(4) Calcolare $\displaystyle\int_0^{\pi/6} \frac{\operatorname{tg}x+1}{(\operatorname{tg}x-1)^2}\,dx = I$

Con la sostituzione $t = \operatorname{tg} x$ si ottiene

$$I = \int_0^{1/\sqrt{3}} \overbrace{\frac{t+1}{(t-1)^2}\,\frac{1}{t^2+1}}^{f(t)}\,dx$$

Usando il metodo dei fratti semplici:

$$f(t) = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{Ct+D}{t^2+1}$$

$$= \frac{A(t^2+1)(t-1) + B(t^2+1) + (Ct+D)(t-1)^2}{(t-1)^2(t^2+1)}$$

$$= \frac{A(t^3-t^2+t-1) + B(t^2+1) + (Ct^3-2Ct^2+Ct) + D(t^2-2t+1)}{(t-1)^2(t^2+1)}$$

$$\Longleftrightarrow \begin{cases} A+C=0 \\ -A+B-2C+D=0 \\ A+C-2D=1 \\ -A+B+D=1 \end{cases} \Longleftrightarrow \begin{cases} A+C=0 \\ \cancel{-A}-2C+1+\cancel{A}=0 \\ -2D=1 \\ -A+B+D=1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} A=D=-C=-1/2 \\ B=1 \end{cases}$$

da cui:

$$I = \left[-\frac{1}{2}\ln|t-1| - (t-1)\right]_0^{1/\sqrt{3}} + \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{t-1}{t^2+1}\,dt$$

$$= -\frac{1}{2}\ln\left|\frac{1}{\sqrt{3}}-1\right| - \frac{1}{\sqrt{3}} + \frac{1}{4}\left[\ln(t^2+1)\right]_0^{1/\sqrt{3}} - \frac{1}{2}\left[\operatorname{atn} t\right]_0^{1/\sqrt{3}}$$