Some applications arising from the interactions between the theory of Catalan-like numbers and the ECO method*

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Abstract

In [FP] the ECO method and Aigner's theory of Catalan-like numbers are compared, showing that it is often possible to translate a combinatorial situation from one theory into the other by means of a standard change of basis in a suitable vector space. In the present work we emphasize the soundness of such an approach by finding some applications suggested by the above mentioned translation. More precisely, we describe a presumably new bijection between two classes of lattice paths and we give a combinatorial interpretation to an integer sequence not appearing in [Sl].

1 Introduction

A common fact in mathematics is that two distinct theories, often developed using very different tools and on the basis of quite distinguished needs, are successively shown to be equivalent or, at least, to be comparable once a common language to describe them has been found. In this work we focus our investigation on the relationship between two combinatorial methods whose origins and tools are rather different, namely the ECO method and Aigner's theory of Catalan-like numbers. In a previous paper [FP] it has been shown that these two theories can be compared provided that they are both expressed in linear algebraic terms, i.e., more precisely,

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using linear operators on polynomials and (infinite) matrices. The main aim of the present work is to demonstrate how the two theories can fruitfully interact to produce new results or suggest research ideas. Typically, one starts from a known fact of one of the theories and "translates" it into the other, thus (hopefully) finding something new. Therefore our paper is intended to mainly provide some applications of the techniques developed in [FP], thus giving evidence to the strength of such a comparative approach.

We start in section 2 by giving the theoretical basis of the successive applications, that is by recalling the basics of the comparison between ECO method and Aigner's theory of Catalan-like numbers as it is exposited in [FP]. Here we have decided to use matrices rather than linear operators, even if we will occasionally need operators in some of our proofs. Section 3 essentially contains an explicit bijection between two classes of paths that has been suggested by two different interpretation of the same sequence in each of the two theories under consideration. Finally, in section 4, we will give a combinatorial interpretation to a sequence of Catalan-like numbers which cannot be suitably dealt with using Aigner's theory: our technique consists of translating the problem into the framework of the ECO method, where we will be able to use its typical tools to obtain the desired interpretation.

2 Preliminaries on Catalan-like numbers and ECO matrices

We start by recalling the main definitions and results of the two theories we are going to work with, together with a sort of canonical way of translating each of them into the other. This transformation, called the *fundamental change of basis*, constitutes the main theoretical tool which justifies the results developed in the present work. The main reference for the material presented in this section is [FP], where the reader can actually find much more than we are going to recall here. Indeed, we will just provide the basics needed to suitably describe our applications, in the effort of making our paper self-contained.

A *generalized Aigner matrix* is an infinite lower triangular matrix $A = (a_{nk})_{n,k\in\mathbf{N}}$ such that $a_{00} = 1$ and the remaining entries satisfy the following recursions:

$$a_{nk} = r_{k-1}a_{(n-1)(k-1)} + s_k a_{(n-1)k} + t_{k+1}a_{(n-1)(k+1)}. \qquad (1)$$

Alternatively, in a generalized Aigner matrix every entry can be computed as a suitable linear combination of the three entries of the above row next to it. Observe that the coefficients of the linear combination in (1) depend only on the column index. Thus a generalized Aigner matrix is completely determined by the three sequences $\rho = (r_n)_{n\in\mathbf{N}}, \sigma = (s_n)_{n\in\mathbf{N}}$ and $\tau = (t_n)_{n\in\mathbf{N}}$, and for this reason we will refer to such a matrix as the generalized Aigner matrix *of type* (ρ, σ, τ).

Specializing the above definition, when $\rho \equiv 1$ we will say that A is the *Aigner matrix of type* (σ, τ), and when also $\tau \equiv 1$ we will call A the *admissible matrix of type* σ. A huge amount of interesting results on Aigner and admissible matrices are reported in Aigner's papers [A1, A2, A3].

A generalized Aigner matrix A defines a sequence of positive integers by simply taking the elements appearing in column 0; these numbers are called the *Catalan-like numbers determined by* A. In the above cited papers by Aigner many elegant characterizations and properties of Catalan-like numbers are given, for example in terms of their Hankel matrices. Here we wish to recall the fact that a Catalan-like sequence can be determined by at most one admissible matrix, whereas there can be several different Aigner matrices associated with the same sequence of Catalan-like numbers.

Example. Consider the three sequences $\rho = (1, 0, 0, \ldots)$, $\sigma = (1, 1, 1, \ldots)$ and $\tau = (2, 1, 1, \ldots)$. It is easy to see that the generalized Aigner matrix of type (ρ, σ, τ) begins as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 2 & 0 & 0 & 0 & \cdots \\ 7 & 5 & 0 & 0 & 0 & \cdots \\ 17 & 12 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and has sequence A001333 of [Sl] (preceded by a 1) as its Catalan-like sequence. Analogously, one can immediately find that the Aigner matrix of type (σ, τ) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & 0 & \cdots \\ 7 & 6 & 3 & 1 & 0 & \cdots \\ 19 & 16 & 10 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and its Catalan-like sequence is given by the trinomial coefficients, whereas the admissible matrix of type σ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & \cdots \\ 4 & 5 & 3 & 1 & 0 & \cdots \\ 9 & 12 & 9 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

having the Motzkin numbers as its Catalan-like sequence. The last two examples are considered in [A3].

In the rest of this section we will recall the basic facts concerning ECO matrices. The main references concerning these topics are [DFR, FP], whereas for the general notion of succession rule, generating tree and for the ECO method we refer the reader to [BDLPP, B *et al.*]. Here we will just illustrate the concepts we need using a concrete example.

Consider the succession rule:

$$\Omega : \begin{cases} (2) \\ (2k) \rightsquigarrow (2)^k(4)(6)\cdots(2k)(2k+2) \end{cases}, \tag{2}$$

defining the central binomial coefficients $\binom{2n}{n}$ as level sums of the associated generating tree. One can encode the generating tree of Ω by means of the infinite matrix F whose (n,k) entry represents the number of nodes labelled k at level n. The first lines of F are therefore the following:

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & 0 & \cdots \\ 10 & 6 & 3 & 1 & 0 & \cdots \\ 35 & 20 & 10 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

F is called the *ECO matrix* associated with Ω. Obviously F contains all the enumerative information provided by the succession rule Ω; for instance, the row sums of F clearly define the sequence determined by Ω. We recall that Ω has been first considered in [BFR], where the authors use it in the enumeration of directed-convex polyominoes.

In [FP] the authors try to find a relationship between the theory of (generalized) Aigner matrices (and Catalan-like numbers) and the ECO method. The main achievement of the above paper is the discovery that the same combinatorial situation can often be described in the two settings by two matrices which only differ by a *standard* change of basis, called the *fundamental change of basis*. More precisely, let Ω be a succession rule and $F = (f_{n,k})_{n,k\in\mathbf{N}}$ its ECO matrix. Consider the matrix $A = (a_{n,k})_{n,k\in\mathbf{N}} = FP$, where $P = \left(\binom{n}{k}\right)_{n,k\in\mathbf{N}}$ is the usual Pascal matrix. Then, it is shown in [FP] that, in a large number of interesting combinatorial situations, A is in fact a generalized Aigner matrix. Actually, it is immediate to see that $a_{n,0} = \sum_k f_{n,k}$, so that *the row sums of F are mapped into the elements appearing in the first column of A*, i.e. the sequence determined by the succession rule Ω is transformed into the sequence of Catalan-like numbers of A: this fact suggest that the proposed change of basis is indeed the best candidate to establish a bridge between the two theories.

Unfortunately, there are cases in which things do not work so well. The next two examples shows how this change of basis sometimes fails to carry out the desired translation.

Example. Let Ω be the following succession rule:

$$\Omega : \begin{cases} (1) \\ (2^k) \rightsquigarrow (1)^{2^{k-1}}(2)^{2^{k-2}}(4)^{2^{k-3}}\cdots(2^{k-1})(2^{k+1}) \end{cases} . \tag{3}$$

It is well known [BPPR] that the sequence determined by the above rule is that of Catalan numbers. The associated ECO matrix is:

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 2 & 2 & 0 & 1 & 0 & \cdots \\ 6 & 4 & 3 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Multiplying F on the right by P gives the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & \cdots \\ 14 & 14 & 9 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A is the well-known matrix of ballot numbers, which is easily seen not to be a generalized Aigner matrix.

Example. Taking $\sigma = (1, 0, 0, 0, \ldots)$, one gets the admissible matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 0 & 0 & \cdots \\ 3 & 3 & 1 & 1 & 0 & \cdots \\ 6 & 4 & 4 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which defines the Catalan-like sequence of the middle binomial coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. If A is multiplied on the right by the inverse Pascal matrix P^{-1}, then the following matrix is obtained:

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 4 & -2 & 1 & 0 & \cdots \\ 6 & -5 & 7 & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is clear that F is not an ECO matrix, simply because some of its entries are negative.

Remark. Concerning the last example, we wish to notice that there is the possibility of defining succession rules having positive and negative labels, so that the matrix F can be formally interpreted as an ECO matrix. However, the combinatorial meaning of such rules is not clear yet, so we prefer to avoid this possibility: for us, an ECO matrix is required to have nonnegative entries.

2.1 A nontrivial example

In spite of the two counterexamples presented above, there are many interesting cases in which the fundamental change of basis does indeed work. A good source of quite classical examples can be found in [FP]. Here we would like to stress the soundness of our approach by exhibiting a nontrivial example in which the two theories interact thus producing a new result.

Consider the admissible matrix A of type $\sigma = (3, 3, 3, \ldots)$, whose first lines are the following:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & \cdots \\ 10 & 6 & 1 & 0 & 0 & \cdots \\ 36 & 29 & 9 & 1 & 0 & \cdots \\ 137 & 132 & 57 & 12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix was first studied in [A1], and defines the Catalan-like sequence of the so-called *restricted hexagonal numbers* [HR]. Performing the fundamental change of basis, i.e. multiplying on the right by P^{-1}, yields the matrix:

$$AP^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 4 & 1 & 0 & 0 & \cdots \\ 15 & 14 & 6 & 1 & 0 & \cdots \\ 51 & 50 & 27 & 8 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From a careful inspection of the first few lines of AP^{-1}, we are led to claim that AP^{-1} is the ECO matrix associated with the succession rule:

$$\Omega : \begin{cases} (3) \\ (k) \rightsquigarrow (3)(4)\cdots(k-1)(k)^2(k+1) \end{cases} . \qquad (4)$$

Therefore, thanks to our fundamental change of basis, we have found a succession rule for the restricted hexagonal numbers. It would be nice to determine a combinatorial construction for some class of structures counted by these numbers encoded by such a rule. We remark that the rule Ω was independently guessed also by E. Deutsch [D] in a completely different manner (i.e. using production matrices [DFR]).

To give a rigorous proof of the above claim, we will settle a more general theorem; as a corollary, we will then be able to deduce the desired result. However, before starting, we need to introduce some notations and known results. For all these facts we refer the reader to [FP].

First of all, in the proof of the next theorem we adopt the following convention: the lines of our matrices will be indexed using positive integers (whereas, in the rest of the paper, we use nonnegative integers). It will be very useful for us to use linear operators and polynomials in our proof. Thus, if $A = (a_{nk})_{n,k}$ and $F = AP^{-1} = (f_{nk})_{n,k}$, then we set $\beta_n(x) = \sum_{k\geq 1} f_{nk}x^k = \sum_{k\geq 1} a_{nk}p_k(x)$, where $p_n(x) = x(x-1)^{n-1}$. We will denote by T the linear operator on polynomials defined by $T(x^n) = 1 + x + x^2 + \cdots + x^{n-1}$. The following technical results can be found in [FP], and will be extensively used in the sequel.

Proposition 2.1 *1.* $[T^k(x^n)]_{x=0} = \binom{n-1}{k-1}$.

2. $a_{nk} = \left[T^k(\beta_n(x))\right]_{x=0}$.

Finally, we wish to remark that all these conventions, notations and results will be used also in the proof of proposition 4.1.

Now we are ready to state and prove our result.

Theorem 2.1 *Let A be the admissible matrix of type $\sigma \equiv s$. Then $F = AP^{-1}$ is the ECO matrix determined by the succession rule*

$$\Omega : \begin{cases} (s) \\ (k) \rightsquigarrow (s)(s+1)(s+2)\cdots(k-1)(k)^{s-1}(k+1) \end{cases}.$$

Proof. We will prove, equivalently, that, if $F = (f_{nk})_{n,k}$ is the ECO matrix associated with Ω, then $A = FP$ is the admissible matrix of type $\sigma \equiv s$. So, if $A = (a_{nk})_{n,k}$, then our thesis is that

$$a_{(n+1)k} = a_{n(k-1)} + sa_{nk} + a_{n(k+1)}.$$

Equivalently, we wish to show that

$$[T^k(\beta_{n+1}(x))]_{x=0} = [(T^{k-1} + sT^k + T^{k+1})(\beta_n(x))]_{x=0}.$$

Applying the above proposition, we then get:

$$\sum_{h=1}^{n+1} \binom{h-1}{k-1} f_{(n+1)h} = \sum_{h=1}^{n} \left(\binom{h-1}{k-2} + s\binom{h-1}{k-1} + \binom{h-1}{k} \right) f_{nh}. \quad (5)$$

Now recall that our hypothesis is that the following recursion holds:

$$\begin{aligned} f_{(n+1)k} &= f_{n(k-1)} + (s-1)f_{nk} + f_{n(k+1)} + \cdots + f_{nn} \\ &= \sum_{i=k-1}^{n} f_{ni} + (s-2)f_{nk}. \end{aligned}$$

Plugging the above formula into the l.h.s. of (5) we obtain:

$$\begin{aligned} &\sum_{h=1}^{n} \left(\sum_{i=1}^{h+1} \binom{i-1}{k-1} + (s-2)\binom{h-1}{k-1} \right) f_{nh} \\ = &\sum_{h=1}^{n} \left(\binom{h-1}{k-2} + 2\binom{h-1}{k-1} + \binom{h-1}{k} + (s-2)\binom{h-1}{k-1} \right) f_{nh}. \end{aligned}$$

Using well-known properties of binomial coefficients, it is not difficult to show that

$$\sum_{i=1}^{h+1} \binom{i-1}{k-1} = \binom{h-1}{k-2} + 2\binom{h-1}{k-1} + \binom{h-1}{k}, \tag{6}$$

and so the theorem is proved. ■

As it is shown in [A1], the generating function $f_s(x)$ of the Catalan-like numbers $C_n^{(s)}$ associated with $A = A^{(s)}$ is

$$f_s(x) = \frac{1 - sx - \sqrt{1 - 2sx + (s^2-4)x^2}}{2x^2}.$$

Moreover, $(C_n^{(s+1)})_n$ is linked to $(C_n^{(s)})_n$ by the following recursion:

$$C_n^{(s+1)} = \sum_{k=0}^{n} \binom{n}{k} C_k^{(s)},$$

i.e. $(C_n^{(s+1)})_n$ is the binomial transform of $(C_n^{(s)})_n$.

It is clear that, for $s = 3$, we get precisely the admissible matrix of the beginning of this section.

In the spirit of the last theorem, we can prove one more general result concerning the fundamental change of basis. Since the proof follows basically the same lines as those of theorem 2.1, we omit it.

Theorem 2.2 *If A_s is the generalized Aigner matrix of type (σ, ρ, τ), with $\rho \equiv s \equiv \tau$ and $\sigma \equiv 2s$, then $F_s = A_s P^{-1}$ is the ECO matrix associated with the succession rule*

$$\Omega : \begin{cases} (2s) \\ (ks) \rightsquigarrow (2s)^s(3s)^s \cdots (ks)^s((k+1)s)^s \end{cases}.$$

As a consequence, we find that the Catalan-like numbers $(C_n^{(s)})_n$ defined by A_s (or, which is the same, the sequence determined by Ω) are strictly linked to the Catalan numbers. Indeed, if $f(x) = \sum_n C_n x^n$ is the generating function of Catalan numbers, we obtain:

$$f_s(x) = \sum_n C_n^{(s)} x^n = \sum_n s^n C_n x^n = f(sx).$$

This result agrees with proposition 3.2 of [DFR].

3 A bijection

Our first application concerns a very classical integer sequence, i.e. the central binomial coefficients $\binom{2n+1}{n}$. It is easy to see that they are indeed a sequence of Catalan-like numbers: they are in fact defined by the admissible matrix A of type $\sigma = (3, 2, 2, 2 \ldots)$ (see [A1]). Now consider the matrix $F = AP^{-1}$: the following proposition can be proved following the argument of theorem 2.1, so it is left to the reader.

Proposition 3.1 *F is the ECO matrix determined by the succession rule*

$$\Omega : \begin{cases} (3) \\ (k) \rightsquigarrow (3)^2(4)(5)\cdots(k)(k+1) \end{cases} .$$

Therefore we have a nice description of the central binomial coefficients both in the theory of Catalan-like numbers and in the framework of the ECO method. But what about the combinatorial interpretations connected with such descriptions?

As far as Aigner's theory is concerned, a general way of giving a combinatorial meaning to each entry of an admissible matrix is exposited in [A1]. More precisely, for an admissible matrix $A = (a_{nk})_{n,k}$ of type $\sigma = (s_k)_k$, a_{nk} is the number of lattice paths from $(0, 0)$ to (n, k) never crossing the x-axis and using up (i.e. $(1, 1)$) steps, down ($(1, -1)$) steps and s_k different types of horizontal ($(1, 0)$) steps at height k. In particular, the associated Catalan-like numbers count paths ending on the x-axis. Thus, for example, Dyck paths and Motzkin paths come from the admissible matrices of types $\sigma \equiv 1$ and $\sigma \equiv 0$, respectively. According to this general setting we have therefore the following combinatorial interpretation for the central binomial coefficients: they count a class of "coloured" Motzkin paths, where horizontal steps are bicoloured, except for those lying on the x-axis, which can assume three colours.

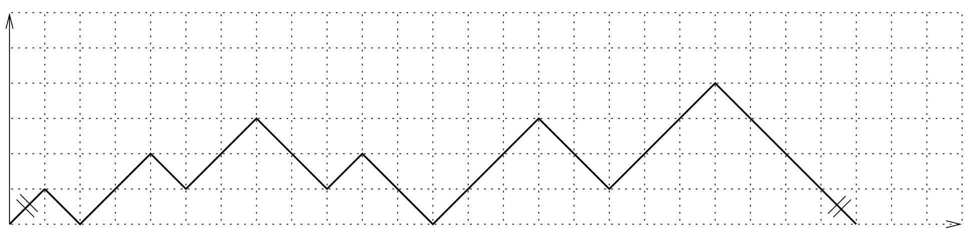
On the other hand, from the point of view of ECO, it is well known that Ω describes an effective construction for the class of Grand Dyck paths starting with an up step (recall that a Grand Dyck path is a Dyck path without the constraint of remaining above the x-axis). For such a construction we refer the reader to [PPR].

Our fundamental change of basis has therefore provided two different combinatorial interpretations of the same integer sequence, thus suggesting the existence of a presumably interesting bijection between the two classes of paths under consideration. The determination of such a bijection will be the object of the rest of this section.

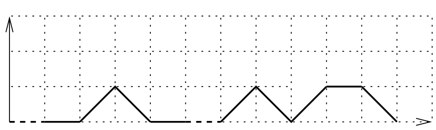
Our bijection is basically a generalization of the bijection given in [DV] between Dyck paths of length $2n$ and bicoloured Motzkin paths of length $n-1$. Such a bijection is obtained by applying the following transformation:

1. the first and the last step of the starting Dyck path (which are clearly an up step and a down step, respectively) are deleted;
2. reading the path from left to right, and denoting by U an up step and by $\overline{U}$ a down step, we perform the following substitutions on pairs of consecutive steps:
   - a pair $U\overline{U}$ is replaced by a black horizontal step;
   - a pair $\overline{U}U$ is replaced by a green horizontal step;
   - a pair UU is replaced by an up step;
   - a pair $\overline{U}\overline{U}$ is replaced by a down step.

   Call such a transformation f (see fig. 1).



(a)



(b)

Figure 1: (a) a Grand Dyck path; (b) the corresponding bicoloured Motzkin path. For simplicity, green steps are represented by means of dashed lines.

Now let P be a Grand Dyck path starting with an up step. P can be decomposed into subpaths $P_1, \dots, P_k$, where each P_i, $1 \leq i \leq k$, is a primitive Dyck path lying above or below the x-axis. In particular, P_1 remains necessarily above the x-axis. Each P_i is of the form $UP'_i\overline{U}$ or $\overline{U}P'_iU$, depending on its position with respect to the x-axis (see fig. 2).

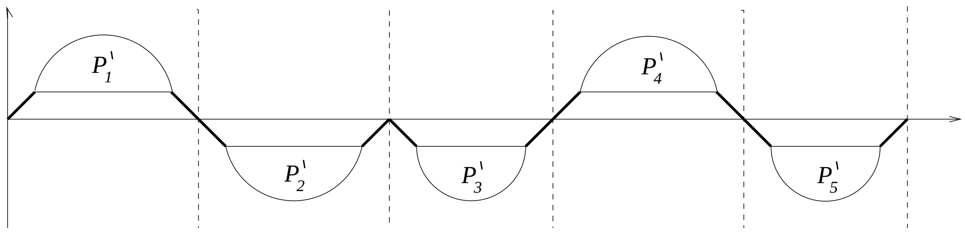


Figure 2: The decomposition of a Grand Dyck path.

Now, if C is a Dyck path, denote by $\overline{C}$ the path obtained from C by interchanging up and down steps. Consider the following operations:

1. apply the bijection of [DV] to P_1, so obtaining a bicoloured Motzkin path;

2. for every i, $2 \leq i \leq k$, if P_i lies above the x-axis, then it becomes the path $f(P_i')$ preceded by a green horizontal step, otherwise it becomes the path $f(\overline{P_i'})$ preceded by a red horizontal step (see fig. 3).

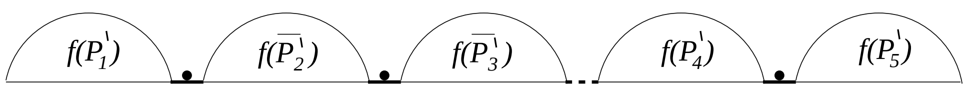


Figure 3: The bicoloured Motzkin path corresponding to the Grand Dyck path sketched in fig. 2; to denote red steps we have used black steps marked with dots.

The above described operations determine a bijection. To show this, consider a bicoloured Motzkin path (with green and black horizontal steps) whose horizontal steps at level zero are possibly tricoloured (they can also be red). Just scan the path from left to right locating the red and green horizontal steps lying on the x-axis; for example, $M = M_1\underline{\text{red}}M_2\underline{\text{red}}M_3\underline{\text{green}}M_4\underline{\text{green}}M_5$, where $\underline{\text{red}}$ and $\underline{\text{green}}$ denote the red and green horizontal steps, respectively. The path M_1 preceding the first red or green horizontal step at level zero is transformed into the path $Uf^{-1}(M_1)\overline{U}$. Let M_i, $2 \leq i \leq k$, a subpath of M as in the above example. If M_i is preceded by a red horizontal step, then it is transformed into $\overline{U}\,\overline{f^{-1}(M_i)}U$; if it is preceded by a green horizontal step, then it is transformed into $Uf^{-1}(M_i)\overline{U}$. Therefore, applying our bijection, the inverse image of the above path is the Grand Dyck path $Uf^{-1}(M_1)\overline{U}\,\overline{U}\,\overline{f^{-1}(M_2)}U\overline{U}\,\overline{f^{-1}(M_3)}UUf^{-1}(M_4)\overline{U}Uf^{-1}(M_5)\overline{U}$.

4 A combinatorial interpretation

In [A3] Aigner succeeds in finding a general way of giving a combinatorial interpretation to several sequences of Catalan-like numbers by considering Aigner matrices whose entries satisfy some kind of nice-looking recurrence. More specifically, he is able to deal with *recurrences of Sheffer type*; here the terminology is borrowed from classical umbral calculus, whose roots can be traced back to Rota and his collaborators [RKO]. On the other hand, no general combinatorial interpretation is known for Catalan-like numbers associated with Aigner matrices not of Sheffer type; only some specific examples are considered by Aigner, along with a few connections with well-known integer sequences.

Here we will provide a new example of an Aigner matrix whose entries do not satisfy a Sheffer-type relation, together with a combinatorial interpretation of the associated Catalan-like numbers. This has been possible thanks to our fundamental change of basis and to the typical tools of the ECO method.

Consider the Aigner matrix A of type (σ, τ), with $\sigma \equiv 1$ and $\tau = (2, 3, 4, \ldots)$. The first lines of A are the following:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & 0 & \cdots \\ 7 & 8 & 3 & 1 & 0 & \cdots \\ 23 & 24 & 15 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We point out that the sequence $1, 1, 3, 7, 23, 71, \ldots$ does not appear in [Sl], so a combinatorial interpretation would be particularly interesting. Applying the fundamental change of basis we get the following matrix F:

$$F = AP^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 5 & 0 & 1 & 0 & \cdots \\ 11 & 2 & 9 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proposition 4.1 *F is the ECO matrix associated with the following succession rule:*

$$\Omega : \begin{cases} (1) \\ (2k-1) \rightsquigarrow (1)(3)(5)\cdots(2k-5)(2k-3)^k(2k+1) \end{cases}. \tag{7}$$

Proof. As we did in theorem 2.1, we will prove the equivalent statement that, if $F = (f_{nk})_{n,k}$ is the ECO matrix defined by Ω, then $A = (a_{nk})_{n,k} = FP$ is the Aigner matrix of type (σ, τ), with $\sigma \equiv 1$ and $\tau = (2, 3, 4, \ldots)$. Therefore we have to show that the following recurrence relation holds:

$$a_{(n+1)k} = a_{n(k-1)} + a_{nk} + (k+1)a_{n(k+1)}. \tag{8}$$

As usual, the idea is to use linear operators, so that (8) is equivalent to

$$[T^k(\beta_{n+1}(x))]_{x=0} = [(T^{k-1} + T^k + (k+1)T^{k+1})(\beta_n(x))]_{x=0}. \tag{9}$$

The application of proposition 2.1 translates (9) into

$$\sum_{h=1}^{n+1} \binom{h-1}{k-1} f_{(n+1)h} = \sum_{h=1}^{n} \left(\binom{h-1}{k-2} + \binom{h-1}{k-1} + (k+1)\binom{h-1}{k} \right) f_{nh}. \tag{10}$$

Now we use our hypothesis, which is essentially the following recursion:

$$f_{(n+1)k} = f_{n(k-1)} + (k+1)f_{n(k+1)} + f_{n(k+2)} + \cdots + f_{nn}.$$

Plugging the above formula into (10) returns:

$$\begin{aligned} &\sum_{h=1}^{n+1} \binom{h-1}{k-1} \left(\sum_{i=k-1}^{n} f_{ni} + h f_{n(h+1)} - f_{nh} \right) \\ = &\sum_{h=1}^{n} \left(\binom{h-1}{k-2} + 2\binom{h-1}{k-1} + \binom{h-1}{k} + k\binom{h-1}{k} - \binom{h-1}{k-1} \right) f_{nh}. \end{aligned}$$

Once again, using the equality (6) of theorem 2.1, we see that the last equality is equivalent to

$$\sum_{h=1}^{n} \left((h-1)\binom{h-2}{k-1} - \binom{h-1}{k-1} \right) f_{nh} = \sum_{h=1}^{n} \left(k\binom{h-1}{k} - \binom{h-1}{k-1} \right) f_{nh}.$$

But the equality $(h-1)\binom{h-2}{k-1} = k\binom{h-1}{k}$ is trivially true, whence the proposition follows. ■

Our next goal is to find a class of objects generated according to Ω. Below we will show that a particular class of coloured steep polyominoes has an ECO construction described by such a rule.

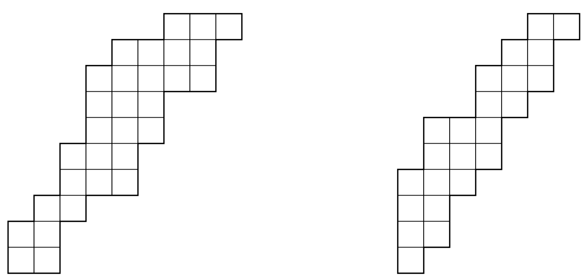
4.1 Coloured steep polyominoes

Let us begin by introducing some basics of the combinatorial objects we are going to deal with, and their main features.

In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square, and a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to translations. For basic definitions on polyominoes we refer to the book of Golomb [G].

A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells whose centers lie on a vertical (horizontal) line.

A particular class of polyominoes are the *parallelogram polyominoes*, defined by two lattice paths that use north and east unit steps, and intersect only at their starting and ending point. These paths are commonly called the *upper* and the *lower path*. Without loss of generality we assume that the upper and lower path of the polyomino start in $(0,0)$. The *area* of a parallelogram polyomino is the number of its cells, and the *semi-perimeter* is given by the sum of the numbers of its rows and columns. Figure 4 (a) depicts a parallelogram polyomino having area 30 and semi-perimeter 19.



(a) (b)

Figure 4: (a) a parallelogram polyomino; (b) a steep polyomino.

It is known [P, St] that the number of parallelogram polyominoes having semi-perimeter $n+1$ is the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

A *lower parallelogram steep polyomino* (briefly, *steep polyomino*) is a parallelogram polyomino whose lower path has no consecutive east steps (see Fig. 4). This class of polyominoes was introduced and studied in [BDFP]; in particular the authors proved that the number of steep polyominoes having semi-perimeter equal to $n+2$ is given by the n-th Motzkin number. A simple proof of this statement can be given using the ECO method. Let $\mathcal{S}_n$ be the set of steep polyominoes having semi-perimeter equal to $n+2$, and let ϑ be an operator such that, for any $n \geq 0$:

$$\vartheta : \mathcal{S}_n \to 2^{\mathcal{S}_{n+1}},$$

and ϑ acts on a polyomino $P \in \mathcal{S}_n$ as follows (see Fig. 5):

i) ϑ glues a column of length h, $1 \leq h \leq k-1$ to the rightmost column of P, with the top of the glued column at the same level as the top of the rightmost column;

ii) ϑ glues a cell on the top of the rightmost column of P.

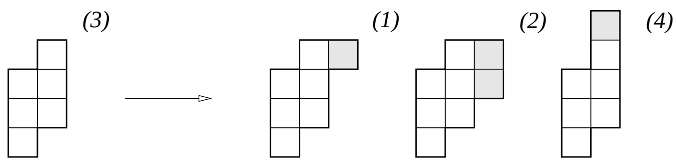


Figure 5: The ECO operator ϑ applied to the steep polyomino on the left produces the three polyominoes on the right. The cells added by ϑ have been shaded.

We easily prove that the operator ϑ satisfies the two fundamental conditions to be an ECO operator [BDLPP]:

1. for each $P' \in \mathcal{S}_{n+1}$, there exists $P \in \mathcal{S}_n$ such that $P' \in \vartheta(P)$,
2. for each $P, P' \in \mathcal{S}_n$ such that $P \neq P'$, $\vartheta(P) \cap \vartheta(P') = \emptyset$.

Thus all the objects of $\mathcal{S}$ are generated, and each object $P' \in \mathcal{S}_{n+1}$ is obtained from a unique $P \in \mathcal{S}_n$. Moreover, the succession rule associated with ϑ is:

$$\begin{cases} (1) \\ (k) \rightsquigarrow (1)\dots(k-1)(k+1), & k \geq 2. \end{cases} \tag{11}$$

We point out that the succession rule (11) has already been studied in [BDLPP], where it was proved that (11) defines Motzkin numbers.

Our aim is now to define a new class of steep polyominoes, namely *2-coloured steep polyominoes* where some columns can have a coloured cell, and to give an ECO construction for such a class. Since, up to now, we have tacitly assumed that a generic parallelogram or steep polyomino has all white coloured cells, then we will allow these new cells to be coloured black.

Let us give a formal definition of the class of 2-coloured steep polyominoes $2\mathcal{S}$. Let P be a steep polyomino having $m > 1$ columns, namely $C_1, \ldots, C_m$ numbered from left to right, and assume that for $1 \leq i \leq m$, $h(i)$ and $\overline{h}(i)$ are the ordinates of the bottom and top cell of C_i, respectively. The polyomino P is in $2\mathcal{S}$ if:

i) the first column of P has all white cells;

ii) for any $1 \leq i \leq m-1$, the column C_{i+1} has all white cells if $h(i+1) > h(i) + 1$. Otherwise, if $h(i+1) = h(i) + 1$, the column C_{i+1} can have at most one black cell, and its height must be between $h(i) + 1$ and $\overline{h}(i)$, (see Fig. 6 (a)).

Figure 6 (b) depicts a 2-coloured steep polyomino, having four black cells, in columns C_3, C_4, C_6, and C_8.

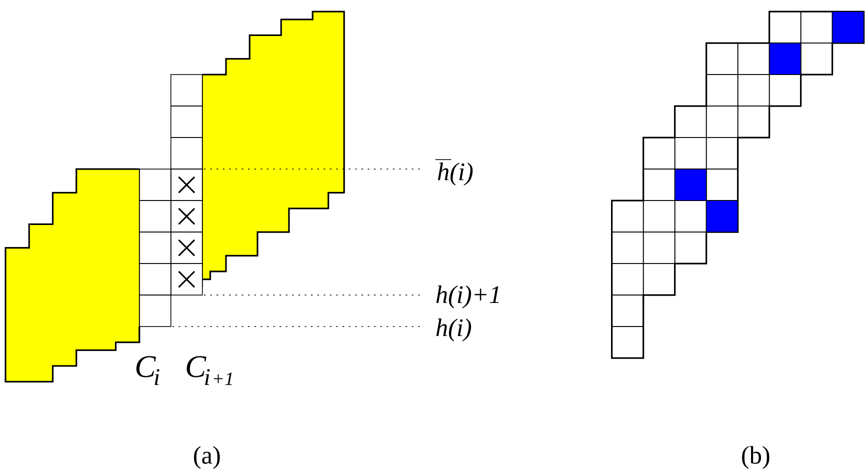


Figure 6: (a) Definition of a 2-coloured steep polyomino. The polyomino can have one black cell in column C_{i+1} at height $h(i) + 1, \ldots, \overline{h}(i)$. The possible positions where a black cell can be placed have been marked with $\times$. (b) A 2-coloured steep polyomino. Columns C_2 and C_5 cannot have a black cell, since $h(2) = 2 > 1 = h(1) + 1$, and $h(5) = 7 > 5 = h(4) + 1$.

Let $2\mathcal{S}_n$ be the set of 2-coloured steep polyominoes having semi-perimeter equal to $n + 2$, and let ϑ_2 be an operator such that, for any $n \geq 0$:

$$\vartheta_2 : 2\mathcal{S}_n \rightarrow 2^{2\mathcal{S}_{n+1}},$$

working as follows on a polyomino $P \in 2\mathcal{S}_n$ (see Fig. 7):

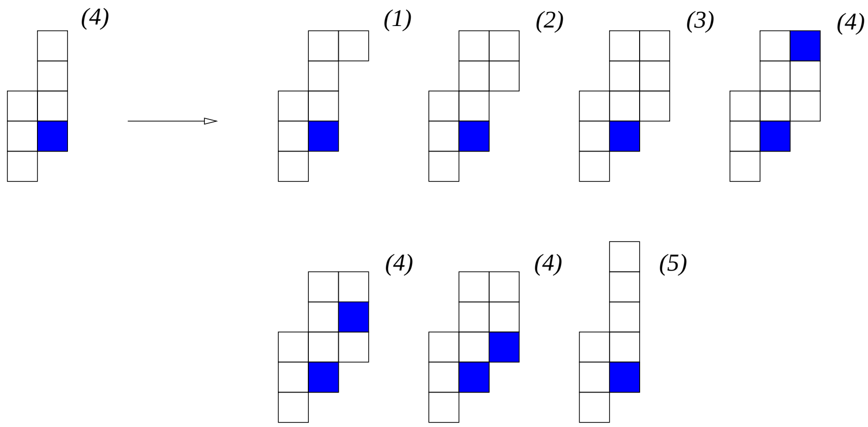


Figure 7: The ECO operator ϑ_2 applied to a 2-colored steep polyomino.

i) ϑ_2 glues a column of length h, $1 \leq h < k-1$ to the rightmost column of P, with the top of the glued column at the same level as the top of the rightmost column;

ii) ϑ_2 glues a column of length $k-1$, with one black cell, at height $1, \ldots, k-1$, to the rightmost column of P, with the top of the glued column at the same level as the top of the rightmost column;

iii) ϑ_2 glues a white cell on the top of the rightmost column of P.

The application of ϑ_2 to a polyomino P having k cells in the rightmost column produces $(k-1)+(k-1)+1 = 2k-1$ polyominoes whose semi-perimeter is increased by 1. It is clear that ϑ_2 is an ECO operator on the class ϑ_2.

The succession rule associated with ϑ_2 is then:

$$\left\{ \begin{array}{ll} (1) & \\ (k) \rightsquigarrow (1) \ldots (k-1)^k (k+1), & \qquad k \geq 2. \end{array} \right. \tag{12}$$

The key point in writing down such a succession rule is that the label (k) denotes the number of cells in the rightmost column of a polyomino.

We point out that the succession rule (12) is equal to (11), except for the exponent k of the label $(k-1)$ in the production of (11). Despite this similarity, and according to the notation introduced in [B *et al.*], (11) is clearly a succession rule of a *factorial form*, leading to an algebraic generating function, whereas (12) is a *transcendental system*, leading to a transcendental generating function (for more details see [B *et al.*], Prop. 6).

We also would like to remark that the rule (12) has not the *consistency property* for succession rules, that any label (k) should produce exactly k labels. It is easy to show that the succession rule (12) is equivalent to the rule in (7), which instead has the consistency property.

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