Convergence of asynchronous variational integrators in linear elastodynamics

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SUMMARY. Within the setting of linear elastodynamics of simple bodies, we prove that the discrete action functional obtained by following the scheme of asynchronous variational integrators converges in time. The convergence in space is assured by standard arguments when the finite element mesh is progressively refined.

1 THE MAIN RESULT

A simple elastic body is placed in a regular region $\mathcal{B}$ of the ambient space. The differentiable map $(x,t) \longmapsto u := u(x,t) \in \mathbb{R}^n$, $n = 1,2,3$, indicate the standard *displacement field* in the point $x \in \mathcal{B}$ and in the instant $t \in [t_0, t_f]$. The map $(x,t) \longmapsto \varepsilon := sym\nabla u(x,t)$ associates the *infinitesimal deformation tensor* ε to each point at a given instant. In linear elastic constitutive setting and infinitesimal deformation regime, the dynamics of a simple body is governed by the action functional

$$\mathcal{A}(\mathcal{B}, [t_0, t_f]; u) := \int_{t_0}^{t_f} \left(\int_{\mathcal{B}} \frac{1}{2} \rho \, |\dot{u}|^2 \, dx - V(\mathcal{B}, u, t) \right) dt \tag{1.1}$$

where ρ is the density of mass and

$$V(\mathcal{B}, u, t) := \int_{\mathcal{B}} \frac{1}{2} (\mathbb{C}\varepsilon) \cdot \varepsilon \, dx - \int_{\mathcal{B}} b \cdot u \, dx - \int_{\partial \mathcal{B}_t} \mathtt{t} \cdot u \, d\mathcal{H}^2 \tag{1.2}$$

the potential, with $\mathbb{C}$ the standard elastic constitutive tensor, b and $\mathtt{t}$ bulk and surface conservative forces respectively, the latter applied over a part $\partial\mathcal{B}_t$ of the boundary $\partial\mathcal{B}$. The fields $x \longmapsto b(x)$ and $x \longmapsto \mathtt{t}(x)$ belong to $L^2(\mathcal{B}, \mathbb{R}^3)$ and $L^2(\partial\mathcal{B}, \mathbb{R}^3)$, respectively. Initial conditions are given by smooth space fields $(x, t_0) \longmapsto u_0(x, t_0)$ and $(x, t_0) \longmapsto \dot{u}_0(x, t_0)$.

Approximate solutions to some boundary value problem are obtained by finite-element procedures. One selects first a suitable tessellation $\mathcal{T}$ of $\mathcal{B}$ of finite elements and, for each element K of it, a discrete time set

$$\Theta_K = \left\{ t_0 = t_K^1 < \ldots < t_K^{N_K - 1} < t_K^{N_K} = t_f \right\}.$$

$\Theta := \cup_{K \in \mathcal{T}} \Theta_K$ is the entire time set for $[t_0, t_f]$. We say that Θ has time size h when the difference between two arbitrary subsequent instants in Θ is lesser or equal to h. When appropriate, we write then Θ_h to underline the time size. As a measure of asynchronicity of Θ, we consider the ratio M_Θ between the time size and the minimum of the differences between subsequent instants in Θ.

Asynchronous variational integrators (AVI) are constructed by means of the direct discretization of the action both in space and time. The sole space discretization leads first to an action $\mathcal{A}_{\mathcal{T}}$ defined by

$$\mathcal{A}_{\mathcal{T}}\left([t_0, t_f]; u\right) := \sum_{K \in \mathcal{T}} \int_{t_0}^{t_f} \mathcal{A}\left(K; u_K\left(t\right)\right)\, dt, \tag{1.3}$$

where, for each $K \in \mathcal{T}$,

$$\mathcal{A}\left(K; u_K\left(t\right)\right) := \sum_{a \in K} \frac{m_{K,a}}{2} \left|\dot{u}_a\left(t\right)\right|^2 - V_K\left(u_K\left(t\right)\right),$$

with $m_{K,a}$ the *nodal mass* associated with the node a in K endowed with velocity $\dot{u}_a\left(t\right)$ and

$$V_K\left(u_K\left(t\right)\right) := \int_K \frac{1}{2}\left(\mathbb{C}\nabla u_K\left(x,t\right)\right) \cdot \nabla u_K\left(x,t\right)\, dx$$
$$- \int_K b \cdot u_K\left(x,t\right)\, dx - \int_{\partial K \cap \partial \mathcal{B}_t} \mathtt{t} \cdot u_K\left(x,t\right)\, d\mathcal{H}^{n-1},$$

with $u_K\left(x,t\right)$ the restriction of $u\left(x,t\right)$ to K. When we select also a time discretization, a discrete action sum

$$\mathcal{A}_{\mathcal{T},\Theta}\left(u\right) := \sum_{K \in \mathcal{T}} \sum_{\left\{j \mid \left[t_K^j, t_K^{j+1}\right) \in I\right\}} \mathcal{A}^j\left(K; u_K\right) \tag{1.4}$$

arises, u_K indicates the nodal displacements in the element K.

The scheme has been developed in [8], [9], [12] and [13] (see previous results in [2] and [4]). Since one may choose time discretization in an element independently of the time steps in the neighboring fellows, in the scheme above one may select time sequences in a way able to assure conservation of local energy and momenta exactly.

The analysis of the convergence of variational integrators has been investigated in [14] with reference to the elementary (zero-dimensional) oscillator. Assumptions of technical nature have been removed in [10]. With this note we enlarge the stage and adapt the technique in [14] and [10] to analyze the convergence of variational integrators in linear elastodynamics of a three-dimensional body. We analyze the relation between $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{A}_{\mathcal{T},\Theta_h}$ as $h \to 0$. Our main result is the theorem below.

Theorem 1 *(Convergence in time [6]) Let Θ_h be the entire time set for $[t_0, t_f]$ with time size h. Let also $u_h\left(t_0\right)$ and $\dot{u}_h\left(t_0\right)$ be initial conditions satisfying*

$$\sup_h \left(M_{\Theta_h} + \left|u_h\left(t_0\right)\right| + \left|\dot{u}_h\left(t_0\right)\right|\right) < +\infty,$$

with $u_h\left(t_0\right) \to u_0\left(t_0\right)$ and $\dot{u}_h\left(t_0\right) \to \dot{u}_0\left(t_0\right)$ as $h \to 0$ in each node of the spatial discretization. Then any sequence $\left(u_h\right)$ of stationary points of the discrete action $\mathcal{A}_{\mathcal{T},\Theta_h}$ is pre-compact in the weak$-*$ $W^{1,\infty}\left(\left(t_0, t_f\right), \mathbb{R}^N\right)$ topology and all its cluster points are stationary points for the action $\mathcal{A}_{\mathcal{T}}$.

In the statement above N is the total number of nodal degrees of freedom in the tessellation $\mathcal{T}$.

Theorem 1 can be extended to the linear elastodynamics of complex bodies, provided that the manifold of substructural shapes is embedded isometrically in a linear space (the paradigmatic case of quasicrystals is treated in [7]; for quasicrystals the standard Cauchy balance is augmented by a balance of substructural actions of parabolic type arising from a d'Alembert-Lagrange type variational principle).

The results contained in this note summarize those of the homonymous paper [6] to which we refer for the full proofs and all details.

2 LINEAR ELASTODYNAMICS OF SIMPLE BODIES: AN ESSENTIAL SUMMARY

Our results hold in $\mathbb{R}^n$. However, for the sake of physical concreteness we restrict the developments below to the three-dimensional ambient space $\mathbb{R}^3$ in which the region occupied by a body is always denoted by $\mathcal{B}$, and is a bounded domain with boundary $\partial\mathcal{B}$ of finite two-dimensional measure, a boundary where the outward unit normal n is defined to within a finite number of corners and/or edges. On $\mathcal{B}$ the standard *displacement field* is defined by $x \longmapsto u := u(x) \in \mathbb{R}^3$, $x \in \mathcal{B}$, and is assumed to be differentiable. The field $x \longmapsto u(x) + x$ is also one-to-one and orientation preserving in the sense that $\det(\nabla u + \mathbf{I}) > 0$ at each x, with $\mathbf{I}$ the unit tensor. When $|\nabla u| << 1$, the natural measure of infinitesimal deformations is given at each point by the value of the field $x \longmapsto \varepsilon(x) := sym\nabla u(x)$ assigning at each point the strain ε.

In a time interval $[t_0, t_f]$, a standard motion is then $(x,t) \longmapsto u := u(x,t) \in \mathbb{R}^n$, $x \in \mathcal{B}$, $t \in [t_0, t_f]$, a field twice differentiable in time.

The (contact) action, power conjugated with the velocity $\dot{u} := \frac{d}{dt}u(x,t)$ on any virtual smooth surface in the body, oriented by the normal n, is the tension t which depends linearly on n through *Cauchy stress tensor* σ, namely $\mathsf{t} = \sigma n$. A tensor field $(x,t) \longmapsto \sigma = \sigma(x,t) \in Hom(\mathbb{R}^3, \mathbb{R}^3)$ is then defined over $\mathcal{B}$ and is assumed to be differentiable.

The invariance with respect to isometric changes in observers of the external power of bulk and surface actions on any subset $\mathfrak{b}$ of $\mathcal{B}$ with non-vanishing volume measure and the same regularity of $\mathcal{B}$ allows one to get pointwise balances between bulk and contact actions. Moreover, the inertial parts of the body forces are identified by making use of the balance between the rate of the kinetic energy and the power of inertial forces. As a result one gets the standard balance equations

$$b + div\sigma = \rho\ddot{u}, \ \ skw\sigma = 0, \tag{2.1}$$

where ρ is the mass density. Natural boundary conditions are given by the prescription of the traction t on a part $\partial\mathcal{B}_t$ of the boundary $\partial\mathcal{B}$ and of the displacement u on another part $\partial\mathcal{B}_u$ provided that $\partial\mathcal{B}_t \cap \partial\mathcal{B}_u = \varnothing$ and $\overline{\partial\mathcal{B}_t \cup \partial\mathcal{B}_u} = \partial\mathcal{B}$. Precisely, in this note it is assumed that $u = 0$ along $\partial\mathcal{B}_u$.

When the material is homogeneous and displays a linear hyperelastic behavior, the standard constitutive relation holds

$$\sigma = \mathbb{C}\varepsilon, \tag{2.2}$$

with $\mathbb{C}$ a constant fourth-rank tensor with minor and major symmetries. In particular it is assumed that

$$\mathbb{C}(\xi \otimes \eta) \cdot (\xi \otimes \eta) > 0$$

for any pair of vectors ξ and η. Such a condition implies that the balance equations above generate a quasi-contractive semigroup in $H^1 \times L^2$ (see, e.g., [1], [11]).

Under the mixed boundary conditions mentioned above and the constitutive structure (2.2), the balance equation (2.1) can be also obtained by imposing that the first variation of the action functional (1.1) vanishes.

3 DISCRETIZATION OF THE ACTION FUNCTIONAL

As anticipated above, a way to obtain algorithms preserving the symplectic structure of the linear elastodynamics of simple bodies consists in constructing a direct discretization of the action functional (1.1) in space and time and attributing different discrete time sequences to each spatial finite element. Discretization in space is obtained by means of standard finite elements given by a tessellation $\mathcal{T}$ of $\mathcal{B}$, a simple triangulation, for example, chosen to be consistent with the partition of the boundary $\partial\mathcal{B}$ into $\partial\mathcal{B}_t$ and $\partial\mathcal{B}_u$ (see, e.g., [3]). A triangulation is adopted here for the sake of simplicity. For this reason we consider $\mathcal{B}$ with polyhedral shape. For each $K \in \mathcal{T}$ a finite number of points is selected, they are integration nodes. The generic node is indicated by a. Here we choose as nodes the vertexes of the elements of the triangulation. We consider then the space $PA(\mathcal{T})$ of linear polynomials on each $K \in \mathcal{T}$ and are interested in a vector subspace $\mathcal{V}_\mathcal{T}$ of $PA(\mathcal{T}) \otimes H^1\left((t_0, t_f), \mathbb{R}^3\right)$ containing all displacement mappings satisfying $u|_{x\in\partial\mathcal{B}_u} = 0$. Any $u \in \mathcal{V}_\mathcal{T}$ is then of the form

$$u(x,t) = \sum_{a\in\mathcal{T}} \mathcal{N}_a(x)\, u_a(t). \tag{3.1}$$

When restricted to a generic finite element K, the map $u(x,t)$ in (3.1) is indicated by $u_K(x,t)$. For each finite element K, $\mathcal{N}_a(x)$ is the nodal shape function corresponding to the node a and $u_a(t)$ is the value of the displacement at the generic node a. The vector of displacements of all nodes in the generic element K is indicated by $u_K(t)$. Of course $u_a(t) = 0$ if $a \in \mathcal{T} \cap \partial\mathcal{B}_u$. As usual, shape functions are selected in a way that they form an orthonormal family in $L^2(\mathcal{B})$.

With a slight abuse of notation, we denote by $u(t)$ a vector in $\mathbb{R}^N$, with N the number of degrees of freedom of all nodal placements at time t. Precisely, $N = 3N_n$, with N_n the total number of nodes in $\mathcal{T}$.

It is possible to check that there exists a constant c such that

$$\sup_{\mathcal{B}} |\nabla u(\cdot,t)| \leq c\,|u(t)|. \tag{3.2}$$

Denote by $\mathcal{E}$ the class of open intervals $I \subseteq (t_0, t_f)$. The spatial semi-discretization $\mathcal{A}_\mathcal{T}$: $H^1((t_0,t_f),\mathbb{R}^n) \times \mathcal{E} \to [0,+\infty]$ is then defined by

$$\mathcal{A}_\mathcal{T}(u,I) := \begin{cases} \mathcal{A}(u,I) & u \in \mathcal{V}_\mathcal{T} \\ +\infty & \text{otherwise} \end{cases}.$$

By means of a straightforward computation one gets (1.3) where, for each $K \in \mathcal{T}$, we rewrite

$$\mathcal{A}(K; u_K(t)) := \sum_{a\in K} \frac{m_{K,a}}{2} |\dot{u}_a(t)|^2 - V_K(u_K(t)), \tag{3.3}$$

and

$$\begin{aligned} V_K(u_K(t)) := &\int_K \frac{1}{2}(\mathbb{C}\nabla u_K(x,t)) \cdot \nabla u_K(x,t)\,dx \\ &- \int_K b \cdot u_K(x,t)\,dx - \int_{\partial K\cap\partial\mathcal{B}_t} \mathsf{t} \cdot u_K(x,t)\,d\mathcal{H}^2. \end{aligned} \tag{3.4}$$

Of course the expansion (3.1) of maps in $\mathcal{V}_\mathcal{T}$ allows one to consider $\mathcal{A}_\mathcal{T}$ as a functional over $H^1\left((t_0,t_f),\mathbb{R}^N\right)$.

We call *stationary point* for $\mathcal{A}_{\mathcal{T}}$ any map $u \in H^1\left(\left(t_0,t_f\right),\mathbb{R}^N\right)$ satisfying for any time interval $I \in \mathcal{E}$ and any $w_a = u_a + H_0^1\left(I,\mathbb{R}^3\right)$ the weak balance

$$\int_I m_a \dot{u}_a(t)\cdot\dot{w}_a(t)\ dt = \int_I \sum_{\{K|a\in K\}}\left(\int_K \frac{1}{2}\left(\mathbb{C}\nabla u_K(x,t)\right)\nabla\mathcal{N}_a(x)\ dx\right)\cdot w_a(t)\ dt$$

$$-\int_I \sum_{\{K|a\in K\}}\left(\int_K \mathcal{N}_a(x)\,b(x)\ dx + \int_{\partial K\cap\partial\mathcal{B}_t}\mathcal{N}_a(x)\,\mathsf{t}(x)\ d\mathcal{H}^2\right)\cdot w_a(t)\ dt,$$

where $m_a := \sum_{\{K|a\in K\}} m_{K,a}$.

Standard results about finite elements imply the theorem below (see [5]).

Theorem 2 *Consider a family $\{\mathcal{T}_m\}_{m>0}$ of regular triangulations of $\mathcal{B}$, with $m>0$ the mesh size of $\mathcal{T}_m$. Let also $u_m \in \mathcal{V}_{\mathcal{T}_m}$ be a stationary point for $\mathcal{A}_{\mathcal{T}_m}$. The sequence $\{u_m\}$ converges in $H^1\left(\mathcal{B}\times\left(t_0,t_f\right),\mathbb{R}^3\right)$ to a stationary point of $\mathcal{A}$.*

The discretization of the time interval follows the guidelines indicated in the first section. A partition $\Theta := \{t_i\}_{i=0,\ldots,N_\Theta}$ of $[t_0,t_f]$ with $t_{N_\Theta}=t_f$ is selected. Its *size* is $h := \max_i\left(t_{i+1}-t_i\right)$. Each $K\in\mathcal{T}$ is endowed with an *elemental time set* which is an ordered subset Θ_K of Θ. By relabeling the elements we write

$$\Theta_K = \left\{t_0 = t_K^1 < \ldots < t_K^{N_K-1} < t_K^{N_K} = t_f\right\}.$$

We assume $\Theta = \cup_{K\in\mathcal{T}}\Theta_K$ and, for the sake of simplicity, we presume also that $\Theta_K\cap\Theta_{K'}\neq\varnothing$ for any K, $K'\in\mathcal{T}$ with $K\neq K'$. The fact that each finite element can be endowed with a different time set is the basic characteristic of *asynchronous variational integrators* as mentioned in Section 1: appropriate choices of elemental time sets allow one to prove conservation of energy in discrete time (see [8]). For a node a in $\mathcal{T}$, elemental time sets define also the relevant *nodal time set* Θ_a by

$$\Theta_a := \cup_{\{K:\,a\in K\}}\Theta_K = \left\{t_0 = t_a^1 < \ldots < t_a^{N_a-1} < t_a^{N_a} = t_f\right\},$$

where $\sqcup$ denotes disjoint union. A measure of the *asynchronicity of* Θ is the ratio

$$M_\Theta = \frac{\max_{K\in\mathcal{T}}\left(\max_{\Theta_K}\left(t_K^{j+1}-t_K^j\right)\right)}{\min_{K\in\mathcal{T}}\left(\min_{\Theta_K}\left(t_K^{j+1}-t_K^j\right)\right)}. \tag{3.5}$$

Amid possible choices, we assume that each node $a\in\mathcal{T}$ follows a *linear* trajectory within each time interval with end points that are consecutive instants in Θ_a. Such a choice characterizes the class of AVI we analyze here. Then we denote by Y_Θ the subspace of functions in $L^2\left(\left(t_0,t_f\right),\mathbb{R}^N\right)$ which are continuous and with piecewise constant time rates in the intervals in Θ_a. Thus, for each $u\in Y_\Theta$, $a\in\mathcal{T}$, $t_a^i\in\Theta_a$ and $t\in\left[t_a^i,t_a^{i+1}\right)$ we have

$$\dot{u}_a(t) = \frac{u_a\left(t_a^{i+1}\right)-u_a\left(t_a^i\right)}{t_a^{i+1}-t_a^i}.$$

Then, by following [8], the discrete action sums in time is defined for $u\in Y_\Theta$ by (1.4) where

$$\mathcal{A}^j\left(K;u_K\right) := \sum_{a\in K}\sum_{\left\{i|t_a^i\in\left[t_K^j,t_K^{j+1}\right)\right\}}\left(\frac{1}{2}m_{K,a}\left(t_a^{i+1}-t_a^i\right)\left|\dot{u}_a\left(t_a^i\right)\right|^2 - \left(t_K^{j+1}-t_K^j\right)V_K\left(u_K\left(t_K^{j+1}\right)\right)\right),$$

with V_K defined by (3.4). Such a choice gives rise to explicit integrators of central-difference type and is only one of the possible schemes that can be used.

It is convenient to define all action sums on the same function space to avoid to link the function space itself to the choice of Θ. For this reason we extend $\mathcal{A}_{\mathcal{T},\Theta}$ to $+\infty$ on $Y\backslash Y_\Theta$, where $Y = L^2\left((t_0,t_f),\mathbb{R}^N\right)$ is endowed with the usual metric.

The discrete variational principle

$$\delta\mathcal{A}_{\mathcal{T},\Theta} = 0,$$

with δ indicating the first variation, implies that for all $a \in \mathcal{T}\backslash\partial\mathcal{B}_u$ and $t_a^i \in (t_0, t_f]$ discrete Euler-Lagrange equations have to be satisfied. They read

$$m_a\left(\dot{u}_a\left(t_a^{i+1}\right) - \dot{u}_a\left(t_a^i\right)\right) = \left(t_K^{j+1} - t_K^j\right)\int_K \frac{1}{2}\left(\mathbb{C}\nabla u_K(x,t)\right)\nabla\mathcal{N}_a(x)\ dx$$

$$- \left(t_K^{j+1} - t_K^j\right)\left(\int_K \mathcal{N}_a(x)\, b(x)\ dx + \int_{\partial K\cap\partial\mathcal{B}_t} \mathcal{N}_a(x)\,\mathsf{t}(x)\ d\mathcal{H}^2\right), \tag{3.6}$$

where K is the sole element in $\mathcal{T}$ for which $t_a^i \in \Theta_K$ and $t_a^i = t_K^{j+1}$.

Given initial conditions $u(t_0)$ and $\dot{u}(t_0)$, the discrete Euler-Lagrange equations (3.6) define inductively a trajectory u piecewise affine in time, a trajectory which is a (discrete) stationary point for $\mathcal{A}_{\mathcal{T},\Theta}$.

4 STRATEGY OF THE PROOF OF THEOREM 1

To prove the convergence of asynchronous variational integrators in linear elastodynamics, we adapt here the strategy used in [14] and [10] for analyzing the convergence of variational integrators for the zero-dimensional oscillator (a mass point connected to an elastic massless spring).

The essential structure of the proof is listed below.

1. *L^∞ estimates for the velocity of stationary points of discrete actions*

As suggested in [10], L^∞ estimates on the velocity of stationary points can be derived by exploiting directly the discrete Euler-Lagrange equations and the growth conditions of the potential energy density. Of course here stationarity does not mean independence of time.

Proposition 1 *Given initial conditions $u(t_0)$ and $\dot{u}(t_0)$, there exists a constant $k > 0$ depending on the initial conditions themselves and on the data of the problem such that, for every entire time set Θ and $u \in Y_\Theta$ solution to the discrete Euler-Lagrange equations, it satisfies the inequality*

$$\|\dot{u}\|_{L^\infty((t_0,t_f),\mathbb{R}^N)} \leq k\exp\left(kM_\Theta\right). \tag{4.1}$$

2. *Stationary points of discrete actions are minimizers in short time intervals*

The next step consists in showing that stationary points of the discrete action sums are minimizers in short-time intervals. This minimality property is crucial to prove that the cluster points of sequences of stationary points of discrete actions are stationary for the continuous action *via* a Γ-convergence type argument.

Proposition 2 *Given initial conditions $u(t_0)$, $\dot{u}(t_0)$, there exists a constant $\kappa > 0$, depending only on the initial condition themselves and the data, such that for every entire time set Θ and*

$u \in Y_\Theta$ solution to the discrete Euler-Lagrange equations, u is a local minimizer of the functional $\mathcal{A}_{\mathcal{T},\Theta}(\cdot, I)$, namely $\mathcal{A}_{\mathcal{T},\Theta}(u, I) \leq \mathcal{A}_{\mathcal{T},\Theta}(v, I)$ for any $v \in u + H_0^1\left(I, \mathbb{R}^N\right)$, provided that $|I| \leq \kappa$.

3. *Conclusion*

Finally we analyze the convergence of such stationary points in the limit $h \to 0^+$, that is as the time size of the discretization goes to zero.

To this aim we state two technical results. The first lemma is analogous to Lemma 4.3 in [14].

Lemma 1 *Let Θ_h be entire time sets for $[t_0, t_f]$ with time size h. For any $I \in \mathcal{E}$ and $u \in H^1\left(I, \mathbb{R}^3\right)$ there exists $u_h \in H^1\left(I, \mathbb{R}^3\right) \cap Y_{\Theta_h}$ such that $u_h \to u$ strongly in $H^1\left(I, \mathbb{R}^3\right)$.*

Lemma 2 *Given entire time sets Θ_h for $[t_0, t_f]$ indexed by the time size h and characterized by $\sup_h M_{\Theta_h} < +\infty$, for every sequence $u_h \in Y_{\Theta_h}$ with $|\dot{u}_h|^2$ weakly convergent in $L^1\left(I, \mathbb{R}^N\right)$, one gets*

$$\lim_{h \to 0^+} \left(\mathcal{A}_{\mathcal{T}}(u_h, I) - \mathcal{A}_{\mathcal{T},\Theta_h}(u_h, I)\right) = 0.$$

The sketch of the proof of Theorem 1 can be then presented.

Proof. Pre-compactness of (u_h) in $W^{1,\infty}\left((t_0, t_f), \mathbb{R}^N\right)$ follows easily from Proposition 1 since the ratios M_{Θ_h} are bounded uniformly with respect to h by assumption. Denote by $u \in W^{1,\infty}\left((t_0, t_f), \mathbb{R}^N\right)$ a cluster point of (u_h). By Ascoli-Arzelà theorem we may suppose $u_h \to u$ uniformly on $\bar{I}$, up to a subsequence not relabeled for convenience.

The proof that u is a stationary point for $\mathcal{A}_{\mathcal{T}}$ follows by showing that, for every $I := (t_1, t_2) \in \mathcal{E}$ with $|I| \leq \kappa$, with κ the constant in Proposition 2, one gets

$$\mathcal{A}_{\mathcal{T}}(u, I) \leq \mathcal{A}_{\mathcal{T}}(w, I),$$

for any $w \in u + H_0^1\left(I, \mathbb{R}^N\right)$.

To this aim Lemma 1 provides a sequence (w_h), with $w_h \in Y_{\Theta_h}$, converging to w strongly in $H^1\left(I, \mathbb{R}^N\right)$. Then assuming $w_h - u_h \in H_0^1\left(I, \mathbb{R}^N\right)$ the use of Proposition 2 and Lemma 2 implies

$$\mathcal{A}_{\mathcal{T}}(u, I) \leq \liminf_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(u_h, I) \leq \lim_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(w_h, I) = \mathcal{A}_{\mathcal{T}}(w, I).$$

Of course we have used the lower semicontinuity of $\mathcal{A}_{\mathcal{T}}(\cdot, I)$ under weak$-*$ convergence in $W^{1,\infty}$, and its continuity under strong H^1 convergence.

To cover cases in which the boundary values are not matched on ∂I, we notice that it is possible to perturb u_h with $p_h \in W^{1,\infty}\left(I, \mathbb{R}^N\right)$ such that p_h is linear componentwise, $p_h \to 0$ strongly in $W^{1,\infty}\left(I, \mathbb{R}^N\right)$ as $h \to 0^+$, and $u_h = w_h + p_h$ on ∂I. The previous argument then applies. □

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