omega)$ by

$$\mathcal{F}_\varepsilon(u, A) := \begin{cases} \displaystyle\int_{A_\varepsilon} |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(S_u \cap A_\varepsilon) & \text{if } u \in SBV^2(A), \\ +\infty & \text{otherwise in } L^2(\Omega). \end{cases} \tag{5.1}$$

**Theorem 5.1.** *For every $A \in \mathcal{A}(\Omega)$ the family $(\mathcal{F}_\varepsilon(\cdot, A))$ $\Gamma$-converges to some functional $\mathcal{F}_{\text{hom}}(\cdot, A)$ with respect to the $L^2$ topology. Moreover, the functional $\mathcal{F}_{\text{hom}}(\cdot, A)$ restricted to $SBV^2(A)$ is given by*

$$\mathcal{F}_{\text{hom}}(u, A) := \int_A f_{\text{hom}}(\nabla u)\, dx + \int_{S_u \cap A} g_{\text{hom}}(\nu_u)\, d\mathcal{H}^{N-1}, \tag{5.2}$$

*where $f_{\text{hom}}$ and $g_{\text{hom}}$ are defined in (1.3) and (1.5), respectively.*

The proof of Theorem 5.1 will be a consequence of several preliminary results (see Propositions 5.1–5.4). The first step is to show the compactness property in the sense of $\Gamma$-convergence of $\mathcal{F}_\varepsilon$ and the integral representation of its $\Gamma$-limit $\mathcal{F}$. These results follow by standard arguments in $\Gamma$-convergence; we will limit ourselves to provide the related references (see Refs. 25 and 14). We recall that $\mathcal{A}(\Omega)$ denotes the family of all open subsets of $\Omega$.

**Proposition 5.1.** *Let $(\varepsilon_n)$ be a positive vanishing sequence. Then there exists a subsequence $(\varepsilon_{j_n})$ of $(\varepsilon_n)$ and a functional $\mathcal{F} : L^2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that for every $A \in \mathcal{A}(\Omega)$*

$$\mathcal{F}(\cdot, A) = \Gamma - \lim_n \mathcal{F}_{\varepsilon_{j_n}}(\cdot, A).$$

*Moreover $\mathcal{F}$ satisfies the following properties*:

(i) *the set function $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure on $\Omega$ for every fixed $u \in SBV^2(\Omega) \cap L^2(\Omega)$, and the functional $\mathcal{F}(\cdot, A)$ is local and $L^2(A)$ lower semicontinuous for every $A \in \mathcal{A}(\Omega)$;*
(ii) *for every $A \in \mathcal{A}(\Omega)$ with $A \subset\subset \Omega$ and for every $y \in \mathbb{R}^N$ such that $y + A \subset \Omega$ and $u \in SBV^2(A)$ we have $\mathcal{F}(u(\cdot - y), A + y) = \mathcal{F}(u, A)$;*
(iii) *for every $z \in \mathbb{R}$, $A \in \mathcal{A}(\Omega)$ and $u \in SBV^2(A) \cap L^2(A)$ we have $\mathcal{F}(u + z, A) = \mathcal{F}(u, A)$.*

By taking into account the integral representation results of Ref. 12 we get the following result.

**Proposition 5.2.** *Assume that $(\mathcal{F}_{\varepsilon_n}(\cdot, A))$ $\Gamma$-converges to a functional $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Then there exist Borel functions $f : \mathbb{R}^N \to [0, +\infty]$ and $g : \mathbb{R} \times \mathbf{S}^{N-1} \to [0, +\infty]$ such that for every $A \in \mathcal{A}(\Omega)$ and $u \in SBV^2(A)$*

$$\mathcal{F}(u, A) = \int_A f(\nabla u) dx + \int_{S_u \cap A} g(u^+ - u^-, \nu)\, d\mathcal{H}^{N-1}.$$

**Proof.** To prove the result we apply the integral representation Theorem 1 Ref. 12. In order to match the assumptions of that result we need to extend $\mathcal{F}(\cdot, A)$, $A \in \mathcal{A}(\Omega)$, to $SBV^2(A)$ by relaxation with respect to the $L^1$ topology and then to use a perturbation argument to enforce the growth condition from below.

In this respect let us consider the functional $\mathcal{F}(\cdot, A)$ extended to $SBV^2(A)$ as follows

$$\mathcal{F}(u, A) := \inf \left\{ \liminf_n \mathcal{F}(u_n, A),\ u_n \to u \text{ in } L^1(A) \right\}.$$

By a truncation argument it is possible to check that this relaxation procedure does not change the value of $\mathcal{F}$ on $SBV^2(A) \cap L^2(A)$.

Clearly $\mathcal{F}$ satisfies properties (ii) and (iii) of Proposition 5.1. Let us briefly check that $\mathcal{F}$ also satisfies property (i), i.e. for every $u \in SBV^2(\Omega)$, $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure on $\Omega$ ($L^1$ lower semicontinuity and locality being trivial). Setting $u_k := u \vee (-k) \wedge k$ for any $k \in \mathbb{N}$, we have $\mathcal{F}(u_k, A) \to \mathcal{F}(u, A)$ for all $A \in \mathcal{A}(\Omega)$, being the $\mathcal{F}_\varepsilon$, and hence $\mathcal{F}$, decreasing by truncations. In turn from this we infer that $\mathcal{F}(u, \cdot)$ is monotone on $\mathcal{A}(\Omega)$, sub-additive, and super-additive on disjoint open sets. Moreover, since for every $A \in \mathcal{A}(\Omega)$

$$\mathcal{F}(u, A) \leq \int_A |\nabla u|^2 dx + \mathcal{H}^{N-1}(S_u \cap A)$$

we deduce the inner-regularity of $\mathcal{F}$ by standard arguments (see Ref. 13, Proposition 11.6). By the De Giorgi-Letta's criterion (see Ref. 8, Theorem 1.53), we conclude that $\mathcal{F}$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure on $\Omega$.

Thanks to Proposition 5.1, conditions $(H1)-(H3)$ in Ref. 12, Theorem 1 are satisfied, namely $\mathcal{F}$ is a variational semicontinuous functional on $SBV^2(\Omega) \times \mathcal{A}(\Omega)$ with respect to the $L^1$ topology.

In order to enforce the growth condition from below $(H4)$ let us fix $\delta > 0$ and consider the functional

$$\mathcal{F}^\delta(u, A) = \mathcal{F}(u, A) + \delta \int_A |\nabla u|^2 dx + \delta \int_{S_u \cap A} (1 + |u^+ - u^-|)\, d\mathcal{H}^{N-1}.$$

According to Theorem 1 of Ref. 12 there exist Borel functions $f^\delta : A \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$, $g^\delta : A \times \mathbb{R} \times \mathbb{R} \times \mathbf{S}^{N-1} \to [0, +\infty]$ for which

$$\mathcal{F}^\delta(u, A) = \int_A f^\delta(x, u, \nabla u) dx + \int_{S_u \cap A} g^\delta(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1}$$

for every $A \in \mathcal{A}(\Omega)$ and $u \in SBV^2(A)$.

Thanks to properties (b) and (c) in Proposition 5.1 we conclude that both $f^\delta$ and $g^\delta$ are independent of $x$, that $f^\delta$ does not depend on $u$, and that $g^\delta$ depends on $(u^+, u^-)$ only through their difference so that we may write $g^\delta = g^\delta(u^+ - u^-, \nu)$. By construction the families $(f^\delta)_{\delta>0}$, $(g^\delta)_{\delta>0}$ are increasing in $\delta$, hence we can set $f = \lim_{\delta\to 0^+} f^\delta$, $g = \lim_{\delta \to 0^+} g^\delta$. To conclude we use the pointwise convergence of $(\mathcal{F}^\delta(\cdot, A))_{\delta>0}$ to $\mathcal{F}(\cdot, A)$ and the Monotone Convergence Theorem. □

In the next proposition we identify the bulk density of all $\Gamma$-cluster points of $(\mathcal{F}_\varepsilon)$ to be $f_{\hom}$. We will use the standard notation $[\,\cdot\,]$ for the integer part.

**Proposition 5.3.** *Assume that $(\mathcal{F}_{\varepsilon_n}(\cdot, A))$ $\Gamma$-converges to a functional $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Then for every $\xi \in \mathbb{R}^N$*

$$f(\xi) = f_{\hom}(\xi),$$

*where $f$ is the bulk energy density of $\mathcal{F}(\cdot, A)$, and $f_{\hom}$ is defined in (1.3).*

**Proof.** For the sake of simplicity we assume that the unitary cube $Q$ is contained in $\Omega$. Fix $\xi \in \mathbb{R}^N$, we begin with proving inequality $f(\xi) \le f_{\rm hom}(\xi)$. To this aim consider any $w \in W^{1,2}_\sharp(Q\setminus K)$, extend it to 0 on $K$ and define $w_n(x) = \varepsilon_n w(x/\varepsilon_n)$. We have $(w_n) \subseteq L^2(Q) \cap W^{1,2}(Q\setminus K_{\varepsilon_n})$ and $(w_n)$ converges to 0 in $L^2(Q)$. Moreover, setting $v_n(x) = w_n(x) + \xi \cdot x$, by periodicity and a change of variables it follows

$$\mathcal{F}_{\varepsilon_n}(v_n, Q) = \int_{Q\setminus K_{\varepsilon_n}} |\nabla w(x/\varepsilon_n) + \xi|^2 dx \le \varepsilon_n^N \left(1 + \left[\frac{1}{\varepsilon_n}\right]\right)^N \int_{Q\setminus K} |\nabla w + \xi|^2 dx.$$

Since $(v_n)$ converges to $\xi \cdot x$ in $L^2(Q)$ we deduce

$$f(\xi) = \mathcal{F}(\xi \cdot x, Q) \le \liminf_n \mathcal{F}_{\varepsilon_n}(v_n, Q) \le \int_{Q\setminus K} |\nabla w + \xi|^2 dx,$$

taking the infimum with respect to $w$ we conclude.

The proof of the opposite inequality $f_{\rm hom}(\xi) \le f(\xi)$ will be split into several steps. Let us first deal with regular perforations $K$, namely we assume that $K$ is the closure of an open set with Lipschitz boundary (with $Q\setminus K$ connected).

Consider a sequence $(w_n) \subset L^2(Q)$ converging to $\xi \cdot x$ in $L^2(Q)$ and such that

$$f(\xi) = \mathcal{F}(\xi \cdot x, Q) = \lim_n \mathcal{F}_{\varepsilon_n}(w_n, Q).$$

Since $\mathcal{F}_{\varepsilon_n}$ decreases by truncation we may also assume $\|w_n\|_{L^\infty(Q)} \le \|\xi \cdot x\|_{L^\infty(Q)}$ for every $n \in \mathbb{N}$. We first use a blow-up type argument in order to get from $(w_n)$ a new sequence whose energy has not increased and whose jump set is vanishing (see Ref. 14, Step 1 in Proposition 5.2).

**Step 1.** *Reduction to a recovery sequence with vanishing jumps.* More precisely, we prove that there exist a diverging sequence $(j_n) \subseteq \mathbb{N}$ and $(v_n) \subseteq L^2(Q)$ such that

(i) $(v_n)$ converges to $\xi \cdot x$ in $L^2(Q)$;
(ii) $\|v_n\|_{L^\infty(Q)} \le \|\xi \cdot x\|_{L^\infty(Q)}$ for every $n \in \mathbb{N}$;
(iii) $\lim_n \mathcal{H}^{N-1}(S_{v_n} \cap (Q\setminus K_{1/j_n})) = 0$;
(iv) $\limsup_n F_{1/j_n}(v_n, Q) \le f(\xi)$.

Fix a sequence $(j_n) \subset \mathbb{N}$ to be chosen later, and let $Q_n^{\mathtt{i}} = j_n\varepsilon_n(\mathtt{i} + Q)$ be a cube among those of type $j_n\varepsilon_n(i + Q) \subset Q$, $i \in \mathbb{Z}^N$, satisfying

$$\left[\frac{1}{j_n\varepsilon_n}\right]^N \mathcal{F}_{\varepsilon_n}(w_n, Q_n^{\mathtt{i}}) \le \mathcal{F}_{\varepsilon_n}(w_n, Q). \tag{5.3}$$

Define $v_n \in L^2(Q)$ to be

$$v_n(x) = \frac{1}{j_n\varepsilon_n} w_n(j_n\varepsilon_n(\mathtt{i} + x)) - \xi \cdot \mathtt{i},$$

then a simple change of variables entails

$$\|v_n - \xi \cdot x\|_{L^2(Q)} \le (j_n\varepsilon_n)^{-(1+N/2)} \|w_n - \xi \cdot x\|_{L^2(Q)}. \tag{5.4}$$

It is easy to check that we may choose $(j_n)$ in such a way that $j_n \to +\infty$, $j_n\varepsilon_n \to 0$ and (5.4) vanishes as $n \to +\infty$. So that (i) is established.

Moreover, the choice of $Q_n^{\mathtt{i}}$ in (5.3) implies by changing variables

$$\begin{aligned}\mathcal{H}^{N-1}(S_{v_n}\backslash K_{1/j_n}) &= (j_n\varepsilon_n)^{1-N}\mathcal{H}^{N-1}(S_{w_n}\cap(Q_n^{\mathtt{i}}\backslash K_{\varepsilon_n}))\\ &\le (j_n\varepsilon_n)^{1-N}\left[\frac{1}{j_n\varepsilon_n}\right]^{-N}\mathcal{F}_{\varepsilon_n}(w_n,Q)\end{aligned}$$

and

$$\begin{aligned}\int_{Q\backslash K_{1/j_n}}|\nabla v_n|^2dx &= (j_n\varepsilon_n)^{-N}\int_{Q_n^{\mathtt{i}}\backslash K_{\varepsilon_n}}|\nabla w_n|^2dx\\ &\le (j_n\varepsilon_n)^{-N}\left[\frac{1}{j_n\varepsilon_n}\right]^{-N}\mathcal{F}_{\varepsilon_n}(w_n,Q),\end{aligned}$$

from which we deduce (iii) and (iv), respectively.

Eventually, statement (ii) follows by truncating $v_n$ at levels $\pm\|\xi\cdot x\|_{L^\infty(Q)}$.

Next we refine the recovery sequence to obtain a sequence with Sobolev regularity. To do this we employ by now standard techniques to truncate gradients.

**Step 2.** *Reduction to a recovery sequence in Sobolev spaces.* In this step we prove that for every fixed cube $Q' \subset\subset Q$ there exists $(u_n) \subseteq W^{1,2}(Q')$ such that

(i′) $(u_n)$ converges to $\xi\cdot x$ in $L^2(Q')$;
(ii′) $\|u_n\|_{L^\infty(Q')} \le \|\xi\cdot x\|_{L^\infty(Q)}$ for every $n\in\mathbb{N}$;
(iv′) $\limsup_n F_{1/j_n}(u_n,Q') \le f(\xi)$.

Reproducing the beginning of the proof of Ref. 38, Lemma 2.1 in a varying Lipschitz domain, we can modify $v_n$ in order to construct a function $\tilde{v}_n \in W^{1,\infty}(Q\backslash K_{1/j_n})$ such that

$$\lim_n \mathcal{L}^N(\{x\in Q\backslash K_{1/j_n} : \tilde{v}_n(x) \ne v_n(x)\}) = 0 \tag{5.5}$$

and

$$\sup_n \int_{Q\backslash K_{1/j_n}} |\nabla\tilde{v}_n|^2dx < +\infty.$$

Up to a truncation argument, thanks to Step 1(ii), we may assume also that $\|\tilde{v}_n\|_{L^\infty(Q)} \le \|\xi\cdot x\|_{L^\infty(Q)}$. Furthermore, by taking advantage of the connectedness of $Q\backslash K$ and of the Lipschitz regularity assumption on $K$ we employ classical extension results to fill the holes (see Theorem 2.1 of Ref. 1, and also Ref. 23). More precisely, with fixed $Q' \subset\subset Q$ we extend $\tilde{v}_n$ to the full $Q'$ (we keep the notation $\tilde{v}_n$ for the extended function) with $\tilde{v}_n \in W^{1,2}(Q')$ and $\sup_n\|\tilde{v}_n\|_{W^{1,2}(Q')} < +\infty$. Then Lemma 1.2 of Ref. 33 provides a sequence $(u_n) \subset W^{1,2}(Q')$ such that

$$\lim_n \mathcal{L}^N(\{x\in Q' :\, \tilde{v}_n(x) \ne u_n(x)\}) = 0, \tag{5.6}$$

and $(|\nabla u_n|^2)$ is equi-integrable on $Q'$. Up to the usual truncation argument we may assume also that $\|u_n\|_{L^\infty(Q')} \le \|\xi\cdot x\|_{L^\infty(Q)}$.

By collecting (5.5) and (5.6) we infer

$$\lim_n \mathcal{L}^N(\{x \in Q'\backslash K_{1/j_n} : u_n(x) \neq v_n(x)\}) = 0. \tag{5.7}$$

Since $(|\nabla u_n|^2)$ is equi-integrable, by Step 1(iii) and (iv) we get

$$\begin{aligned}\limsup_n \int_{Q'\backslash K_{1/j_n}} |\nabla u_n|^2 dx &= \limsup_n \int_{(Q'\backslash K_{1/j_n})\backslash\{u_n\neq v_n\}} |\nabla u_n|^2 dx \\ &= \limsup_n \int_{(Q'\backslash K_{1/j_n})\backslash\{u_n\neq v_n\}} |\nabla v_n|^2 dx \\ &\leq \limsup_n \int_{Q\backslash K_{1/j_n}} |\nabla v_n|^2 dx \\ &\leq f(\xi),\end{aligned}$$

so that (iv′) is established.

Let us pass to the proof of (i′). Given any subsequence of $(u_n)$ by Sobolev embedding we may extract a further subsequence $(u_{j_n})$ converging to a function $u$ in $L^2(Q')$. Set $\varphi_n = \chi_{(Q'\backslash K_{1/j_n})\backslash\{u_{j_n}\neq v_{j_n}\}}$, then by (5.7) (see also Remark 3.1) $(\varphi_n)$ converges to $1 - \mathcal{L}^N(K)$ weak$^*$ $L^\infty(Q')$. By taking into account Step 1(i), $(\varphi_n(u_{j_n} - v_{j_n}))$ converges to $(1 - \mathcal{L}^N(K))(u - \xi \cdot x)$ weak $L^1(Q')$, and since $\varphi_n(u_{j_n} - v_{j_n}) = 0\, \mathcal{L}^N$ a.e. on $Q'$ we deduce that $u = \xi \cdot x\, \mathcal{L}^N$ a.e. on $Q'$. Furthermore, Urysohn property implies (i′), i.e. the whole sequence $(u_n)$ converges to $\xi \cdot x$ in $L^2(Q')$. This concludes the proof of Step 2.

**Step 3.** *Conclusion.* Let us first prove $f_{\text{hom}}(\xi) \leq f(\xi)$ for $K$ Lipschitz regular. In this case the classical homogenization result for Sobolev spaces in perforated domains (see Ref. 13, Theorem 19.1) and Step 2 entail

$$\mathcal{L}^N(Q') f_{\text{hom}}(\xi) \leq \liminf_n F_{1/j_n}(u_n, Q') \leq f(\xi).$$

The thesis follows as $\mathcal{L}^N(Q') \to 1^-$.

Finally we recover the general case (without assuming further regularity on $K$) through an approximation argument. More precisely consider a generic closed set $K$ (with $Q\backslash K$ connected) and let $(C^m)$ be a sequence as in Lemma 4.3. Let $f^m_{\text{hom}} : \mathbb{R}^N \to [0, +\infty]$ be defined as $f_{\text{hom}}$ in (1.3) with $K$ there substituted by $C^m$, i.e.

$$f^m_{\text{hom}}(\xi) = \inf\left\{\int_{Q\backslash C^m} |\nabla w + \xi|^2 : w \in W^{1,2}_\sharp(Q\backslash C^m)\right\}.$$

It is clear that $f^m_{\text{hom}} \leq f^{m+1}_{\text{hom}} \leq f_{\text{hom}}$, we claim that

$$\sup_m f^m_{\text{hom}} = f_{\text{hom}}. \tag{5.8}$$

Indeed, for every $m \in \mathbb{N}$ let $w_m \in W^{1,2}_\sharp(Q\backslash C^m)$, with $\int_{Q\backslash C^1} w_m dx = 0$, be such that

$$\int_{Q\backslash C^m} |\nabla w_m + \xi|^2 dx \leq f^m_{\text{hom}}(\xi) + \frac{1}{m}.$$

Note that for every fixed $M > 0$

$$\sup_{m\geq M}\int_{Q\setminus C^M}|\nabla w_m+\xi|^2dx\leq f_{\hom}(\xi)+1<+\infty.$$

In particular, the sequence $(w_m)_{m\geq M}$ is bounded in $W^{1,2}(Q\setminus C^M)$ by Poincaré–Wirtinger inequality for every $M$. Then a diagonal argument implies the existence of a subsequence $(w_{j_m})$ weakly pre-compact in $W^{1,2}(Q\setminus C^M)$ for every $M$. Denote by $w$ a cluster point, then $w\in W^{1,2}_\sharp(Q\setminus C^M)$ for every $M$ and

$$\int_{Q\setminus C^M}|\nabla w+\xi|^2dx\leq\liminf_m\int_{Q\setminus C^M}|\nabla w_{j_m}+\xi|^2dx\leq\sup_m f^m_{\hom}(\xi).$$

By letting $M\to+\infty$ we infer that actually $w\in L^2_{\mathrm{loc}}(Q\setminus K)$, $\nabla w\in L^2(Q\setminus K,\mathbb{R}^N)$ and

$$\int_{Q\setminus K}|\nabla w+\xi|^2dx\leq\sup_m f^m_{\hom}(\xi). \tag{5.9}$$

In particular, it is easy to check that the truncated functions $w^j=(w\wedge j)\vee(-j)$ belong to $W^{1,2}_\sharp(Q\setminus K)$ and for every $M$

$$\begin{aligned}f_{\hom}(\xi)&\leq\int_{Q\setminus K}|\nabla w^j+\xi|^2dx\\&=\int_{(Q\setminus K)\setminus\{|w|\geq j\}}|\nabla w+\xi|^2dx+|\xi|^2\mathcal{L}^N((Q\setminus K)\cap\{|w|\geq j\})\\&\leq\int_{Q\setminus K}|\nabla w+\xi|^2dx+|\xi|^2(\mathcal{L}^N((Q\setminus C^M)\cap\{|w|\geq j\})+\mathcal{L}^N(C^M\setminus K)).\end{aligned}$$

Since $w\in L^2(Q\setminus C^M)$ we have $\mathcal{L}^N((Q\setminus C^M)\cap\{|w|\geq j\})\to 0$ as $j\to+\infty$, so that

$$f_{\hom}(\xi)\leq\int_{Q\setminus K}|\nabla w+\xi|^2dx+|\xi|^2\mathcal{L}^N(C^M\setminus K). \tag{5.10}$$

Eventually from (5.9) and (5.10) we deduce equality (5.8) as $M\to+\infty$.

Finally denote by $\mathcal{F}^m_{\varepsilon_n}$ the functional defined in (5.1) with $C^m$ in place of $K$, then $\mathcal{F}^m_{\varepsilon_n}\leq\mathcal{F}_{\varepsilon_n}$. Up to extracting a further subsequence we assume that $(\mathcal{F}^m_{\varepsilon_n}(\cdot,A))$ $\Gamma$-converges to a functional $F^m(\cdot,A)$ for every $A\in\mathcal{A}(\Omega)$. By Steps 1 and 2 we know that the bulk energy density of $F^m$ is $f^m_{\hom}$, and by construction $f^m_{\hom}(\xi)\leq f(\xi)$ for every $m\in\mathbb{N}$. Hence, we derive $f_{\hom}(\xi)\leq f(\xi)$ from (5.8). □

**Remark 5.1.** The argument above entails the existence of a minimizer for the minimum problem defining $f_{\hom}$ in (1.3) in a suitable Deny–Lions type space (see Ref. 31).

In order to prove the counterpart of Proposition 5.3 for the surface term we first show that the limit defining $g_{\hom}$ exists. To this aim we introduce some extension procedure.

Given any $\nu \in \mathbf{S}^{N-1}$, let $\{\nu_1, \dots, \nu_{N-1}\}$ any collection of unitary vectors such that $\{\nu_1, \dots, \nu_{N-1}, \nu\}$ form an orthonormal basis of $\mathbb{R}^N$ with unit cell $Q^\nu$. Given $w \in P(Q^\nu \backslash K_\varepsilon)$ such that $w = u_{0,1,\nu}$ (defined in (1.4)) on a neighborhood of $\partial Q^\nu$, we regard $w$ as extended to $\mathbb{R}^N$ as follows. First we extend it on $Q^\nu$ by setting $w \equiv u_{0,1,\nu}$ in $K_\varepsilon$, then on the strip $\mathcal{S} = \{x \in \mathbb{R}^N : |\langle x, \nu\rangle| \le 1/2\}$ by 1-periodicity in directions $\nu_1, \dots, \nu_{N-1}$, and finally we set $w \equiv u_{0,1,\nu}$ on $\{x \in \mathbb{R}^N : |\langle x, \nu\rangle| \ge 1/2\}$.

**Lemma 5.1.** *For every $\nu \in \mathbf{S}^{N-1}$ there exists the limit as $\varepsilon \to 0^+$ of $m_\varepsilon(\nu)$, where*

$$m_\varepsilon(\nu) = \inf_{w \in P(Q^\nu \backslash K_\varepsilon)} \{\mathcal{H}^{N-1}(S_w \backslash K_\varepsilon) : w = u_{0,1,\nu} \text{ on a neighborhood of } \partial Q^\nu\}.$$

**Proof.** Let $\varepsilon, \sigma, \eta > 0$ be fixed, with $\sigma \le \varepsilon$, and let $\nu_1, \dots, \nu_{N-1}$ be unitary vectors as above. Fix $w \in P(Q^\nu \backslash K_\varepsilon)$ such that $w = u_{0,1,\nu}$ on a neighborhood of $\partial Q^\nu$, and regard it as extended to $\mathbb{R}^N$ as explained above.

Consider the strip $\mathcal{S}_{\sigma/\varepsilon} = \{x \in \mathbb{R}^N : |\langle x, \nu\rangle| \le \sigma/(2\varepsilon)\}$ and its decomposition into cubes of the family $\Lambda = \left\{\frac{\sigma}{\varepsilon}(i + Q^\nu) : i \in \oplus_{k=1}^{N-1} \nu_k \mathbb{Z}\right\}$, where $\oplus_{k=1}^{N-1} \nu_k \mathbb{Z}$ is the $N-1$-dimensional integer lattice generated by $\nu_1, \dots, \nu_{N-1}$. Moreover let $\mathcal{I} = \left\{i \in \oplus_{k=1}^{N-1} \nu_k \mathbb{Z} : \frac{\sigma}{\varepsilon}(i + Q^\nu) \subset \eta Q^\nu\right\}$, then a simple counting argument gives

$$\#\mathcal{I} \le \left(\frac{\varepsilon\eta}{\sigma}\right)^{N-1}. \tag{5.11}$$

Define $w_\sigma : Q^\nu \to \{0, 1\}$ by $u_{0,1,\nu}$ on $Q^\nu \backslash \mathcal{S}_{\sigma/\varepsilon}$ and on each cube of the family $\Lambda$ intersecting $Q^\nu \backslash \eta Q^\nu$, and let $w_\sigma(x) = w(\varepsilon x/\sigma)$ otherwise on $\mathcal{S}_{\sigma/\varepsilon}$.

By construction $w_\sigma \in P(Q^\nu \backslash K_\sigma)$ and $w_\sigma = u_{0,1,\nu}$ on $Q^\nu \backslash \eta Q^\nu$, and since $S_{w_\sigma} \cap (Q^\nu \backslash \eta Q^\nu) \subseteq \{x \in \mathbb{R}^N : \langle x, \nu\rangle = 0\} \cap (Q^\nu \backslash \eta Q^\nu)$, we have $\mathcal{H}^{N-1}(S_{w_\sigma} \cap (Q^\nu \backslash \eta Q^\nu)) \le 1 - \eta^{N-1}$. Furthermore (5.11), the 1-periodicity of $w$ in directions $\nu_1, \dots, \nu_{N-1}$, and a scaling argument imply

$$\begin{aligned} \mathcal{H}^{N-1}(S_{w_\sigma} \backslash K_\sigma) &\le \#\mathcal{I}\left(\frac{\sigma}{\varepsilon}\right)^{N-1} \mathcal{H}^{N-1}(S_w \backslash K_\varepsilon) + 1 - \eta^{N-1} \\ &\le \eta^{N-1} \mathcal{H}^{N-1}(S_w \backslash K_\varepsilon) + 1 - \eta^{N-1}. \end{aligned} \tag{5.12}$$

Passing to the infimum on the class of admissible functions on both sides of (5.12) and then on the superior limit as $\sigma \to 0^+$ and the inferior limit as $\varepsilon \to 0^+$ we infer

$$\limsup_{\sigma \to 0^+} m_\sigma(\nu) \le \eta^{N-1} \liminf_{\varepsilon \to 0^+} m_\varepsilon(\nu) + 1 - \eta^{N-1},$$

and the thesis follows as $\eta \to 1^-$. □

In the next proposition we identify the surface density of all Γ-cluster points of $(\mathcal{F}_\varepsilon)$ to be $g_{\text{hom}}$.

**Proposition 5.4.** *Assume that $(\mathcal{F}_{\varepsilon_n}(\cdot, A))$ Γ-converges to a functional $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Then for every $(a, b, \nu) \in \mathbb{R} \times \mathbb{R} \times \mathbf{S}^{N-1}$*

$$g(b - a, \nu) = g_{\text{hom}}(\nu),$$

*where $g$ is the surface energy density of $\mathcal{F}(\cdot, A)$ and $g_{\text{hom}}$ is defined in (1.5).*

**Proof.** Fix $(a,b,\nu) \in \mathbb{R} \times \mathbb{R} \times \mathbf{S}^{N-1}$. We start with inequality $g(b-a,\nu) \le g_{\rm hom}(\nu)$.

To this aim fixed $\varepsilon > 0$ consider any $w \in P(Q^\nu \setminus K_\varepsilon)$ such that $w = u_{0,1,\nu}$ on a neighborhood of $\partial Q^\nu$, regarded as extended to $\mathbb{R}^N$ with the convention adopted before Lemma 5.1. Define $w_n(x) = a + (b-a)w(\varepsilon x/\varepsilon_n)$, then a simple change of variables gives

$$\|w_n - u_{a,b,\nu}\|_{L^2(Q^\nu)} \le |b-a| \left(\frac{\varepsilon_n}{\varepsilon}\right)^{N/2} \left(1 + \left[\frac{\varepsilon}{\varepsilon_n}\right]\right)^{(N-1)/2} \|w - u_{0,1,\nu}\|_{L^2(Q^\nu)},$$

so that $(w_n)$ converges to $u_{a,b,\nu}$ in $L^2(Q^\nu)$. Moreover, a straightforward calculation implies

$$\begin{aligned} \mathcal{F}_{\varepsilon_n}(w_n, Q^\nu) &= \mathcal{H}^{N-1}(S_{w_n} \cap (Q^\nu \setminus K_{\varepsilon_n})) \\ &= \mathcal{H}^{N-1}(S_{w_n} \cap \{x \in Q^\nu \setminus K_{\varepsilon_n} : |\langle x, \nu\rangle| \le \varepsilon_n/(2\varepsilon)\}) \\ &\le \left(\frac{\varepsilon_n}{\varepsilon}\right)^{N-1} \left(1 + \left[\frac{\varepsilon}{\varepsilon_n}\right]\right)^{N-1} \mathcal{H}^{N-1}(S_w \cap (Q^\nu \setminus K_\varepsilon)). \end{aligned}$$

Taking the limit as $n \to +\infty$ we infer

$$\mathcal{F}(u_{a,b,\nu}, Q^\nu) \le \liminf_n \mathcal{F}_{\varepsilon_n}(w_n, Q^\nu) \le \mathcal{H}^{N-1}(S_w \cap (Q^\nu \setminus K_\varepsilon)),$$

by passing first to the infimum on all such $w$'s and then to the limit as $\varepsilon \to 0^+$ inequality $g(b-a,\nu) \le g_{\rm hom}(\nu)$ follows by Lemma 5.1.

The proof of the opposite inequality $g_{\rm hom}(\nu) \le g(b-a,\nu)$ will be split into three steps. To fix notations we will assume $a \le b$. Consider a sequence $(w_n) \subset L^2(Q^\nu)$ converging to $u_{a,b,\nu}$ in $L^2(Q^\nu)$ and such that

$$g(b-a,\nu) = \mathcal{F}(u_{a,b,\nu}, Q^\nu) = \lim_n \mathcal{F}_{\varepsilon_n}(w_n, Q^\nu).$$

By a truncation argument we may also assume $a \le w_n \le b$ for every $n \in \mathbb{N}$. First we use a blow-up type argument as in Proposition 6.2 of Ref. 14, in order to get from $(w_n)$ a new sequence whose energy has not increased in the limit and whose gradient energy is vanishing.

**Step 1.** *Reduction to a recovery sequence with vanishing gradients.* We prove that there exist a diverging sequence $(j_n) \in \mathbb{N}$ and $(v_n) \in L^2(Q^\nu)$ such that

(i) $(v_n)$ converges to $u_{a,b,\nu}$ in $L^2(Q^\nu)$;
(ii) $a \le v_n \le b$ for every $n \in \mathbb{N}$;
(iii) $\lim_n \int_{Q^\nu \setminus K_{1/j_n}} |\nabla v_n|^2 dx = 0$;
(iv) $\limsup_n F_{1/j_n}(v_n, Q^\nu) \le g(b-a,\nu)$.

Fix a sequence $(j_n) \subset \mathbb{N}$ to be chosen later, and let $Q_n^{\mathtt{i}} = j_n \varepsilon_n(\mathtt{i} + Q^\nu)$ be a cube among those of type $j_n \varepsilon_n(i + Q^\nu) \subset Q^\nu$, $i \in \oplus_{k=1}^{N-1} \nu_k \mathbb{Z}$, satisfying

$$\left[\frac{1}{j_n \varepsilon_n}\right]^{N-1} \mathcal{F}_{\varepsilon_n}(w_n, Q_n^{\mathtt{i}}) \le \mathcal{F}_{\varepsilon_n}(w_n, Q^\nu). \tag{5.13}$$

Define $v_n \in L^2(Q^\nu)$ to be $v_n(x) = w_n(j_n\varepsilon_n(\mathtt{i}+x))$, then a simple change of variables entails

$$\|v_n - u_{a,b,\nu}\|_{L^2(Q^\nu)} \leq (j_n\varepsilon_n)^{-N/2}\|w_n - u_{a,b,\nu}\|_{L^2(Q^\nu)}. \tag{5.14}$$

It is easy to check that we may choose $(j_n)$ in such a way that $j_n \to +\infty$, $j_n\varepsilon_n \to 0$ and (5.14) vanishes as $n \to +\infty$. So that (i) is established.

Moreover, the choice of $Q_n^{\mathtt{i}}$ in (5.13) implies by changing variables

$$\begin{aligned}\mathcal{H}^{N-1}(S_{v_n}\backslash K_{1/j_n}) &= (j_n\varepsilon_n)^{1-N}\mathcal{H}^{N-1}(S_{w_n}\cap(Q_n^{\mathtt{i}}\backslash K_{\varepsilon_n}))\\ &\leq (j_n\varepsilon_n)^{1-N}\left[\frac{1}{j_n\varepsilon_n}\right]^{1-N}\mathcal{F}_{\varepsilon_n}(w_n, Q)\end{aligned}$$

and

$$\begin{aligned}\int_{Q\backslash K_{1/j_n}}|\nabla v_n|^2dx &= (j_n\varepsilon_n)^{2-N}\int_{Q_n^{\mathtt{i}}\backslash K_{\varepsilon_n}}|\nabla w_n|^2dx\\ &\leq (j_n\varepsilon_n)^{2-N}\left[\frac{1}{j_n\varepsilon_n}\right]^{1-N}\mathcal{F}_{\varepsilon_n}(w_n, Q),\end{aligned}$$

from which we deduce (iii) and (iv), respectively. Eventually, statement (ii) follows straightforward.

In the next step the $BV$ Coarea Formula (see Ref. 8, Theorem 3.40) allows us to select suitable sublevels of the sequence in Step 1 whose perimeters is controlled by the energy functionals (see Ref. 14, Proposition 6.2). Subsequently we use a geometric truncation argument, similar to that called *transfer of jump set* performed in Ref. 34, in order to obtain a sequence in $SBV_0$ matching the boundary conditions.

**Step 2.** *Reduction to a recovery sequence in $SBV_0(Q^\nu)$ satisfying the boundary conditions.* We prove that there exists $(\widehat{v}_n) \in SBV_0(Q^\nu)$ such that

(i′) $(\widehat{v}_n)$ converges to $u_{a,b,\nu}$ in $L^2(Q^\nu)$;
(ii′) $\widehat{v}_n$ assumes only the values $a, b$ for every $n \in \mathbb{N}$;
(iii′) $\widehat{v}_n = u_{a,b,\nu}$ on a neighborhood of $\partial Q^\nu$;
(iv′) $\limsup_n F_{1/j_n}(\hat{v}_n, Q^\nu) \leq g(b-a, \nu)$.

Indeed, let us consider the sets $E_t^n = \{x \in Q^\nu : v_n(x) < t\}$, $E_t = \{x \in Q^\nu : u_{a,b,\nu}(x) < t\}$. Thanks to property (i) of Step 1 $E_t^n \to E_t$ in measure for $\mathcal{H}^1$ a.e. $t$ and the $BV$ Coarea Formula (see Ref. 8, Theorem 3.40) yields

$$\int_a^b \mathcal{H}^{N-1}(\partial^* E_s^n\backslash K_{1/j_n})\,ds \leq |Dv_n|(Q^\nu\backslash K_{1/j_n}). \tag{5.15}$$

Note that the absolute continuous part of $|Dv_n|(Q^\nu\backslash K_{1/j_n})$ can be estimated by using the Hölder inequality, while for the singular part is sufficient to take into account that thanks to property (ii) of Step 1 $|v_n^+ - v_n^-| \leq (b-a)$. Hence we can refine inequality (5.15) and obtain

$$\int_a^b \mathcal{H}^{N-1}(\partial^* E_s^n \backslash K_{1/j_n})\, ds \le (b-a) F_{1/j_n}(v_n, Q^\nu) + \left( \int_{Q^\nu \backslash K_{1/j_n}} |\nabla v_n|^2 dx \right)^{1/2}. \tag{5.16}$$

By using the Mean Value Theorem in (5.15) and by using property (iii) of Step 1 in (5.16), we may choose $s_n \in (a,b)$ such that we have convergence in measure of the sublevels $E_{s_n}^n$ and

$$\limsup_n \mathcal{H}^{N-1}(\partial^* E_{s_n}^n \backslash K_{1/j_n}) \le \limsup_n F_{1/j_n}(v_n, Q^\nu). \tag{5.17}$$

Set $E_n = E_{s_n}^n$. Taking into account that $u_{a,b,\nu}$ is piecewise constant in $Q^\nu$ we easily infer that $E_n$ tends in measure to the lower half cube. Let us now fix $\eta \in (0, 1/2)$ and set

$$Q_\eta^- := \{x \in Q^\nu :\ -\eta < \langle x, \nu \rangle < 0\}, \quad Q_\eta^+ := \{x \in Q^\nu : 0 < \langle x, \nu \rangle < \eta\}.$$

Since $E_n \cap Q_\eta^-$ tends to $Q_\eta^-$ in measure and $E_n \cap Q_\eta^+$ tends to zero in measure, recalling that $\mathcal{L}^N(Q_\eta^-) = \mathcal{L}^N(Q_\eta^+) = \eta$, for $n$ large enough, we have that

$$\mathcal{L}^N(E_n \cap Q_\eta^-) > \eta - \eta^2, \quad \mathcal{L}^N(E_n \cap Q_\eta^+) \le \eta^2.$$

Therefore, thanks to Fubini's Theorem we may find $s^-$, $s^+$ in a set of positive measure in $(0, \eta)$ such that

$$\mathcal{H}^{N-1}(E_n \cap \{x \in Q^\nu : \langle x, \nu \rangle = -s^-\}) \ge 1 - \eta,$$

$$\mathcal{H}^{N-1}(E_n \cap \{x \in Q^\nu : \langle x, \nu \rangle = s^+\}) \le \eta.$$

Finally set

$$H_n^- := \{x \in Q^\nu : \operatorname{dist}(x, \partial Q^\nu) \le \eta,\ -s^- \le \langle x, \nu \rangle \le 0\},$$
$$H_n^+ := \{x \in Q^\nu : \operatorname{dist}(x, \partial Q^\nu) \le \eta,\ 0 < \langle x, \nu \rangle < s^+\},$$

and consider functions $\widehat{v}_n^\eta$ defined by

$$\widehat{v}_n^\eta(x) := \begin{cases} a & \text{in } \{x \in Q^\nu : \langle x, \nu \rangle \le -s^-\} \cup (Q_\eta^- \cap E_n) \cup ((Q_\eta^+ \cap E_n) \backslash H_n^+) \cup H_n^-, \\ b & \text{everywhere else in } Q^\nu. \end{cases}$$

By construction $\widehat{v}_n^\eta \in SBV_0(Q^\nu)$ and property (i′), (ii′), (iii′) are satisfied. In addition, by taking into account (5.17) we get

$$\begin{aligned} \limsup_n F_{1/j_n}(\hat{v}_n^\eta, Q^\nu) &= \limsup_n \mathcal{H}^{N-1}(S_{\hat{v}_n^\eta} \backslash K_{1/j_n}) \\ &\le \limsup_n \mathcal{H}^{N-1}(\partial^* E_{s_n}^n \backslash K_{1/j_n}) + O(\eta) \\ &\le \limsup_n F_{1/j_n}(v_n, Q^\nu) + O(\eta), \end{aligned}$$

where $O(\eta) \to 0$ as $\eta \to 0^+$. Finally we get a sequence $(\widehat{v}_n)$ satisfying property (iv′), as well as (i′), (ii′), (iii′), by taking a positive vanishing sequence $(\eta_n)$ and a standard diagonalization argument.

**Step 3.** *Conclusion.* Let $u_n = (\widehat{v}_n - a)/(b-a)$. Then $u_n$ coincides with $u_{0,1,\nu}$ on a neighborhood of $\partial Q^\nu$ and converges to $u_{0,1,\nu}$ in $L^2(Q^\nu)$. Eventually

$$g_{\text{hom}}(\nu) \leq \limsup_n \mathcal{H}^{N-1}(S_{u_n} \backslash K_{1/j_n}) = \limsup_n F_{1/j_n}(\hat{v}_n, Q^\nu) \leq g(b-a, \nu). \qquad \square$$

Summarizing, we have proved that, if $(\mathcal{F}_{\varepsilon_n})$ $\Gamma$-converges to $\mathcal{F}$, then the bulk and the surface energy densities of $\mathcal{F}$ are given by the functions $f_{\text{hom}}$ and $g_{\text{hom}}$ defined in (1.3) and (1.5), respectively. This clearly implies that the $\Gamma$-limit of the whole family $(\mathcal{F}_\varepsilon)$ is the functional $\mathcal{F}_{\text{hom}}$ defined in (5.2). Indeed, the cluster points of $(\mathcal{F}_\varepsilon)$ do not depend on the particular $\Gamma$-convergent subsequence, and so Urysohn's property enforces the conclusion. The proof of Theorem 5.1 is then complete.

## 6. Matching Boundary Conditions

In this section we extend our asymptotic analysis adding a Dirichlet boundary condition on the fixed boundary $\partial\Omega$. We present a $\Gamma$-convergence result for (suitable restrictions of) the functionals $\mathcal{F}_\varepsilon^\psi$ defined in (3.1) and prove the convergence of the associated minimum problems. This last result will be a consequence of standard $\Gamma$-convergence theory once the equicoercivity of the associated minimum configurations is proved (see Ref. 25, Theorem 7.4).

Since we are interested mainly in the asymptotic behavior of minimizers we restrict ourselves to the domain $SBV^2(\Omega) \cap L^2(\Omega)$. Indeed, as already mentioned at the beginning of Sec. 4, the functionals $\mathcal{F}_\varepsilon^\psi$ are decreasing by truncation, and thus we can limit our analysis to functions equibounded in $L^\infty(\Omega)$. According to this, we investigate the $\Gamma$-convergence of $(\mathcal{F}_\varepsilon^\psi)$ on the $L^1$-subspace $SBV^2(\Omega) \cap L^2(\Omega)$. In this respect, it is also clear that the convergence property is not affected by the choice of any $L^p$ topology in which the study of the $\Gamma$-limit is set.

We begin with the $\Gamma$-convergence analysis. It exploits the result in the unconstrained case proved in Theorem 5.1.

**Theorem 6.1.** *The family $(\mathcal{F}_\varepsilon^\psi)$ $\Gamma$-converges to some functional $\mathcal{F}_{\text{hom}}^\psi$ with respect to the $L^2(\Omega)$ topology. Moreover, the functional $\mathcal{F}_{\text{hom}}^\psi$ restricted to $SBV^2(\Omega)$ is given by*

$$\mathcal{F}_{\text{hom}}^\psi(u) := \int_\Omega f_{\text{hom}}(\nabla u)\, dx + \int_{S_u^\psi} g_{\text{hom}}(\nu_u)\, d\mathcal{H}^{N-1},$$

*where $S_u^\psi := S_u \cup \{x \in \partial\Omega : \psi(x) \neq u(x)\}$, and $f_{\text{hom}}$, $g_{\text{hom}}$ are defined in (1.3) and (1.5), respectively.*

**Proof.** Consider an open set $\tilde{\Omega}$ with $\Omega \subset\subset \tilde{\Omega}$, and let $\tilde{\mathcal{F}}_\varepsilon : SBV^2(\tilde{\Omega}) \cap L^2(\tilde{\Omega}) \to [0, +\infty]$ be defined as in (5.1) with $A$ replaced by $\tilde{\Omega}$. By Theorem 5.1 we have that the functionals $\tilde{\mathcal{F}}_\varepsilon$ $\Gamma$-converge to the functional $\tilde{\mathcal{F}}_{\text{hom}}$ defined as in (5.2) with $A$ replaced by $\tilde{\Omega}$.

In order to prove the Γ-lim inf inequality for the functionals $\mathcal{F}^{\psi}_{\varepsilon}$, let $u_{\varepsilon} \to u$ in $L^2(\Omega)$, and set $\tilde{u}_{\varepsilon}$ (respectively $\tilde{u}$) equal to $u_{\varepsilon}$ (respectively $u$) in $\Omega$, and equal to $\psi$ in $\tilde{\Omega}\setminus\Omega$. Taking into account that $\psi \in W^{1,2}(\tilde{\Omega})$, we have that

$$\tilde{\mathcal{F}}_{\varepsilon}(\tilde{u}_{\varepsilon}) \le \mathcal{F}^{\psi}_{\varepsilon}(\tilde{u}_{\varepsilon}) + \int_{\tilde{\Omega}\setminus\Omega} |\nabla\psi|^2\, dx,$$

and thus by the Γ-lim inf inequality for the functionals $\tilde{\mathcal{F}}_{\varepsilon}$ we get

$$\mathcal{F}^{\psi}_{\text{hom}}(u) \le \tilde{\mathcal{F}}_{\text{hom}}(\tilde{u}) \le \liminf \mathcal{F}^{\psi}_{\varepsilon}(\tilde{u}_{\varepsilon}) + \int_{\tilde{\Omega}\setminus\Omega} |\nabla\psi|^2\, dx.$$

We deduce the Γ-lim inf inequality for the family $(\mathcal{F}^{\psi}_{\varepsilon})$ by absolute continuity of Lebesgue integral by letting $\tilde{\Omega}$ decrease to $\Omega$.

Let us pass to the Γ-lim sup inequality. To this aim let $u \in SBV^2(\Omega) \cap L^2(\Omega)$ and $\tilde{u}$ be its extension to $\tilde{\Omega}$ defined to be equal to $\psi$ in $\tilde{\Omega}\setminus\Omega$. Taking into account the fundamental estimate (see Ref. 14, Proposition 3.1) it is easy to infer the existence of a recovery sequence $(\tilde{u}_{\varepsilon})$ for the functionals $\tilde{\mathcal{F}}_{\varepsilon}$ satisfying

$$\lim \tilde{\mathcal{F}}_{\varepsilon}(\tilde{u}_{\varepsilon}) = \tilde{\mathcal{F}}_{\text{hom}}(\tilde{u}),$$

with $\tilde{u}_{\varepsilon} \equiv \psi$ on $\tilde{\Omega}\setminus\Omega$. Therefore, setting $u_{\varepsilon}$ to be the restriction of $\tilde{u}_{\varepsilon}$ to $\Omega$ we have

$$\limsup \mathcal{F}^{\psi}_{\varepsilon}(u_{\varepsilon}) \le \lim \tilde{\mathcal{F}}_{\varepsilon}(\tilde{u}_{\varepsilon}) = \tilde{\mathcal{F}}_{\text{hom}}(\tilde{u}) = \mathcal{F}^{\psi}_{\text{hom}}(u) + \int_{\tilde{\Omega}\setminus\Omega} f_{\text{hom}}(\nabla\psi)dx.$$

Again, since the term $\int_{\tilde{\Omega}\setminus\Omega} f_{\text{hom}}(\nabla\psi)dx$ can be chosen arbitrarily small, we deduce the Γ-lim sup inequality for the functionals $\mathcal{F}^{\psi}_{\varepsilon}$. □

Before investigating the convergence of the minimum problems associated to $\mathcal{F}^{\psi}_{\varepsilon}$, we recall that for any $u \in L^1(\Omega)$ the value $\mathcal{F}^{\psi}_{\varepsilon}(u)$ (as well as $\mathcal{F}_{\varepsilon}(u)$) is not affected by that of $u$ in the sets $\Omega\setminus\Omega_{\varepsilon}$. Due to this fact, a real compactness result for sequences of minimizers cannot hold unless $K$ is negligible. Hence, in the general case, the next theorem can be thought as a selection principle of compact minimizing sequences in $L^1(\Omega)$. We recall also that, since the energy functionals decrease by truncations, we can always assume that the minimizers $u_{\varepsilon}$ satisfy $\|u_{\varepsilon}\|_{L^{\infty}(\Omega)} \le \|\psi\|_{L^{\infty}(\Omega)}$.

**Theorem 6.2.** *For any $\varepsilon > 0$ let $u_{\varepsilon} \in L^1(\Omega_{\varepsilon})$ be a minimizer for $\mathcal{F}^{\psi}_{\varepsilon}$ with $\|u_{\varepsilon}\|_{L^{\infty}(\Omega)} \le \|\psi\|_{L^{\infty}(\Omega)}$. Then there exists a family $(w_{\varepsilon}) \subset L^1(\Omega)$ which is compact in $L^1(\Omega)$ and such that $w_{\varepsilon} \equiv u_{\varepsilon}$ in $\Omega_{\varepsilon}$ for any $\varepsilon > 0$ (in particular $w_{\varepsilon}$ are minimizers for $\mathcal{F}^{\psi}_{\varepsilon}$). Moreover, any cluster point $u$ of $w_{\varepsilon}$ is a minimizer for $\mathcal{F}^{\psi}_{\text{hom}}$.*

**Proof.** We can apply Theorem 4.2, obtaining the desired sequence $(w_{\varepsilon}) \subset L^1(\Omega)$. The fact that any cluster point $u$ of $w_{\varepsilon}$ is a minimizer for $\mathcal{F}^{\psi}_{\text{hom}}$ is a direct consequence of the Γ-convergence result given in Theorem 6.1 (see Ref. 25, Theorem 7.4). □

## 7. Further Results

In the present section we extend the asymptotic analysis performed in Secs. 5 and 6 for the Mumford−Shah energy in periodically perforated domains to more general free-discontinuity energies. We limit ourselves to state the generalizations of Theorems 5.1, 6.1, 6.2, being the proofs analogous and only technically more demanding (e.g. in the coercive case see Ref. 14, Sec. 8).

In the following we keep the notation fixed in Secs. 5 and 6. Furthermore, let $p \in (1, +\infty)$ and consider $f : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty)$, $g : \mathbb{R}^N \times \mathbf{S}^{N-1} \to [0, +\infty)$ two Borel functions. We suppose that $f$ satisfies

(f1) $f(\cdot, \xi)$ is 1-periodic for every $\xi \in \mathbb{R}^N$,
(f2) there exist two constants $c_1, c_2 > 0$ such that for every $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$

$$c_1|\xi|^p \le f(x, \xi) \le c_2(1 + |\xi|^p),$$

and that $g$ satisfies

(g1) $g(\cdot, \nu)$ is 1-periodic for every $\nu \in \mathbf{S}^{N-1}$,
(g2) $g(x, -\nu) = g(x, \nu)$ for every $(x, \nu) \in \mathbb{R}^N \times \mathbf{S}^{N-1}$,
(g3) there exist two constants $c_3, c_4 > 0$ such that for every $(x, \nu) \in \mathbb{R}^N \times \mathbf{S}^{N-1}$

$$c_3 \le g(x, \nu) \le c_4.$$

Then we introduce the family of functionals $\mathcal{G}^\psi_\varepsilon : L^p(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{G}^\psi_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f\Big(\frac{x}{\varepsilon}, \nabla u\Big) dx + \int_{S^{\psi,\varepsilon}_u} g\Big(\frac{x}{\varepsilon}, \nu_u\Big)\, d\mathcal{H}^{N-1} & u \in SBV^p(\Omega), \\ +\infty & \text{otherwise in } L^p(\Omega). \end{cases}$$

We are now in a position to extend the results of Theorems 5.1, 6.1, 6.2 to the family $(\mathcal{G}^\psi_\varepsilon)$.

**Theorem 7.1.** *The family* $(\mathcal{G}^\psi_\varepsilon)$ $\Gamma$*-converges to some functional* $\mathcal{G}^\psi_{\mathrm{hom}}$ *with respect to the* $L^p(\Omega)$ *topology. Moreover, the functional* $\mathcal{G}^\psi_{\mathrm{hom}}$ *restricted to* $SBV^p(\Omega)$ *is given by*

$$\mathcal{G}^\psi_{\mathrm{hom}}(u) := \int_\Omega f_{\mathrm{hom}}(\nabla u)\, dx + \int_{S^\psi_u} g_{\mathrm{hom}}(\nu_u)\, d\mathcal{H}^{N-1},$$

*where the bulk energy density* $f_{\mathrm{hom}} : \mathbb{R}^N \to [0, +\infty)$ *is the convex function given by*

$$f_{\mathrm{hom}}(\xi) = \lim_{\varepsilon \to 0^+} \inf \left\{ \int_{Q \setminus K_\varepsilon} f\Big(\frac{x}{\varepsilon}, \nabla w + \xi\Big) dx : w \in W^{1,p}_\sharp(Q \setminus K_\varepsilon) \right\}, \tag{7.1}$$

*and the surface energy density* $g_{\mathrm{hom}} : \mathbf{S}^{N-1} \to [0, +\infty)$ *is the BV-elliptic function given by*

$$g_{\mathrm{hom}}(\nu) = \lim_{\varepsilon \to 0^+} \inf \left\{ \int_{S_w \setminus K_\varepsilon} g\Big(\frac{x}{\varepsilon}, \nu_w\Big)\, d\mathcal{H}^{N-1} : w \in P(Q^\nu \setminus K_\varepsilon) \right.$$
$$\left. w = u_{0,1,\nu} \text{ on a neighborhood of } \partial Q^\nu \right\}.$$

*Moreover, if* $u_\varepsilon$ *are minimizers for* $\mathcal{G}^\psi_\varepsilon$ *satisfying* $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|\psi\|_{L^\infty(\Omega)}$, *then there exists a family* $(w_\varepsilon) \subset L^p(\Omega)$ *which is compact in* $L^p(\Omega)$ *and such that* $w_\varepsilon \equiv u_\varepsilon$ *in* $\Omega_\varepsilon$

*for any* $\varepsilon > 0$ (*in particular* $w_\varepsilon$ *are minimizers for* $\mathcal{G}_\varepsilon^\psi$). *Any cluster point* $u$ *of* $w_\varepsilon$ *is a minimizer for* $\mathcal{G}_{\mathrm{hom}}^\psi$.

**Remark 7.1.** In case $f(x, \cdot)$ is convex for all $x \in \mathbb{R}^N$ formula (7.1) can be specialized (see Ref. 13, Remark 19.2), and reduces to the cell minimization formula

$$f_{\mathrm{hom}}(\xi) = \inf\left\{ \int_{Q\setminus K} f(x, \nabla w + \xi)dx : \, w \in W_\sharp^{1,p}(Q\setminus K) \right\}.$$

**Remark 7.2.** Let us point out that the analogue of Proposition 5.2 to prove Theorem 7.1 needs a different argument. Indeed in this case the Γ-cluster points $\mathcal{G}^\psi$ of $(\mathcal{G}_\varepsilon^\psi)$ are not decreasing by truncations in general.

Nevertheless, the growth conditions (f2) and (g3) and a well known argument enable us to perform truncations of families of functions in such a way that the $\mathcal{G}_\varepsilon^\psi$ energies of the truncations are controlled in terms of the original energies plus an error term which can be made arbitrarily small (see Ref. 14, Lemma 3.5). This property provides the continuity of the relaxation in $L^1$ of $\mathcal{G}^\psi$ along truncations.

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