$\boxed{\text{ESERCIZIO 4}}$ Sia $f \in C^3((-1,1))$ t.c. $f(0)=0$ e

$$\lim_{x\to 0} \frac{\operatorname{sen}(f(x)) - x\cos(f(x))}{\operatorname{tg}(x^3)} = \frac{4}{3}$$

determinare il polinomio di Taylor di grado 3 centrato in $x=0$.

Poiché $f \in C^3((-1,1))$ si ha

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + o(x^3)$$

con $a_0 = f(0) = 0$; $a_1 = f'(0)$; $a_2 = \frac{f''(0)}{2}$, $a_3 = \frac{f'''(0)}{6}$

Poiché $f(0)=0$ si ha quindi:

$$\operatorname{sen}(f(x)) = f(x) - \frac{f^3(x)}{6} + o(f^3(x)) =$$

$$= a_1 x + a_2 x^2 + a_3 x^3 - \frac{a_1^3}{6} x^3 + o(x^3)$$

$$x\cos(f(x)) = x\left(1 - \frac{f^2(x)}{2} + o(f^2(x))\right) =$$

$$= x\left(1 - \frac{1}{2}(a_1 x + o(x))^2 + o(x^2)\right)$$

$$= x - \frac{a_1^2}{2} x^3 + o(x^3)$$

ed essendo:

$$\operatorname{tg}(x^3) = x^3 + o(x^3)$$

segue:

$$\frac{\operatorname{sen}(f(x)) - x\cos(f(x))}{\operatorname{tg}(x^3)} = \frac{4}{3} + o(1) \iff$$

$$\frac{(a_1 - 1)x + a_2 x^2 + \left(a_3 - \frac{a_1^3}{6} + \frac{a_1^2}{2}\right)x^3 + o(x^3)}{x^3 + o(x^3)} = \frac{4}{3} + o(1)$$

$$\iff a_1 = 1;\ a_2 = 0;\ a_3 = 1 \implies f(x) = x + x^3 + o(x^3)$$