Cerchiamo i $t\in[-1,1]$ t.c. $\varphi(t)<0$:

$$\varphi'(t) = 2k e^{2t} + \frac{1}{2} = 0 \iff e^{2t} = -\frac{1}{4k}$$

quindi:

$\boxed{k>0} \Rightarrow \varphi \nearrow\nearrow$ con $\max\limits_{[-1,1]} \varphi = \varphi(1) = 2ke^{2} + \frac{3}{4} > 0$
$\min\limits_{[-1,1]} \varphi = \varphi(-1) = 2ke^{-2} - \frac{1}{4}$

e quindi: $k \geq \frac{e^2}{8} \Rightarrow \varphi \geq 0 \ \forall t\in[-1,1] \Rightarrow \tilde{I}_k = \emptyset$

$k\in\left(0, \frac{e^2}{8}\right) \Rightarrow \varphi < 0 \ \forall t \in [-1, \alpha_k) = \tilde{I}_k,\ \varphi(\alpha_k)=0$

$\boxed{k\leq 0} \Rightarrow \varphi \nearrow\nearrow\ t < t_k = \frac{1}{2}\ln\left(-\frac{1}{4k}\right),\ \searrow\searrow\ t>t_k$

$\boxed{t_k \leq -1}$: $(\iff k \leq -e^2/4) \Rightarrow \varphi \searrow\searrow$ in $[-1,1] \Rightarrow$
$\max\limits_{[-1,1]}\varphi = \varphi(-1) = 2ke^{-2} - \frac{1}{4}$ ed è $<0$ se $k < -\frac{e^2}{4}$
quindi: $k \leq -e^2/4 \Rightarrow \varphi < 0\ [-1,1] = \tilde{I}_k$

$\boxed{t_k \geq 1}$: $(\iff k \geq -e^{-2}/4) \Rightarrow \varphi \nearrow\nearrow$ in $[-1,1] \Rightarrow$
$\max\limits_{[-1,1]}\varphi = \varphi(1) = 2ke^{2} + \frac{3}{4} > 0$, $\min\limits_{[-1,1]}\varphi = \varphi(-1) = 2ke^{-2} - \frac{1}{4} < 0$
quindi: $-\frac{e^{-2}}{4} \leq k \leq 0 \Rightarrow \varphi<0\ [-1,\alpha_k) = \tilde{I}_k,\ \varphi(\alpha_k)=0$

$\boxed{t_k\in(-1,1)}$: $\left(\iff -\frac{e^2}{4} < k < -\frac{e^{-2}}{4}\right) \Rightarrow \max\limits_{[-1,1]}\varphi = \max\limits_{\mathbb{R}}\varphi$
$= \varphi(t_k) \overset{\varphi'(t_k)=0}{=} \frac{t_k}{2}$, $t_k<0 \iff k<-1/4$, $\varphi(-1) \lessgtr 0$,
$\varphi(1) = 2ke^2 + \frac{3}{4}$

$k\in\left(-\frac{e^2}{4}, -\frac{1}{4}\right) \Rightarrow \varphi<0\ [-1,1] = \tilde{I}_k$

$k\in\left[-\frac{1}{4}, -\frac{3}{8}e^{-2}\right] \Rightarrow \varphi<0\ [-1,\alpha_k)\cup(\beta_k,1] = \tilde{I}_k,\ \varphi(\alpha_k)=\varphi(\beta_k)=0$

$k\in\left(-\frac{3}{8}e^{-2}, -\frac{e^{-2}}{4}\right) \Rightarrow \varphi<0\ [-1,\alpha_k) = \tilde{I}_k,\ \varphi(\alpha_k)=0$

Per determinare $I_k$ basta ora imporre $\cos x \in \tilde{I}_k$ e scegliere il punto iniziale.