ESERCIZIO 1: Calcolare

$$\lim_{x\to+\infty} \frac{\frac{\pi}{2}\cos\frac{1}{x} - \operatorname{atn}\left(\frac{4}{\pi}x^2\right) - \frac{1}{48x}\operatorname{sh}\left(\frac{\pi}{x^3}\right)}{\left(x^{1/x^3}-1\right)^2 \ln\left(1+\frac{1}{\ln^2 x}\right)}$$

Utilizzando la notazione esponenziale ed i limiti notevoli $\frac{e^t-1}{t} = 1+o(1)\ \ t\to 0$, $\frac{\ln(1+t)}{t} = 1+o(1)\ t\to 0$ il denominatore si sviluppa come segue:

$$\left(x^{1/x^3}-1\right)^2 \ln\left(1+\frac{1}{\ln^2 x}\right) = \left(e^{\frac{\ln x}{x^3}}-1\right)^2 \left(\frac{1}{\ln^2 x}+o\left(\frac{1}{\ln^2 x}\right)\right)$$

$\frac{\ln x}{x^3} \xrightarrow{x\to+\infty} 0$

$$= \left(\frac{\ln x}{x^3}+o\left(\frac{\ln x}{x^3}\right)\right)^2 \left(\frac{1}{\ln^2 x}+o\left(\frac{1}{\ln^2 x}\right)\right) =$$

$$= \frac{1}{x^6}\left(1+o(1)\right)^2\left(1+o(1)\right) = \frac{1}{x^6}+o\left(\frac{1}{x^6}\right) \quad x\longrightarrow+\infty$$

Per il numeratore invece si utilizzano gli sviluppi di Taylor di $\cos t$, $\operatorname{sh} t$, $\operatorname{atn} t$ in $t=0$:

$$\frac{\pi}{2}\cos\frac{1}{x} = \frac{\pi}{2}\left(1-\frac{1}{2x^2}+\frac{1}{24x^4}-\frac{1}{6!\,x^6}+o\left(\frac{1}{x^7}\right)\right) \quad x\to+\infty$$

$$-\operatorname{atn}\left(\frac{4}{\pi}x^2\right) = -\frac{\pi}{2}+\operatorname{atn}\left(\frac{\pi}{4x^2}\right) = -\frac{\pi}{2}+\frac{\pi}{4x^2}-\frac{\pi^3}{3\cdot 4^3 x^6}+o\left(\frac{1}{x^8}\right) \quad x\to+\infty$$

$$-\frac{1}{48x}\operatorname{sh}\left(\frac{\pi}{x^3}\right) = -\frac{1}{48x}\left(\frac{\pi}{x^3}+\frac{\pi^3}{6x^9}+o\left(\frac{1}{x^{12}}\right)\right) \quad x\to+\infty$$

da cui

$$\frac{\pi}{2}\cos\frac{1}{x} - \operatorname{atn}\left(\frac{4}{\pi}x^2\right) - \frac{1}{48x}\operatorname{sh}\left(\frac{\pi}{x^3}\right) = \frac{1}{x^6}\left(-\frac{\pi}{2\cdot 6!}-\frac{\pi^3}{3\cdot 4^3}+o\left(\frac{1}{x}\right)\right)$$

$\Longrightarrow$ il valore del limite è $-\frac{\pi}{2\cdot 6!}-\frac{\pi^3}{3\cdot 4^3}$

ESERCIZIO 2: Sia $f(x) = \sin^{21}(\ln x) - \operatorname{sh}^{21}(\ln x)$, determinare $f^{(23)}(1)$.

Poiché $f \in C^{23}((0,+\infty))$, in realtà $C^\infty((0,+\infty))$, basta trovare un polinomio $p(x) = \sum_{k=0}^{23} a_k (x-1)^k$ t.c.

$$f(x) = p(x) + o((x-1)^{23}) \quad x \longrightarrow 1,$$

dato che dall'unicità del polinomio di Taylor segue:
$$a_{23} = \frac{f^{(23)}(1)}{23!}.$$

Per determinare $p$ si usano gli sviluppi di $\sin t$, $\operatorname{sh} t$, $\ln(1+t)$, $(1+t)^{21}$ in $t=0$ e l'algebra degli o-piccoli:

$$f(x) = \left(\ln x - \frac{\ln^3 x}{6} + o(\ln^4 x)\right)^{21} - \left(\ln x + \frac{\ln^3 x}{6} + o(\ln^4 x)\right)^{21}$$

$$= \left((x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^3}{6} + o((x-1)^3)\right)^{21} +$$

$$- \left((x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \frac{(x-1)^3}{6} + o((x-1)^3)\right)^{21} \quad x \longrightarrow 1$$

$$= (x-1)^{21}\left[\left(1 \overbrace{- \frac{(x-1)}{2} + \frac{(x-1)^2}{6} + o((x-1)^2)}^{Y}\right)^{21} - \left(1 \overbrace{- \frac{(x-1)}{2} + \frac{(x-1)^2}{2} + o((x-1)^2)}^{Z}\right)^{21}\right]$$

$$= (x-1)^{21}\left[\cancel{1} + 21Y + 21 \cdot \frac{\overset{10}{\cancel{20}}}{\cancel{2}} Y^2 + o(Y^2) - \cancel{1} - 21Z - 21 \cdot \frac{\overset{10}{\cancel{20}}}{\cancel{2}} Z^2 + o(Z^2)\right]$$

$$= (x-1)^{21}\left[21 \cdot \left(-\frac{(x-1)^2}{3}\right) + 210 \overset{0}{\cancel{\left(\frac{(x-1)^2}{4} - \frac{(x-1)^2}{4}\right)}} + o((x-1)^2)\right]$$

$$= -7(x-1)^{23} + o((x-1)^{23}) \quad x \longrightarrow 1$$

$$\Longrightarrow f^{(23)}(1) = -7 \cdot 23!$$

$\boxed{\text{ESERCIZIO 3:}}$ Determinare le primitive di $\dfrac{1}{(1-x^2)^3}$ ③

<u>1° MODO</u>: Poichè $1-x^2=(1-x)(1+x)$ si cerca una fattorizzazione del tipo:

$$\frac{1}{(1-x^2)^3}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{(1-x)^3}+\frac{D}{1+x}+\frac{E}{(1+x)^2}+\frac{F}{(1+x)^3}.$$

Essendo $x\longrightarrow(1-x^2)^{-3}$ pari, si deduce che $A=D$, $B=E$, $C=F$.

Quindi:

$$C=\lim_{x\to1}\frac{\cancel{(1-x)^3}}{\cancel{(1-x^2)^3}}\overset{\frac{1}{(1+x)^3}}{}=\frac{1}{8}\Rightarrow\boxed{C=F=\frac{1}{8}}$$

$$B=\lim_{x\to1}(1-x)^2\left[\frac{1}{(1-x^2)^3}-\frac{1}{8(1-x)^3}\right]=\lim_{x\to1}\frac{8-(1+x)^3}{8(1+x)^3(1-x)}$$

$$=\lim_{x\to1}\frac{2-\cancel{(1+x)}}{\cancel{8(1-x)}}\,\frac{1}{8}\,\frac{4+(1+x)^2+2(1+x)}{(1+x)^3}=\frac{3}{16}\Rightarrow\boxed{B=E=\frac{3}{16}}$$

$$A=\lim_{x\to1}(1-x)\left[\frac{1}{(1-x^2)^3}-\frac{5}{32}\frac{1}{(1-x)^2}-\frac{1}{8}\frac{1}{(1-x)^3}\right]$$

$$=\lim_{x\to1}(1-x)\left[\frac{4+(1+x)^2+2(1+x)}{8(1+x)^3(1-x)^2}-\frac{3}{16}\frac{1}{(1-x)^2}\right]$$

$$=\lim_{x\to1}\frac{8+2(1+x)^2+4(1+x)-3(1+x)^3}{16(1+x)^3(1-x)}$$

$t=1+x$

$$=\lim_{t\to2}\frac{8+2t^2+4t-3t^3}{16t^3(2-t)}=\lim_{t\to2}\frac{+3t^2+4t+4}{16t^3}=\frac{3}{16}$$

$$\Longrightarrow\boxed{A=D=\frac{3}{16}}$$

Da questo segue:

$$\int \frac{1}{(1-x^2)^3}\,dx = \frac{3}{16}\ln\left|\frac{1+x}{1-x}\right| + \frac{3}{16}\frac{1}{1-x} + \frac{1}{16}\frac{1}{(1-x)^2} +$$

$$-\frac{3}{16}\frac{1}{1+x} - \frac{1}{16}\frac{1}{(1+x)^2} + C$$

$$= \frac{3}{16}\ln\left|\frac{1+x}{1-x}\right| + \frac{3}{\cancel{16}_8}\frac{2x}{1-x^2} + \frac{1}{4}\cdot\frac{x}{(1-x^2)^2} + C$$

<u>2° MODO</u>: Si noti che: $\displaystyle \frac{1}{(1-x^2)^3} \overset{\pm x^2}{=} \frac{1}{(1-x^2)^2} + \frac{x^2}{(1-x^2)^3} =$

$$\overset{\pm x^2}{=} \frac{1}{1-x^2} + \frac{x^2}{(1-x^2)^2} + \frac{x^2}{(1-x^2)^3}$$

D'altra parte, integrando per parti:

$$\int \frac{x^2}{(1-x^2)^3}\,dx = x\int \frac{x}{(1-x^2)^3}\,dx - \int\left(\int \frac{x}{(1-x^2)^3}\,dx\right)dx,$$

poiché $\displaystyle \int \frac{x}{(1-x^2)^3}\,dx \overset{t=x^2}{=} \frac{1}{2}\int \frac{1}{(1-t)^3}\,dt = \frac{1}{4}\frac{1}{(1-x^2)^2} + C$

si ottiene quindi:

$$\int \frac{x^2}{(1-x^2)^3}\,dx = \frac{x}{4(1-x^2)^2} - \int \frac{1}{4}\frac{1}{(1-x^2)^2}\,dx$$

$$= \frac{x}{4(1-x^2)^2} - \frac{1}{4}\int \frac{1}{1-x^2}\,dx - \frac{1}{4}\int \frac{x^2}{(1-x^2)^2}\,dx$$

Da ciò segue:

$$\int \frac{1}{(1-x^2)^3}\,dx = \frac{x}{4(1-x^2)^2} + \frac{3}{4}\int \frac{1}{1-x^2}\,dx + \frac{3}{4}\int \frac{x^2}{(1-x^2)^2}\,dx$$

Infine: $\dfrac{1}{1-x^2} = \dfrac{1}{2}\,\dfrac{1}{1-x} + \dfrac{1}{2}\,\dfrac{1}{1+x}$

$$\Rightarrow \int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c\,,$$

ed inoltre integrando per parti

$$\int \frac{x^2}{(1-x^2)^2}\,dx = x\underbrace{\int \frac{x}{(1-x^2)^2}\,dx}_{t=x^2} - \int\left(\int \frac{x}{(1-x^2)^2}\,dx\right)dx$$

$$\frac{1}{2}\int \frac{1}{(1-t)^2}\,dt = \frac{1}{2(1-t)} + c = \frac{1}{2(1-x^2)} + c$$

$$= \frac{x}{2(1-x^2)} - \int \frac{1}{2(1-x^2)}\,dx = \frac{x}{2(1-x^2)} - \frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| + c$$

In conclusione:

$$\int \frac{1}{(1-x^2)^3}\,dx = \frac{x}{4(1-x^2)^2} + \frac{3}{8}\ln\left|\frac{1+x}{1-x}\right| + \frac{3x}{8(1-x^2)} + $$

$$- \frac{3}{16}\ln\left|\frac{1+x}{1-x}\right| + c$$

$$= \frac{x}{4(1-x^2)^2} + \frac{3x}{8(1-x^2)} + \frac{3}{16}\ln\left|\frac{1+x}{1-x}\right| + c$$

$\boxed{\text{ESERCIZIO 4:}}$ Calcolare

$$I = \int_{-\ln 2}^{\ln 2} e^{3x} \left( \operatorname{atn} e^{x} + \ln(e^{2x}+2) \right) dx$$

Effettuando la sostituzione $t = e^{x}$ si ottiene

$$I = \int_{+\frac{1}{2}}^{2} t^2 \left( \operatorname{atn} t + \ln(t^2+2) \right) dt = \underbrace{\int_{+\frac{1}{2}}^{2} t^2 \operatorname{atn} t \, dt}_{\overset{!!}{I_1}} + \underbrace{\int_{+\frac{1}{2}}^{2} t^2 \ln(t^2+2) \, dt}_{\overset{!!}{I_2}}$$

Integrando per parti si ottengono i valori $I_1, I_2$.

Infatti:

$$I_1 = \frac{t^3}{3} \operatorname{atn} t \Big|_{\frac{1}{2}}^{2} - \int_{\frac{1}{3}}^{2} \frac{t^3}{3} \frac{1}{1+t^2} dt = \frac{8}{3} \operatorname{atn} 2 - \frac{1}{24} \operatorname{atn} \frac{1}{2} + I_3$$

con

$$I_3 = -\frac{1}{3} \int_{\frac{1}{2}}^{2} \frac{t^3}{1+t^2} dt = -\frac{1}{3} \int_{\frac{1}{2}}^{2} \left( \frac{t^3+t}{t^2+1} - \frac{t}{t^2+1} \right) dt$$

$$= -\frac{1}{3} \left( \frac{t^2}{2} \Big|_{\frac{1}{2}}^{2} - \frac{1}{2} \ln(1+t^2) \Big|_{\frac{1}{2}}^{2} \right)$$

$$= -\frac{1}{3} \left( 2 - \frac{1}{8} - \frac{1}{2} \ln 5 + \frac{1}{2} \ln \frac{5}{4} \right)$$

$$= -\frac{1}{3} \left( \frac{15}{8} - \frac{1}{2} \ln 4 \right) = -\frac{5}{8} + \frac{\ln 2}{3}$$

$$\Longrightarrow \quad I_1 = \frac{8}{3} \operatorname{atn} 2 - \frac{1}{24} \operatorname{atn} \frac{1}{2} - \frac{5}{8} + \frac{\ln 2}{3}$$

Analogamente:

$$I_2 = \frac{t^3}{3} \ln(t^2+2) \Big|_{\frac{1}{2}}^{2} - \int_{\frac{1}{2}}^{2} \frac{t^3}{3} \frac{2t}{t^2+2} dt \Rightarrow$$

$$I_2 = \frac{8}{3}\ln 6 - \frac{1}{24}\ln\frac{9}{4} + I_4 = \frac{\cancel{33}^{11}}{\cancel{12}_{4}}\ln 2 + \frac{31}{12}\ln 3 + I_4$$

con

$$I_4 = -\frac{2}{3}\int_{1/2}^{2} \frac{t^4}{t^2+2}\,dt = -\frac{2}{3}\int_{1/2}^{2}\left(\frac{\cancel{t^4-4}^{\,t^2-2}}{\cancel{t^2+2}} + \frac{4}{t^2+2}\right)dt$$

$$= -\frac{2}{3}\left(\left(\frac{t^3}{3}-2t\right)\Bigg|_{1/2}^{2} + \frac{4}{\sqrt{2}}\,\mathrm{atn}\left(\frac{t}{\sqrt{2}}\right)\Bigg|_{1/2}^{2}\right)$$

$$= -\frac{2}{3}\left(\underbrace{\frac{8}{3}-4-\frac{1}{24}+1}_{-\frac{9}{24}=-\frac{3}{8}} + \frac{4}{\sqrt{2}}\,\mathrm{atn}\sqrt{2} - \frac{4}{\sqrt{2}}\,\mathrm{atn}\left(\frac{1}{2\sqrt{2}}\right)\right)$$

$$= \frac{1}{4} - \frac{4}{3}\sqrt{2}\,\mathrm{atn}\sqrt{2} + \frac{4}{3}\sqrt{2}\,\mathrm{atn}\left(\frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow I_2 = \frac{11}{4}\overset{\times}{\ln} 2 + \frac{31}{12}\ln 3 + \overset{\checkmark}{\frac{1}{4}} - \frac{4}{3}\sqrt{2}\,\mathrm{atn}\sqrt{2} + \frac{4}{3}\sqrt{2}\,\mathrm{atn}\left(\frac{1}{2\sqrt{2}}\right)$$

In conclusione:

$$I = -\frac{3}{8} + \frac{37}{12}\ln 2 + \frac{31}{12}\ln 3 + \frac{8}{3}\,\mathrm{atn}\,2 - \frac{1}{24}\,\mathrm{atn}\,\frac{1}{2}$$

$$-\frac{4}{3}\sqrt{2}\,\mathrm{atn}\sqrt{2} + \frac{4}{3}\sqrt{2}\,\mathrm{atn}\left(\frac{1}{2\sqrt{2}}\right)$$