ESERCIZIO 3: Calcolare

$$\int_0^2 \left(1+\sqrt{x+2}+\sqrt[3]{x+2}+\sqrt[6]{x+2}\right)^{-1} dx =: I$$

Si pone $t=\sqrt[6]{x+2} \iff t^6 = x+2$, allora

$$I = \int_{\sqrt[6]{2}}^{\sqrt[6]{4}} \left(1+t^3+t^2+t\right)^{-1} \cdot 6t^5\, dt$$

Usando l'algoritmo della divisione si ottiene

$$\frac{t^5}{t^3+t^2+t+1} = t^2-t+\frac{t}{t^3+t^2+t+1}$$

Si noti anche che $t^3+t^2+t+1 = (t+1)(t^2+1)$, da cui si decompone

$$\frac{t}{t^3+t^2+t+1} = \frac{At+B}{t^2+1}+\frac{C}{t+1} \iff \begin{cases} A=B=1/2 \\ C=-1/2 \end{cases}$$

quindi:

$$I = \frac{t^3}{3}-\frac{t^2}{2}\Bigg|_{\sqrt[6]{2}}^{\sqrt[6]{4}} + \frac{1}{2}\int_{\sqrt[6]{2}}^{\sqrt[6]{4}} \frac{t+1}{t^2+1}\,dt - \frac{1}{2}\ln(t+1)\Bigg|_{\sqrt[6]{2}}^{\sqrt[6]{4}}$$

$$\Rightarrow I = \frac{2-\sqrt{2}}{3}+\frac{\sqrt[6]{4}-\sqrt[3]{4}}{2}+\frac{1}{2}\ln\left(\frac{1+\sqrt[6]{2}}{1+\sqrt[6]{4}}\right)+ \\ +\frac{1}{2}\left(\operatorname{atn}\sqrt[6]{4}-\operatorname{atn}\sqrt[6]{2}\right)+\frac{1}{4}\ln\left(\frac{1+\sqrt[3]{4}}{1+\sqrt[6]{4}}\right)$$