ESERCIZIO 2: Siano $\alpha \in \mathbb{R}$ e

$$f_\alpha(x) = \frac{\pi \ln x + \sin(\sin(\pi x)) + \frac{\pi}{2}(x-1)^2}{\left|\cos\left(\frac{\pi}{2}x\right)\right|^\alpha}$$

allora $f_\alpha \in C^0([1,2])$ se $\alpha \leq 0$.

Se $\alpha > 0$:

$$\cos\left(\frac{\pi}{2}x\right) = \cos\left(\frac{\pi}{2}(x-1)+\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}(1-x)\right)$$

$$\Rightarrow \frac{\cos\left(\frac{\pi}{2}x\right)}{\frac{\pi}{2}(1-x)} \xrightarrow{x \to 1} 1$$

quindi il denominatore converge a zero come $|1-x|^\alpha$.

Per quanto riguarda il numeratore, dagli sviluppi notevoli:

$$\ln x = \ln(1+(x-1)) = x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+o((x-1)^3)$$

$$\sin(\sin(\pi x)) = \sin(\pi x) - \frac{1}{6}(\sin(\pi x))^3 + o((\sin(\pi x))^4)$$

$$\sin(\pi x) = \sin(\pi(x-1)+\pi) = \sin(\pi(1-x)) =$$
$$\pi(1-x) - \frac{\pi^3}{6}(1-x)^3 + o((1-x)^4)$$

da cui:

$$\sin(\sin(\pi x)) = \pi(1-x) - \frac{\pi^3}{6}(1-x)^3 + o((1-x)^4)$$
$$-\frac{\pi^3}{6}(1-x)^3 = \pi(1-x) - \frac{\pi^3}{3}(1-x)^3 + o((1-x)^4)$$

In conclusione il numeratore ha parte principale

$$-\frac{\pi}{3}(1+\pi^2)(1-x)^3 + o((1-x)^4)$$

e quindi l'integrale converge se e solo se

$$-3+\alpha < 1 \Leftrightarrow \alpha < 4$$