ESERCIZIO 3: Provare che

$$f(x) = e^{\ln^2(1+x^2)} - 2\,\mathrm{ch}(x^2) + \cos(x^4) + x^3 \operatorname{atn}(x^3)$$

ha minimo relativo in $x=0$.

Dagli sviluppi notevoli si ottiene:

$$\ln^2(1+x^2) = \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} + o(x^6)\right)^2$$

$$= x^4 + \frac{x^8}{4} - x^6 + \frac{2}{3}x^8 + o(x^8)$$

$$e^{\ln^2(1+x^2)} = 1 + x^4 - x^6 + \left(\frac{1}{4}+\frac{2}{3}\right)x^8 + o(x^8) +$$

$$\frac{1}{2}\left(x^4 - x^6 + \left(\frac{1}{4}+\frac{2}{3}\right)x^8 + o(x^8)\right)^2$$

$$= 1 + x^4 - x^6 + \left(\frac{1}{4}+\frac{2}{3}+\frac{1}{2}\right)x^8 + o(x^8)$$

$$-2\,\mathrm{ch}(x^2) = -2 - x^4 - \frac{x^8}{12} + o(x^8)$$

$$\cos(x^4) = 1 - \frac{x^8}{2} + o(x^8)$$

$$x^3 \operatorname{atn}(x^3) = x^3\left(x^3 + o(x^6)\right) = x^6 + o(x^9)$$

quindi

$$f(x) = 1 + x^4 - x^6 + \left(\frac{1}{4}+\frac{2}{3}+\cancel{\frac{1}{2}}\right)x^8 - 2 - x^4 - \frac{x^8}{12} + 1 - \cancel{\frac{x^8}{2}}$$

$$+ x^6 + o(x^8)$$

$$= \frac{11}{12}x^8 + o(x^8) \geq 0 \quad \text{per } x \text{ sufficientemente piccolo}$$