RELAXATION OF FREE-DISCONTINUITY ENERGIES WITH OBSTACLES

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Abstract. Given a Borel function ψ defined on a bounded open set Ω with Lipschitz boundary and $\varphi \in L^1(\partial\Omega, \mathcal{H}^{n-1})$, we prove an explicit representation formula for the L^1 lower semicontinuous envelope of Mumford-Shah type functionals with the obstacle constraint $u^+ \geq \psi$ $\mathcal{H}^{n-1}$ a.e. on Ω and the Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$.

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1. Introduction

Weak formulations of Fracture Mechanics theories for brittle hyperelastic media have been studied in the last years in the framework of free-discontinuity problems (see [1],[10] and [2] for a more exhaustive list of references). In these models the state of a brittle body is described by a pair displacement-crack with total energy given by the sum of a bulk and a surface term, related to the (approximate) gradient and the set of (approximate) discontinuities of the deformation, respectively.

In particular, homogenization of brittle media with reinforcements may involve minimum problems for free-discontinuity energies with an obstacle condition. In case of bodies with a periodic distribution of perforations, intended in the sense of holes on which a Dirichlet or a unilateral obstacle condition is imposed, one is interested in analyzing the behaviour of the energy as the diameter of the perforations tends to 0.

In case of antiplane setting and selecting the Mumford-Shah energy as a prototype, one investigates the asymptotics as ε tends to 0 of Dirichlet boundary value problems for functionals of the type

$$\int_\Omega |\nabla u(x)|^p \, dx + \mathcal{H}^{n-1}(S_u) + \text{lower order terms} \qquad u^+ \geq 0 \ \mathcal{H}^{n-1} \text{ a.e. on } \mathbf{E}_\varepsilon \tag{1.1}$$

where the open set $\Omega \subseteq \mathbf{R}^n$ represents a section in the cylindrical reference configuration of the body $\Omega \times \mathbf{R}$, $u \in GSBV(\Omega)$ is the antiplane displacement, and the set $\mathbf{E}_\varepsilon$ is obtained periodically perforating the domain Ω with a rescaled copy of the reference hole E.

We remark that the formulation of the obstacle condition in (1.1) as an $\mathcal{H}^{n-1}$ constraint is consistent with perforations $\mathcal{L}^n$ negligible, and it is a natural generalization for such sets of the usual unilateral inequality in the $\mathcal{L}^n$ sense (see Remark 4.2 [15]).

The homogenization problem above was addressed in the paper [15] via Γ-convergence methods. The convergence of the minimum problems associated to (1.1) to the corresponding problem for the Γ-limit is a byproduct of such analysis. The coercivity of functionals as in (1.1) is ensured by a well known result of Ambrosio (see Theorem 2.1), instead the L^1 lower semicontinuity of free-discontinuity energies subject to an $\mathcal{H}^{n-1}$ constraint has to be investigated.

In this paper we characterize the relaxed functional associated to an energy as in (1.1) under a general unilateral constraint. Namely, given a Borel function $\psi : \Omega \to \mathbf{R} \cup \{\pm\infty\}$, $p > 1$, we consider the functional

$$F_\psi(u,\Omega) = \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u) \quad \text{if } u \in GSBV(\Omega),\ u^+ \geq \psi \ \ \mathcal{H}^{n-1} \text{ a.e. on } \Omega,$$

and $+\infty$ otherwise in $L^1(\Omega)$.

In order to deal with this problem we introduce a variational measure σ following the approach of De Giorgi for parametric Plateau problems with an obstacle (see definitions (3.1),(3.2)). The main result proved in this paper is that the relaxed functional of F_ψ can be written in the form

$$\begin{aligned}\mathcal{F}_\psi(u,\Omega) = &\int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u)\\ &+\frac{1}{2}\sigma\left(\{x \in S_u : u^+(x) < \psi(x)\}\right) + \sigma\left(\{x \in \Omega \setminus S_u : u^+(x) < \psi(x)\}\right)\end{aligned}$$

if $u \in GSBV(\Omega)$, $+\infty$ otherwise in $L^1(\Omega)$.

In particular, we show that the measure σ introduced above coincides with the analogous measure originally defined by De Giorgi for minimal surfaces with obstacles (see Subsection 2.3 and Section 3 for more exhaustive details).

An outline of the paper is as follows. In Section 2 we review some prerequisites needed in the sequel: We recall some properties of sets with finite perimeter, BV functions and De Giorgi's measure σ. In Section 3 we introduce and study the properties of a variational measure which is naturally involved in the relaxation process. In particular, we compare it with De Giorgi's one. In Section 4 we state and prove the main result justifying the relaxation formula above. The result is shown to be consistent with the addition of a Dirichlet boundary condition in Section 5. All results illustrated for the Mumford-Shah energy are extended in Section 6 to more general free-discontinuity energies.

2. Notation and preliminaries

In the sequel $n \geq 1$ will be a fixed integer, and $p \in (1, +\infty)$ will be a fixed exponent.

2.1. **Relaxation.** We recall the notion of relaxation of a functional $F : X \to [0, +\infty]$ in a generic metric space (X, d) endowed with the topology induced by d (see [9],[5]). The *relaxed functional* $\overline{F} : X \to [0, +\infty]$ is the lower semicontinuous envelope of F, that is

$$\overline{F}(u) = \sup\{G(u) :\ G \leq F,\ G\ d\text{-lower semicontinuous}\}.$$

A different characterization holds for $\overline{F}$, namely

$$\overline{F}(u) = \inf\{\liminf_{j\to+\infty} F(u_j) : \ u_j \to u\}.$$

Thus, given a candidate $\mathcal{F}$ for the lower semicontinuous envelope of F, to show that it equals $\overline{F}$ it suffices to prove the following two inequalities

(i) (*lower bound*) for every (u_j) converging to u in X, we have $\liminf_j F(u_j) \geq \mathcal{F}(u)$;

(ii) (*upper bound*) there exists (u_j) converging to u in X such that $\limsup_j F(u_j) \leq \mathcal{F}(u)$.

2.2. **BV functions.** In this subsection we recall some basic definitions and results on sets of finite perimeter, BV, SBV and GSBV functions. We refer to the book [2] for all the results used throughout the whole paper, for which we will give a precise reference.

Let $A \subseteq \mathbf{R}^n$ be an open set, for every $u \in L^1(A)$ and $x \in A$, we define

$$u^+(x) = \inf\left\{t \in \mathbf{R} : \lim_{r\to 0^+} r^{-n}\mathcal{L}^n(\{y \in B_r(x) : u(y) > t)\} = 0\right\}$$
$$u^-(x) = \sup\left\{t \in \mathbf{R} : \lim_{r\to 0^+} r^{-n}\mathcal{L}^n(\{y \in B_r(x) : u(y) < t)\} = 0\right\},$$

with the convention $\inf\emptyset = +\infty$ and $\sup\emptyset = -\infty$. We remark that u^+, u^- are Borel functions uniquely determined by the $\mathcal{L}^n$-equivalence class of u. If $u^+(x) = u^-(x)$ the common value is denoted by $\tilde{u}(x)$ or ap-$\lim_{y\to x} u(y)$ and it is said to be the *approximate limit* of u in x.

In particular, for every $\mathcal{L}^n$ measurable set $E \subseteq \mathbf{R}^n$ it holds $(\chi_E)^+ = \chi_{E^+}$, where

$$E^+ = \{x \in \mathbf{R}^n : \limsup_{r\to 0^+} r^{-n}\mathcal{L}^n(E \cap B_r(x)) > 0\}.$$

We remark that for any $u \in L^1(A)$ and $s \leq t$, it holds

$$\{x \in A : \ u(x) \geq s\}^+ \supseteq \{x \in A : \ u^+(x) \geq t\}. \tag{2.1}$$

The set $S_u = \{x \in A : u^-(x) < u^+(x)\}$ is called the *set of approximate discontinuity points* of u and it is well known that $\mathcal{L}^n(S_u) = 0$. Let $x \in A \setminus S_u$ be such that $\tilde{u}(x) \in \mathbf{R}$, we say that u is *approximately differentiable* at x if there exists $L \in \mathbf{R}^n$ such that

$$\text{ap-}\lim_{y\to x} \frac{|u(y) - \tilde{u}(x) - L(y-x)|}{|y-x|} = 0. \tag{2.2}$$

If u is approximately differentiable at a point x, the vector L uniquely determined by (2.2), will be denoted by $\nabla u(x)$ and will be called the *approximate gradient* of u at x.

A function $u \in L^1(A)$ is said to be of *Bounded Variation* in A, in short $u \in BV(A)$, if its distributional derivative Du is a $\mathbf{R}^n$-valued finite Radon measure on A with mass $\|Du\|(A)$, called the *total variation* of u on A. If $u \in BV(A)$ denote by $D^a u$, $D^s u$ the absolutely and singular part of the Lebesgue's decomposition of Du w.r.t. $\mathcal{L}^n \llcorner A$, respectively. Then u turns out to be approximately differentiable a.e. on A (see Theorems 3.83 [2]), S_u to be *countably* $\mathcal{H}^{n-1}$*-rectifiable* (see Theorem 3.78 [2]), and the values $u^+(x)$, $u^-(x)$ are finite and specified $\mathcal{H}^{n-1}$ a.e. in A (see Remark 3.79 [2]). Moreover, it holds

$$D^a u = \nabla u \, \mathcal{L}^n \llcorner A, \quad D^s u \llcorner S_u = (u^+ - u^-)\nu_u \, \mathcal{H}^{n-1} \llcorner S_u,$$

where $\nu_u \in \mathbf{R}^n$ is an orientation for S_u.

We say that a $\mathcal{L}^n$ measurable set $E \subseteq \mathbf{R}^n$ is of *finite perimeter* in A if $\chi_E \in BV(A)$, and we call the total variation of χ_E in A the *perimeter* of E in A, denoting it by $\mathrm{Per}(E, A)$ and simply by $\mathrm{Per}(E)$ if $A \equiv \mathbf{R}^n$. Setting for $t \in [0,1]$

$$E^t = \left\{ x \in \mathbf{R}^n : \lim_{r \to 0^+} \frac{\mathcal{L}^n(E \cap B_r(x))}{\omega_n r^n} = t \right\},$$

and $\partial^* E = E \setminus (E^1 \cup E^0)$, it is well known that the set $\partial^* E$ is countably $\mathcal{H}^{n-1}$-rectifiable, and letting $\nu_{\partial^* E}$ be an orientation for it we have $D\chi_E = D\chi_E \llcorner \partial^* E = \nu_{\partial^* E} \mathcal{H}^{n-1} \llcorner \partial^* E$ (see Theorem 3.59 [2]).

We say that $u \in BV(A)$ is a *Special Function of Bounded Variation* in A if $D^s u \equiv D^j u$ on A, in short $u \in SBV(A)$.

We say that $u \in L^1(A)$ is a *Generalized Special Function of Bounded Variation* in A, in short $u \in GSBV(A)$, if for every $M > 0$ the truncated function $(u \wedge M) \vee (-M) \in SBV(A)$.

Functions in $GSBV$ inherit from BV ones many properties: a generalized distributional derivative can be defined, they are approximately differentiable a.e. on A, and S_u turns out to be countably $\mathcal{H}^{n-1}$-rectifiable (see Theorem 4.34 [2]).

The space $(G)SBV$ has been introduced by De Giorgi and Ambrosio [13] in connection with the weak formulation of the image segmentation model proposed by Mumford and Shah (see [18]). If $u \in GSBV(A)$ and $p \in (1, +\infty)$ the *Mumford-Shah energy* of u is defined as

$$MS_p(u, A) = \int_A |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u). \tag{2.3}$$

We recall the $GSBV$ compactness theorem due to Ambrosio in a form needed for our purposes (see Theorem 4.8 and Theorem 5.22 [2]).

Theorem 2.1. *Let $(u_j) \subset GSBV(A)$ and assume that for some $p \in (1, +\infty)$*

$$\sup_j \left(MS_p(u_j, A) + \|u_j\|_{L^1(A)} \right) < +\infty.$$

Then, there exist a subsequence (u_{j_k}) and a function $u \in GSBV(A)$ such that $u_{j_k} \to u$ a.e. in A, $\nabla u_{j_k} \to \nabla u$ weakly in $L^p(A; \mathbf{R}^n)$, $D^s u_{j_k} \llcorner S_{u_{j_k}} \to D^s u \llcorner S_u$ weakly $*$ in the sense of measures.

In particular, if $\sup_j \|u_j\|_{L^\infty(A)} < +\infty$ then the cluster point u belongs to $SBV(A)$.

Eventually, if $\varphi : \mathbf{R}^n \to [0, +\infty)$ is a norm, then

$$\int_{S_u} \varphi(\nu_u) d\mathcal{H}^{n-1} \le \liminf_k \int_{S_{u_{j_k}}} \varphi(\nu_{u_{j_k}}) d\mathcal{H}^{n-1}.$$

Eventually, in case $u \in GSBV(A)$ and $MS_p(u,A) < +\infty$ the values $u^+(x)$, $u^-(x)$ are finite and specified $\mathcal{H}^{n-1}$ a.e. in A (see Theorem 4.40 [2]).
To conclude the preliminaries on $GSBV$ functions we recall their characterization *via* restrictions to one-dimensional subspaces. For more details on the so called "slicing techniques" we refer both to Section 3.11 [2] and Chapter 4 [4].
Let $\xi \in \mathbf{S}^{n-1}$ be a fixed direction, denote by Π^ξ the orthogonal space to ξ. If $y \in \Pi^\xi$ and $E \subset \mathbf{R}^n$ define $E^\xi_y = \{t \in \mathbf{R} : y + t\xi \in E\}$ and $E_\xi = \{y \in \Pi^\xi : E^\xi_y \neq \emptyset\}$. Moreover, given $g : E \to \mathbf{R}$ define, for any $y \in E_\xi$, $g_{\xi,y} : E^\xi_y \to \mathbf{R}$ by $g_{\xi,y}(t) := g(y + t\xi)$.

Theorem 2.2. *Let $u \in GSBV(A)$, then $u_{\xi,y} \in GSBV\left(A^\xi_y\right)$ for all $\xi \in \mathbf{S}^{n-1}$ and $\mathcal{H}^{n-1}$ a.e. $y \in A_\xi$. For such y we have*

(i) $u_{\xi,y}(t) = \nabla u\,(y + t\xi)\,\xi$ *for $\mathcal{L}^1$ a.e. $t \in A^\xi_y$;*
(ii) $S_{u_{\xi,y}} = (S_u)^\xi_y$*;*
(iii) $u^\pm_{\xi,y}(t) = u^\pm\,(y + t\xi)$ *or* $u^\pm_{\xi,y}(t) = u^\mp\,(y + t\xi)$ *according to the cases $\langle \nu_u, \xi\rangle > 0$, $\langle \nu_u, \xi\rangle < 0$ (the case $\langle \nu_u, \xi\rangle = 0$ being negligible).*

We conclude the subsection recalling a consequence of the Coarea formula (see Theorem 2.93 [2]).

Proposition 2.3. *For any $u \in GSBV(A)$, for every $\xi \in \mathbf{S}^{n-1}$ and every open set $A' \subseteq A$ it holds*

$$\int_{A' \cap S_u} |\langle \nu_u(x), \xi\rangle| d\mathcal{H}^{n-1}(x) = \int_{A'_\xi} \mathcal{H}^0((S_u)^\xi_y) d\mathcal{H}^{n-1}(y). \tag{2.4}$$

2.3. **De Giorgi's measure.** In this subsection we recall the definition of an $(n-1)$-dimensional geometric measure which has been introduced in the study of obstacle problems for area-like functionals (see [14],[12],[8],[19],[6],[7]).
Following the original definition by De Giorgi [14], for any open set $A \subseteq \mathbf{R}^n$ and any set $E \subseteq \mathbf{R}^n$, we consider the set functions

$$\sigma^\varepsilon(E,A) = \inf\left\{\mathrm{Per}(D,A) + \frac{1}{\varepsilon}\mathcal{L}^n(D \cap A) : D = D^+,\ \ D \supseteq E \cap A\right\},$$

and

$$\sigma(E,A) = \sup_{\varepsilon > 0} \sigma^\varepsilon(E,A).$$

We collect below some properties of σ summarizing Theorems 2.3, 2.7, 2.8 and 4.10 of Chapter 4 [14].

Theorem 2.4. *Let $A \subseteq \mathbf{R}^n$ be an open set and $E \subseteq \mathbf{R}^n$.*

(a) *σ is a regular Borel measure such that*

$$c_1(n)\mathcal{H}^{n-1}(E \cap A) \le \sigma(E,A) \le c_2(n)\mathcal{H}^{n-1}(E \cap A) \tag{2.5}$$

for two positive constants c_1, c_2 depending only on n.

(b) *If* $E \subseteq A$, *then*

$$\sigma(E, A) = \sigma(E). \tag{2.6}$$

In particular, for every set $F \subseteq \mathbf{R}^n$ *it holds* $\sigma(F, A) = \sigma(F \cap A)$.

(c) *If* E *is a* $\mathcal{H}^{n-1}$*-rectifiable set, then*

$$\sigma(E, A) = 2\mathcal{H}^{n-1}(E \cap A). \tag{2.7}$$

Remark 2.5. *The papers* [16],[17] *study in details the relationship between* σ *and* $\mathcal{H}^{n-1}$. *In particular, an example disproves the equality in* (2.7) *in general.*
Moreover, the inequality $c_2(n) \leq n\omega_n/\omega_{n-1}$ *is established, with* ω_k *the* $\mathcal{L}^k$ *measure of the unit ball in* $\mathbf{R}^k$. *A further example shows the optimality of that bound for* $n = 2$, *and some hints are given in order to generalize such a result for arbitrary* $n \geq 2$. *No lower bound for* $c_1(n)$ *is to our knowledge explicit.*

We now state alternative characterizations of σ, the first proved in [8] the others in [6].

Proposition 2.6. *For any open set* $A \subseteq \mathbf{R}^n$ *and any set* $E \subseteq \mathbf{R}^n$, *we have*

$$\begin{aligned}
\sigma(E, A) &= \sup_{\varepsilon>0} \left(\inf \left\{ \mathrm{Per}(D, A) + \frac{1}{\varepsilon}\mathcal{L}^n(D \cap A) : D \text{ open}, \ D \supseteq E \cap A \right\} \right) \\
&= \sup_{\varepsilon>0} \left(\inf \left\{ \mathrm{Per}(D, A) + \frac{1}{\varepsilon}\mathcal{L}^n(D \cap A) : D \ \mathcal{L}^n \text{ measurable}, \ \mathcal{H}^{n-1}(E \cap A \setminus D^+) = 0 \right\} \right) \\
&= \sup_{\varepsilon>0} \left(\inf \left\{ \|Du\|(A) + \frac{1}{\varepsilon}\int_A |u|\, dx \ : u \in BV(A), \ u^+ \geq 1 \ \ \mathcal{H}^{n-1} \text{ a.e. on } E \cap A \right\} \right).
\end{aligned}$$

Remark 2.7. *The first characterization of* σ *provided in Proposition 2.6 entails that for any set* E *for which* $\sigma(E, A) < +\infty$ *we can find a family of open sets* (D_ε) *admissible for the minimum problems* $\sigma^\varepsilon(E, A)$ *satisfying* $\mathcal{L}^n(D_\varepsilon) = o(\varepsilon^2)$ *and*

$$\sigma(E, A) = \mathrm{Per}(D_\varepsilon, A) + o(1).$$

The following result clarifies how De Giorgi's measure σ arises in the relaxation of obstacle problems with linear growth (see Theorem 3.4 Chapter 4 [14], Theorem 6.1 [6] and Theorem 7.1 [6] when a Dirichlet boundary datum is added). To avoid technicalities we state it in the simplest case.

Theorem 2.8. *Given an open set* $A \subseteq \mathbf{R}^n$ *and a Borel function* $\psi : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$, *consider*

$$G_\psi(u, A) = \int_A |\nabla u| dx, \text{ if } u \in W^{1,1}(A), \ \tilde{u} \geq \psi \ \mathcal{H}^{n-1} \text{ a.e. on } A, \tag{2.8}$$

and $+\infty$ *otherwise in* $L^1(A)$. *Then, the lower semicontinuos envelope of* G_ψ *in the* L^1 *topology is given by*

$$\mathcal{G}_\psi(u, A) = \|Du\|(A) + \int_A [(\psi - u^+) \vee 0] d\sigma$$

if $u \in BV(A)$, $+\infty$ *otherwise in* $L^1(A)$.

3. A variational measure

In this section we introduce a regular Borel measure on any open set $A \subseteq \mathbf{R}^n$ following one of the characterizations of the measure σ provided in Proposition 2.6.

According to the definition given by [6], for any $\varepsilon > 0$ and for any set $E \subseteq \mathbf{R}^n$ we consider the set functions $\sigma^\varepsilon_{MS}(E, A)$ defined by

$$\inf \left\{ MS_p(u, A) + \frac{1}{\varepsilon} \int_A |u|^p \, dx \, : u \in SBV(A), \; u^+ \geq 1 \, \mathcal{H}^{n-1} \text{ a.e. on } E \cap A \right\} \tag{3.1}$$

and

$$\sigma_{MS}(E, A) = \sup_{\varepsilon > 0} \sigma^\varepsilon_{MS}(E, A), \tag{3.2}$$

with the convention of dropping the dependence on A when $A = \mathbf{R}^n$.

Remark 3.1. *Similarly to Remark 2.7 the very definition of σ_{MS} entails that for any set E for which $\sigma_{MS}(E, A) < +\infty$ we can find a family of functions $(v_\varepsilon) \subseteq SBV(A)$ admissible for the minimum problems $\sigma^\varepsilon_{MS}(E, A)$ satisfying $\|v_\varepsilon\|^p_{L^p(A)} = o(\varepsilon^2)$ and*

$$\sigma_{MS}(E, A) = MS_p(v_\varepsilon, A) + o(1).$$

It turns out that the set function σ_{MS} introduced above coincides with the measure σ. To explain this fact we notice that the penalization of the L^p norm forces minimizing functions for $\sigma^\varepsilon_{MS}(E, A)$ to make a transition from 1 to 0 in a thinner and thinner set enclosing E. Therefore the superlinearity in the bulk term makes energetically more convenient for minimizing functions to have a discontinuity in a neighbourhood of E rather than having a high gradient energy. Finally, note that the Mumford-Shah and the total variation functionals coincide on sets of finite perimeter.

Proposition 3.2. *For any open set $A \subseteq \mathbf{R}^n$, for any set $E \subseteq \mathbf{R}^n$, we have*

$$\sigma_{MS}(E, A) = \sigma(E, A).$$

Proof. Let A be a fixed open set throughout all the proof. Given any set E, taking into account that for any measurable set $D \subseteq \mathbf{R}^n$ with finite perimeter which is admissible for $\sigma(E, A)$, the function $u = \chi_D \in SBV(A)$ is admissible for $\sigma_{MS}(E, A)$ and $MS_p(u, A) = \mathrm{Per}(D, A)$, we have $\sigma_{MS}(E, A) \leq \sigma(E, A)$.

In order to get the opposite inequality it suffices to consider a set E such that $\sigma_{MS}(E, A) < +\infty$. Fixed a family of functions (v_ε) as in Remark 3.1, the strategy to prove the inequality $\sigma_{MS} \geq \sigma$ relies on finding suitable superlevel sets of v_ε such that their perimeters are bounded above by the Mumford-Shah energies of v_ε, their $\mathcal{L}^n$ measures are negligible with respect to ε, and the set $E \cap A$ is contained $\mathcal{H}^{n-1}$ a.e. in such superlevel sets.

Let $\eta > 0$, then by Remark 3.1 we can find $v_\varepsilon \in SBV(A)$ such that $v_\varepsilon^+(x) \geq 1$ $\mathcal{H}^{n-1}$ a.e. on $E \cap A$ and

$$MS_p(v_\varepsilon, A) + \frac{1}{\varepsilon^2}\int_A |v_\varepsilon|^p \, dx \leq \sigma_{MS}(E, A) + \eta. \tag{3.3}$$

Up to passing to $0 \vee v_\varepsilon \wedge 1$ we may also assume that $0 \leq v_\varepsilon \leq 1$. By the BV Coarea formula (see Theorem 3.40 [2]) we may choose $z_\varepsilon \in (\varepsilon^{\frac{1}{2p}}, 1)$ such that

$$\begin{aligned}(1-\varepsilon^{\frac{1}{2p}})&\mathrm{Per}(\{x \in A : \, v_\varepsilon(x) > z_\varepsilon\}, A)\\ &\leq \int_{\varepsilon^{\frac{1}{2p}}}^1 \mathrm{Per}(\{x \in A : \, v_\varepsilon(x) > t\}, A) \, dt \leq \|Dv_\varepsilon\|(\{x \in A : \, v_\varepsilon(x) > \varepsilon^{\frac{1}{2p}}\}).\end{aligned} \tag{3.4}$$

Letting $D_\varepsilon := \{x \in A : \, v_\varepsilon(x) > z_\varepsilon\}$ and $A_\varepsilon := \{x \in A : \, v_\varepsilon(x) > \varepsilon^{\frac{1}{2p}}\}$, Hölder inequality, the fact that $|v_\varepsilon^+(x) - v_\varepsilon^-(x)| \leq 1$ $\mathcal{H}^{n-1}$ a.e., and (3.4) imply

$$(1-\varepsilon^{\frac{1}{2p}})\mathrm{Per}(D_\varepsilon, A) \leq \int_{A_\varepsilon} |\nabla v_\varepsilon| \, dx + \mathcal{H}^{n-1}(S_{v_\varepsilon}) \leq \mathcal{L}^n(A_\varepsilon)^{\frac{p-1}{p}} \|\nabla v_\varepsilon\|_{L^p(A)} + \mathcal{H}^{n-1}(S_{v_\varepsilon}). \tag{3.5}$$

Moreover, by (3.3)

$$\mathcal{L}^n(A_\varepsilon)\varepsilon^{\frac{1}{2}} \leq \int_A |v_\varepsilon|^p \, dx \leq (\sigma_{MS}(E, A) + \eta)\varepsilon^2,$$

from which we infer $\mathcal{L}^n(A_\varepsilon) = o(\varepsilon)$ and $\mathcal{L}^n(D_\varepsilon) = o(\varepsilon)$, and thus D_ε has finite perimeter in A. In particular, using (3.3), for ε small enough (3.5) rewrites as

$$\mathrm{Per}(D_\varepsilon, A) \leq \sigma_{MS}(E, A) + 2\eta. \tag{3.6}$$

Furthermore, $D_\varepsilon^+ \supseteq \{x \in A : \, v_\varepsilon^+(x) \geq 1\}$ by (2.1), and thus $\mathcal{H}^{n-1}((E \cap A) \setminus D_\varepsilon^+) = 0$. Hence, D_ε is admissible for $\sigma^\varepsilon(E, A)$ and, taking (3.6) into account, for ε small enough it holds

$$\sigma^\varepsilon(E, A) \leq \mathrm{Per}(D_\varepsilon, A) + \frac{\mathcal{L}^n(D_\varepsilon)}{\varepsilon} \leq \sigma_{MS}(E, A) + 3\eta.$$

Taking first the supremum on ε and then letting $\eta \to 0^+$ we get the desired inequality. □

Remark 3.3. *Following* [8] *and* [6] *one could equivalently define $\sigma_{MS}(E, A)$ as the supremum of the set functions*

$$\inf\left\{MS_p(u, A) + \frac{1}{\varepsilon}\int_A |u|^p \, dx \, : u \in SBV(A), \; u \geq 1 \, \mathcal{L}^n \text{ a.e. on an open set } U \supseteq E \cap A\right\}. \tag{3.7}$$

Actually, by using the previous proposition and exploiting the equivalence between the two definitions already proven for the measure σ one gets that the final measure is the same (see Proposition 2.6).

Remark 3.4. *The proof of Proposition 3.2 shows that the measure σ coincides also with the one obtained by substituting in definitions* (3.1), (3.2) *the Mumford-Shah energy with any of the form*

$$\int_A f(\nabla u) dx + \mathcal{H}^{n-1}(S_u \cap A),$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is such that

$$c_1|\xi|^p \leq f(\xi) \leq c_2|\xi|^p$$

for every $\xi \in \mathbf{R}^n$, for some constants $c_1, c_2 > 0$ (see also Section 6).

We now introduce a Borel measure accounting for a generic obstacle. Let $\psi : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$, A an open set in $\mathbf{R}^n$, and $E \subseteq \mathbf{R}^n$, for any $\varepsilon > 0$ define $\sigma^\varepsilon_{MS}(E, A, \psi)$ as

$$\inf\left\{ MS_p(u, A) + \frac{1}{\varepsilon}\int_A |u|^p\,dx \,:\, u \in SBV(A),\ u^+ \geq \psi\ \mathcal{H}^{n-1} \text{ a.e. on } E \cap A \right\}, \tag{3.8}$$

and as usual set $\sigma_{MS}(E, A, \psi) = \sup_{\varepsilon>0} \sigma^\varepsilon_{MS}(E, A, \psi)$. With a slight abuse of notation when $\psi(x) \equiv c$ we denote $\sigma_{MS}(\cdot, A, \psi)$ simply by $\sigma_{MS}(\cdot, A, c)$. With this notation then $\sigma_{MS}(\cdot, A, 1) = \sigma_{MS}(\cdot, A)$.

Assuming ψ to be a Borel function, one can push forward the arguments used in Proposition 3.2 and prove the following description of $\sigma_{MS}(\cdot, A, \psi)$ on Borel sets.

Proposition 3.5. *For any open set A, for any Borel function ψ, and any Borel set $E \subseteq \mathbf{R}^n$ we have*

$$\sigma_{MS}(E, A, \psi) = \sigma_{MS}(\{x \in E : \psi(x) > 0\}, A).$$

Proof. The open set A and the Borel function ψ will be fixed throughout the whole proof.
Given a Borel set E we first prove that $\sigma_{MS}(E, A, \psi) \geq \sigma_{MS}(\{x \in E : \psi(x) > 0\}, A)$. Thus, it is not restrictive to assume $\sigma_{MS}(E, A, \psi) < +\infty$. With $\lambda \in (0, 1)$ fixed, we claim that

$$\sigma_{MS}(E, A, \psi) \geq \sigma_{MS}(\{x \in E : \psi(x) > \lambda\}, A). \tag{3.9}$$

It is clear that the required inequality will easily follow letting $\lambda \to 0^+$ and using the fact already proved that $\sigma_{MS}(\cdot, A)$ is a regular Borel measure.
In order to get (3.9) we will exploit the same construction and arguments introduced in the proof of Proposition 3.2 complemented with the Borel regularity assumptions.
Let $\eta > 0$ be fixed, reasoning as in Remark 3.1 one can consider functions $w_\varepsilon \in SBV(A)$ such that $w^+_\varepsilon(x) \geq \psi(x)\ \mathcal{H}^{n-1}$ a.e. on $E \cap A$ and

$$MS_p(w_\varepsilon, A) + \frac{1}{\varepsilon^2}\int_A |w_\varepsilon|^p\,dx \leq \sigma_{MS}(E, A, \psi) + \eta. \tag{3.10}$$

Arguing as in the proof of Proposition 3.2 with respect to the functions $v_\varepsilon := 0 \vee (w_\varepsilon/\lambda) \wedge 1$, one can find superlevel sets $D_\varepsilon = \{x \in A : w_\varepsilon(x) > z'_\varepsilon\}$ with $z'_\varepsilon \in (\lambda\varepsilon^{\frac{1}{2p}}, \lambda)$ such that

$$\mathrm{Per}(D_\varepsilon, A) + \frac{1}{\varepsilon}\mathcal{L}^n(D_\varepsilon) \leq \sigma_{MS}(E, A, \psi) + 3\eta. \tag{3.11}$$

Indeed, taking into account (3.10) and the fact that

$$\frac{1}{\varepsilon^2}\int_A |v_\varepsilon|^p\,dx \leq \frac{1}{\lambda^p\varepsilon^2}\int_A |w_\varepsilon|^p\,dx$$

if we set $D_\varepsilon = \{x \in A : v_\varepsilon(x) > z_\varepsilon\}$ with any $z_\varepsilon \in (\varepsilon^{\frac{1}{2p}}, 1)$, we get $\mathcal{L}^n(D_\varepsilon) = o(\varepsilon)$. We now choose $z_\varepsilon \in (\varepsilon^{\frac{1}{2p}}, 1)$ such that (3.5) holds true for v_ε defined as above. Since $\mathcal{L}^n(D_\varepsilon) = o(\varepsilon)$, $\|\nabla v_\varepsilon\|_{L^p(A)} \leq \lambda^{-1}\|\nabla w_\varepsilon\|_{L^p(A)}$ and $\mathcal{H}^{n-1}(S_{v_\varepsilon}) \leq \mathcal{H}^{n-1}(S_{w_\varepsilon})$, it is enough to take $z'_\varepsilon := \lambda z_\varepsilon$ to obtain superlevel sets of the initial functions w_ε with the property (3.11).
Moreover, since for $\mathcal{H}^{n-1}$ a.e. $x \in A \cap \{x \in E : \psi(x) > \lambda\}$ it holds $w^+_\varepsilon(x) \geq \psi(x) > \lambda$, by definition $v^+_\varepsilon(x) = 1$. Hence, taking (2.1) into account, $D^+_\varepsilon \supseteq \{x \in A : v^+_\varepsilon(x) \geq 1\} \supseteq A \cap \{x \in E : \psi(x) > \lambda\}$

and thus the functions $u_\varepsilon = \chi_{D_\varepsilon}$ are admissible for $\sigma^\varepsilon(\{x \in E : \psi(x) > \lambda\}, A)$. Letting eventually $\eta \to 0^+$ in (3.11), we get (3.9).

Notice that the same argument implies that for any positive constant c we have

$$\sigma_{MS}(E, A, c) = \sigma_{MS}(E, A). \tag{3.12}$$

In order to prove the inverse inequality let us consider a Borel set E such that $\sigma_{MS}(\{x \in E : \psi(x) > 0\}, A) < +\infty$. This condition implies at once that the set $A \cap \{x \in E : \psi(x) = +\infty\}$ is $\mathcal{H}^{n-1}$ negligible (see (2.5)). Setting $E_i = \{x \in E : i+1 \geq \psi(x) > i\}$ for $i \in \mathbf{N}$, by the standard additivity property of the Borel measure $\sigma_{MS}(\cdot, A)$ we have

$$\sigma_{MS}(\{x \in E : \psi(x) > 0\}, A) = \sum_{i=0}^{+\infty} \sigma_{MS}(E_i, A).$$

For $\eta > 0$ fixed, let v_ε^i be almost optimal for $\sigma_{MS}(E_i, A, i+1)$, that is $(v_\varepsilon^i)^+(x) \geq i+1$ $\mathcal{H}^{n-1}$ a.e. on E_i and

$$MS_p(v_\varepsilon^i, A) + \frac{1}{\varepsilon}\int_A |v_\varepsilon^i|^p \, dx \leq \sigma_{MS}(E_i, A) + \frac{\eta}{2^i}, \tag{3.13}$$

recalling that $\sigma_{MS}(E_i, A, i+1) = \sigma_{MS}(E_i, A)$ by (3.12).

Set $u_\varepsilon^k := \sup_{0 \leq i \leq k} v_\varepsilon^i$. Then $u_\varepsilon^k \in SBV(A)$ and

$$MS_p(u_\varepsilon^k, A) + \frac{1}{\varepsilon}\int_A |u_\varepsilon^k|^p \, dx \leq \sum_{i=0}^{k}(MS_p(v_\varepsilon^i, A) + \frac{1}{\varepsilon}\int_A |v_\varepsilon^i|^p \, dx) \leq \sum_{i=0}^{k} \sigma_{MS}(E_i, A) + 2\eta. \tag{3.14}$$

In particular, (u_ε^k) is a non-decreasing sequence satisfying the hypotheses of the GSBV compactness theorem 2.1, so that there exists a function $u_\varepsilon \in SBV(A)$ such that $u_\varepsilon^k \to u_\varepsilon$ in $L^1(A)$. Thus, from (3.14) it follows

$$\begin{aligned} MS_p(u_\varepsilon, A) &+ \frac{1}{\varepsilon}\int_A |u_\varepsilon|^p \, dx \leq \\ &\liminf_{k \to +\infty} \left(MS_p(u_\varepsilon^k, A) + \frac{1}{\varepsilon}\int_A |u_\varepsilon^k|^p \, dx \right) \leq \sigma_{MS}(\{x \in E : \psi(x) > 0\}, A) + 2\eta. \end{aligned}$$

Moreover, since $A \cap \{x \in E : \psi(x) > 0\} = \cup_{i \geq 0}(A \cap E_i)$, for $\mathcal{H}^{n-1}$ a.e. $z \in A \cap \{x \in E : \psi(x) > 0\}$ there exists $i \in \mathbf{N}$ such that $z \in A \cap E_i$, and then $u_\varepsilon^+(z) \geq (u_\varepsilon^{i+1})^+(z) \geq i+1 \geq \psi(z)$. Eventually, u_ε is admissible as a test function for $\sigma_{MS}^\varepsilon(E, A, \psi)$ and the inequality follows as usual. $\square$

4. Relaxation result

Given an open bounded set Ω and a Borel function $\psi : \Omega \to \mathbf{R} \cup \{\pm\infty\}$, we study the lower semicontinuous envelope of the functional $F_\psi : L^1(\Omega) \to [0, +\infty]$ defined by

$$F_\psi(u, \Omega) = \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u) \quad \text{if } u \in GSBV(\Omega),\ u^+ \geq \psi\ \ \mathcal{H}^{n-1} \text{ a.e. on } \Omega, \tag{4.1}$$

and $+\infty$ otherwise in $L^1(\Omega)$.

Building on what has been shown in Section 3 we are able to prove the main result of the paper.

Theorem 4.1. *Let F_ψ be as in (4.1), then its lower semicontinuous envelope in the L^1 topology is given by*

$$\mathcal{F}_\psi(u,\Omega) = \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u)$$
$$+\frac{1}{2}\sigma\left(\{x \in S_u : u^+(x) < \psi(x)\}\right) + \sigma\left(\{x \in \Omega \setminus S_u : u^+(x) < \psi(x)\}\right) \tag{4.2}$$

if $u \in GSBV(\Omega)$, $+\infty$ otherwise in $L^1(\Omega)$.

In the sequel it is not restrictive to presume the existence of $w \in GSBV(\Omega)$ such that $F_\psi(w,\Omega) < +\infty$, being otherwise $F_\psi \equiv \mathcal{F}_\psi \equiv +\infty$.
For such w's we have $\{x \in \Omega : \psi(x) = +\infty\} \subseteq \{x \in \Omega : w^+(x) = +\infty\}$, which implies

$$\mathcal{H}^{n-1}(\{x \in \Omega : \psi(x) = +\infty\}) = 0 \tag{4.3}$$

(see Theorem 4.40 [2]).
To prove Theorem 4.1 we address separately the lower and upper bound inequalities.

Proposition 4.2. *For every u and (u_j) in $L^1(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$ we have*

$$\liminf_j F_\psi(u_j,\Omega) \geq \mathcal{F}_\psi(u,\Omega).$$

Proof. First notice that we may assume $\liminf_j F_\psi(u_j,\Omega)$ to be finite being the result trivial otherwise; then by Ambrosio's theorem 2.1 we have $u \in GSBV(\Omega)$. Moreover, we may assume the inferior limit above to be a limit up to extracting a subsequence which we do not relabel for convenience.
We claim the following three estimates to hold true for every open set $A \subseteq \Omega$

$$\liminf_j F_\psi(u_j,A) \geq MS_p(u,A), \tag{4.4}$$

$$\liminf_j F_\psi(u_j,A) \geq \sigma(A \cap \{x \in S_u : u^+(x) < \psi(x)\}), \tag{4.5}$$

$$\liminf_j F_\psi(u_j,A) \geq \sigma\left(A \cap \{x \in \Omega \setminus S_u : u^+(x) < \psi(x)\}\right). \tag{4.6}$$

Given them for granted the result follows by standard measure theoretic arguments (see Proposition 1.16 [4]). Indeed, set $\Sigma_u = \{x \in \Omega : u^+(x) < \psi(x)\}$, then from (4.4), (4.5) and (4.6) and taking into account (2.7), for any $\lambda, \mu \in [0,1]$, $\lambda + \mu \leq 1$ it follows

$$\liminf_j F_\psi(u_j,A) \geq \lambda \int_A |\nabla u|^p dx + \lambda \mathcal{H}^{n-1}(A \cap (S_u \setminus \Sigma_u))$$
$$+(\lambda + 2\mu)\mathcal{H}^{n-1}(A \cap S_u \cap \Sigma_u) + (1 - \lambda - \mu)\sigma(A \cap (\Sigma_u \setminus S_u)).$$

Being the left hand side above a superadditive set function on disjoint open sets of Ω and the right hand side sum of orthogonal Radon measures, we can pass to the supremum on λ, μ separately on each term and infer

$$\liminf_j F_\psi(u_j,\Omega) \geq \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u \setminus \Sigma_u) + 2\mathcal{H}^{n-1}(S_u \cap \Sigma_u) + \sigma(\Sigma_u \setminus S_u)$$

$$= \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u) + \frac{1}{2}\sigma(S_u \cap \Sigma_u) + \sigma(\Sigma_u \setminus S_u),$$

which gives the thesis.

Since (4.4) follows immediately by Ambrosio's theorem 2.1, to conclude the proof we are left with showing the validity of (4.5) and (4.6).

Step 1: Proof of (4.5). We begin with proving the inequality in the one-dimensional case which reads as follows

$$\liminf_j \int_\Omega |\dot{u}_j|^p dt + \mathcal{H}^0(S_{u_j}) \geq 2\mathcal{H}^0(S_u \cap \Sigma_u).$$

We notice that the approximating functions are forced to make a transition from the trace values $u^\pm(\bar{t})$ to the obstacle constraint $\psi(\bar{t})$ in any neighbourhood I of a discontinuity point $\bar{t}$ of u where the constraint is violated, that is $u^+(\bar{t}) < \psi(\bar{t})$. Hence, to prove the estimate above we will quantify the cost of this transition, and show that it is energetically convenient for the approximating functions to have asymptotically at least 2 discontinuity points in I.

Recall that we have assumed

$$\liminf_j \int_\Omega |\dot{u}_j|^p dt + \mathcal{H}^0(S_{u_j}) = \lim_j \int_\Omega |\dot{u}_j|^p dt + \mathcal{H}^0(S_{u_j}) \leq M < +\infty, \tag{4.7}$$

which gives $u_j \in SBV(\Omega)$ for every $j \in \mathbf{N}$; moreover, for a subsequence not relabeled for convenience, we suppose $u_j \to u$ $\mathcal{L}^1$ a.e. in Ω.

We claim that for j sufficiently big

$$\mathcal{H}^0\left(A \cap S_{u_j}\right) \geq 2\mathcal{H}^0\left(A \cap S_u \cap \Sigma_u\right). \tag{4.8}$$

With fixed $\bar{t}$ in the finite set $S_u \cap \Sigma_u$, there exists $\delta > 0$ such that $I_\delta = (\bar{t} - \delta, \bar{t} + \delta) \subset\subset A$ and $I_\delta \cap S_u = \{\bar{t}\}$. Furthermore, being u a Sobolev function on $(\bar{t}-\delta, \bar{t})$ and $(\bar{t}, \bar{t}+\delta)$ separately, we choose $\delta > 0$ sufficiently small such that

$$u(t) \leq u^+(\bar{t}) + \varepsilon$$

for every $t \in I_\delta$, with $\varepsilon \in (0, (\psi(\bar{t}) - u^+(\bar{t}))/4)$.

It is clear that (4.8) follows provided we show

$$\liminf_j \mathcal{H}^0\left(I_\delta \cap S_{u_j}\right) \geq 2. \tag{4.9}$$

Arguing by contradiction we first observe that, up to a subsequence, $I_\delta \cap S_{u_j} = \{t_j\}$ since by lower semicontinuity $\liminf_j \mathcal{H}^0\left(I_\delta \cap S_{u_j}\right) \geq 1$ (see Theorem 2.1). In the sequel we show that this implies

$$\liminf_j \int_{I_\delta} |\dot{u}_j|^p dt \geq \delta^{1-p}\left(\frac{\psi(\bar{t}) - u^+(\bar{t})}{4}\right)^p. \tag{4.10}$$

Select two points $s_1 \in (\bar{t}-\delta, \bar{t}) \setminus \cup_j \{t_j\}$ and $s_2 \in (\bar{t}, \bar{t}+\delta) \setminus \cup_j \{t_j\}$ such that $u_j(s_i) \to u(s_i)$ for $i = 1, 2$, then for j sufficiently big and $i = 1, 2$

$$u_j(s_i) \leq u(s_i) + \varepsilon \leq u^+(\bar{t}) + 2\varepsilon.$$