⑤ Se $g(t) = \begin{cases} \frac{1}{|t|}\left(\operatorname{atn}^2 \frac{1}{t} - \frac{\pi^2}{4}\right) & t \neq 0 \\ -\pi & t = 0 \end{cases}$

allora:

$$f(x,y) = g(x \cdot y)$$

Quindi se $x_0 y_0 = 0 \Rightarrow x \cdot y \to 0$ per $(x,y) \to (x_0,y_0)$

e:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{t\to 0} g(t)$$

$$g(t) = \frac{1}{|t|}\left(\operatorname{atn}\frac{1}{t} - \frac{\pi}{2}\right)\left(\operatorname{atn}\frac{1}{t} + \frac{\pi}{2}\right)$$

$$= \frac{1}{|t|}\left(-\operatorname{atn} t + \frac{\pi}{2}\operatorname{sgn} t - \frac{\pi}{2}\right)\left(-\operatorname{atn} t + \frac{\pi}{2}\operatorname{sgn} t + \frac{\pi}{2}\right)$$

$t \to 0^+$: $g(t) = \frac{1}{t}(-\operatorname{atn} t)(+\pi + o(1)) \longrightarrow -\pi$

$t \to 0^-$: $g(t) = -\frac{1}{t}(-\pi + o(1))(-\operatorname{atn} t) \longrightarrow -\pi$

Poiché $f(x,0) = f(0,y) = -\pi \Rightarrow f_x(0,0) = f_y(0,0) = 0$

Per la diff.tà resta da studiare

$$\frac{f(x,y) + \pi}{\sqrt{x^2+y^2}} = \underbrace{\frac{xy}{\sqrt{x^2+y^2}}}_{\text{infinitesimo}} \; \frac{f(x,y)+\pi}{xy}$$

$$\lim \frac{f(x,y)+\pi}{xy} = \lim_{t\to 0} \frac{g(t)+\pi}{t}$$

$t > 0$: $\frac{g(t)+\pi}{t} = \frac{\left(-1 + o(t)\right)(\pi - \operatorname{atn} t) + \pi}{t}$

$$= \frac{o(t) + \operatorname{atn} t}{t} \longrightarrow 1$$

$t < 0$: $\frac{g(t)+\pi}{t} = \frac{(1 + o(t))(-\pi - \operatorname{atn} t) + \pi}{t} \longrightarrow 1$