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Sustainability, indeterminacy and oscillations in a growth model with environmental assets

Angelo Antoci[a], Luigi Brugnano[b,*], Marcello Galeotti[c]

[a] *Dipartimento di Economia, Impresa e Regolamentazione, Università di Sassari, Via Sardegna 58, 07100 Sassari, Italy*
[b] *Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy*
[c] *Dipartimento di Matematica per le Decisioni, Università di Firenze, Via Lombroso 6/17, 50134 Firenze, Italy*

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Abstract

We consider an economy where welfare of a continuum of (identical) agents depends on three goods: leisure, a free access renewable environmental asset and a produced good which can be consumed or saved to accumulate physical capital. We assume that aggregate consumption depletes the natural resource and that economic agents take as exogenously given the negative impact on the environmental asset by aggregate consumption. In this context, we show that indeterminacy and oscillating behavior may arise.

Keywords: Economic growth models; Environmental assets; Optimal control problems; Dynamic systems; Hopf bifurcation; Periodic orbits

1. Introduction

Optimal economic growth models usually predict an initial phase during which protection activities with respect to environmental goods are negligible (see e.g. [2,7]): in the early stages of growth, the consumption level is low and environmental degradation is also (usually) low. Consequently, people prefer to increase consumption as quickly as possible, even at the expense of a rapid increase in environmental degradation; when the level of consumption is high enough, people take more care of environmental assets

* Corresponding author. Fax: +39-055-422-2605.
E-mail addresses: antoci@uniss.it (A. Antoci), brugnano@math.unifi.it (L. Brugnano), marcello.galeotti@dmd.unifi.it (M. Galeotti).

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and are willing to renounce part of their consumption in order to protect the environment. Thus, when the economy becomes sufficiently wealthy, the actions in defence of the environment are intensified; the intensification of environmental protection makes a high level of economic activity compatible with a fairly high level of environmental quality.

However, some recent theoretical works have brought to light mechanisms that may protract the *duration* of the stage in which no significant intervention is undertaken in defense of environmental assets, averting it from what is considered optimal (see e.g. [6]). In particular, the possibility is focused that policy-makers' choices pursue objectives which do not coincide with welfare maximization of the entire community, but only of a small part of it. In fact, as Lopez and Mitra [6] pointed out, the effectiveness and optimality of environmental policies may be heavily jeopardized by the rent-seeking behavior of public officials, whose choices are conditioned (e.g., through corruption) by firms seeking to "ease up" environmental protection, thus obtaining personal advantages to the detriment of the community welfare. The results of Lopez and Mitra [6] are supported by many case studies; see, e.g., the works collected by Desai [4].

In [4] other factors are also pointed out, which may cause nonoptimality of environmental policy; the more relevant ones are the high cost of environmental monitoring necessary to implement optimal environmental policy and the low efficiency of public officers in detecting economic agents who do not comply with policy restrictions. Such factors mainly act in developing countries and contribute to extend the phase of low environmental protection beyond the optimal threshold.

The general insight that can be drawn from the above literature is that the duration of the stage in which no significant intervention is undertaken in defence of environmental assets may extend well beyond what is considered optimal.

As one acknowledges such a possibility, the dynamic analysis of an economy, in which no policy reducing environmental impact is implemented, acquires theoretical significance. This is the context of the present work, which places itself as diametrically opposed with respect to the economic growth models based on optimal control of environmental impact.

In this paper, we analyze a model in which there is a continuum of economic agents whose consumption choices negatively affect a renewable environmental good. [1] We assume that each agent considers as negligible his own contribution to environmental degradation and takes as exogenously given the negative impact of aggregate consumption on environment; furthermore, we assume that no environmental policy intervention is implemented during the growth process. We study Nash equilibrium dynamics arising under such assumptions. In such a context, we aim to analyze the sustainability of economic growth. In particular, we give the conditions of existence and stability of fixed points where the stock of environmental goods is strictly positive. In addition we prove that an attracting limit cycle may arise via a Hopf bifurcation. The existence of (possibly large) oscillations could heavily jeopardize the sustainability of the

[1] We could relax such an assumption by supposing that also production activity of the consumption good affects the environmental good, thus obtaining similar results.

growth process of the economy. In fact, oscillations may bring environmental degradation to such a high level that the environmental asset would be dangerously exposed to exogenous shocks.

Finally, it is shown that a problem of indeterminacy can also arise: i.e., under certain conditions on parameters, there exists a continuum of (Nash) equilibrium orbits that the economy can follow (for a review of literature on indeterminacy see [3]). This implies that, given the initial values of the state variables, the model may produce a plurality (a continuum) of environmental dynamics, depending on the expectations of economic agents.

2. The model

We consider an economy with a continuum of (identical) agents whose welfare depends on three goods: leisure $(1-l)$, a renewable environmental asset (E) and a produced good which can be consumed (c) or saved to accumulate physical capital (K).

We assume that aggregate consumption depletes the natural resource. In particular, we assume that the representative agent solves the following decision problem:

$$\max_{l,c} \int_0^{\infty} [\ln c + a \ln E + d \ln(1-l)]\, \mathrm{e}^{-rt}\, \mathrm{d}t,$$

s.t.

$$\dot{K} = l^{\alpha} K^{1-\alpha} A - c, \tag{1}$$

$$\dot{E} = E[\beta(\bar{E} - E) - \gamma \bar{c}], \tag{2}$$

where labor l and consumption c are control variables.

In Eq. (1), $l^{\alpha}K^{1-\alpha}A$ is the production function of the representative agent with

$$A := \bar{l}^{\delta}\bar{K}^{\varepsilon}$$

denoting a positive externality that the representative agent takes as exogenously determined; $\bar{l}$ and $\bar{K}$ indicate the average values of l and K, respectively.

In Eq. (2), $\beta E(\bar{E} - E)$ is the regeneration function of the environmental asset E; the parameter $\bar{E}$ represents the value that E would approach if there would be no consumption activity in the economy. Hence, $\bar{E}$ can be interpreted as the endowment of the environmental asset in the economy. The parameter β measures the regeneration rate of the environmental good.

The expression $\gamma\bar{c}$ indicates the negative impact of the average consumption $\bar{c}$ on the rate of growth of E. Since the consumption choice of a single agent (among a continuum of agents) does not modify $\bar{c}$, the representative agent takes it as exogenously given. Consequently, the representative agent does not take into consideration the negative impact of his consumption choice on the environment.

The parameters a, d, r, $\bar{E}$, α, β, γ, δ and ε are strictly positive. We assume $0 < \alpha < 1$; hence the production function exhibits constant returns to scale (i.e., it is homogeneous of degree one) without the effect of A.

3. Dynamics

The Hamiltonian function of our problem is

$$H(c,l,K,E,\lambda,\theta)$$
$$=\ln c + a\ln E + d\ln(1-l) + \lambda(l^{\alpha}K^{1-\alpha}A - c) + \theta E[\beta(\bar{E}-E) - \gamma\bar{c}].$$

Since the representative agent considers $\bar{c}$ as exogenous, it does not take into account the “price” θ of E when he chooses the values of c and l. Therefore, the dynamics of K, E and λ do not depend on θ.

In fact, applying the maximum principle, they are described by the equations

$$\dot{K} = \frac{\partial H}{\partial \lambda} = l^{\alpha}K^{1-\alpha}A - c, \tag{3}$$

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial K} = \lambda[r - (1-\alpha)l^{\alpha}K^{-\alpha}A], \tag{4}$$

$$\dot{E} = \frac{\partial H}{\partial \theta} = E[\beta(\bar{E}-E) - \gamma\bar{c}], \tag{5}$$

where l and c are defined as follows

$$\frac{\partial H}{\partial l} = -\frac{d}{1-l} + \alpha\lambda l^{\alpha-1}K^{1-\alpha}A = 0, \tag{6}$$

$$\frac{\partial H}{\partial c} = \frac{1}{c} - \lambda = 0. \tag{7}$$

The average values $\bar{l}$, $\bar{K}$ and $\bar{c}$ coincide (ex-post) with l, K and c, respectively.

The orbits of system (3)–(5) do not represent growth paths where economic agents’ intertemporal utility is maximized; they are Nash equilibrium orbits that the economy follows if there is no policy maker’s monitoring. No single economic agent is motivated to modify unilaterally his choices [defined by (6)–(7)].

4. Fixed points

Let us look for the fixed points under dynamics (3)–(5). It is easily seen that there always exists a fixed point $(K,\lambda,E) = (\tilde{K},\tilde{\lambda},0)$ (where the environmental asset is completely depleted) whose coordinates are given by

$$K = \tilde{K} := \left[\frac{1-\alpha}{r}\left(\frac{\alpha}{\alpha+d}\right)^{\alpha+\delta}\right]^{1/(\alpha-\varepsilon)},$$

$$\lambda = \tilde{\lambda} := \frac{1-\alpha}{r}\frac{1}{\tilde{K}}.$$

In the region $\{E > 0\}$ the fixed point, at most unique, exists if and only if

$$\bar{E} > \frac{\gamma}{\beta}\left(\frac{1-\alpha}{r}\right)^{1/(\alpha-\varepsilon)-1}\left(\frac{\alpha}{\alpha+d}\right)^{(\alpha+\delta)/(\alpha-\varepsilon)}. \tag{8}$$

Its coordinates are $K = \tilde{K}$, $\lambda = \tilde{\lambda}$ and

$$E = \tilde{E} := \bar{E} - \frac{\gamma r}{\beta(1-\alpha)}\tilde{K}.$$

Notice that, by (8), such a fixed point exists if (coeteris paribus) the endowment $\bar{E}$ of the environmental asset, the discount rate r and the regeneration rate β of the environmental good are high enough, while the environmental impact of consumption (measured by γ) is low enough. The role played by $\bar{E}$, β and γ is obvious. In an economy where the discount rate is high agents accumulate little capital; so, their consumption cannot deplete the environmental good completely. If the discount rate is low, the existence of the fixed point with $E > 0$ requires high enough values of $\bar{E}$, β and a sufficiently low value of γ.

5. Local stability

Let us now analyze the stability properties of the fixed points. Note that system (3)–(5) is decoupled in the planar system given by (3) and (4) and the (non-autonomous) differential equation (5). So, by plugging $1/\lambda$ in the place of $\bar{c}$, we can see that the Jacobian matrix of (3)–(5) has the eigenvalue

$$\bar{e} := \left[\beta(\bar{E} - E) - \frac{\gamma}{\lambda}\right] - \beta E$$

corresponding to the direction of the E-axis. Since the expression in square brackets is zero at the fixed point where $E > 0$, then at such a point $\bar{e} = -\beta\tilde{E} < 0$. At the fixed point where $E = 0$ it holds $\bar{e} = \beta\bar{E} - \gamma/\tilde{\lambda} > 0$ if $\bar{E} > \gamma/\beta\tilde{\lambda}$; therefore $\bar{e} > 0$ when the fixed point with $E > 0$ is existing.

The other eigenvalues of the Jacobian matrix of system (3)–(5) coincide with those of the Jacobian matrix of the planar system given by (3) and (4). By the change of coordinates

$$u := \ln(\lambda K^{1-\alpha}), \quad v := \ln(\lambda K), \tag{9}$$

system (3)–(4) becomes

$$\begin{aligned} \dot{u} &= r - (1-\alpha)\,\mathrm{e}^{-v}, \\ \dot{v} &= r + \left(\frac{dl}{1-l} - 1\right)\mathrm{e}^{-v}, \end{aligned} \tag{10}$$

with a unique fixed point $(\tilde{u}, \tilde{v})$, corresponding to $(\tilde{K}, \tilde{\lambda})$. The Jacobian matrix at $(\tilde{u}, \tilde{v})$ is computed to be

$$J(\tilde{u}, \tilde{v}) = \begin{pmatrix} 0 & (1-\alpha)\,\mathrm{e}^{-\tilde{v}} \\ p & q \end{pmatrix}$$

with

$$p := \frac{r(\alpha - \varepsilon)(\alpha + d)}{(1-\alpha)[d(1-\alpha-\delta) + \alpha]}, \tag{11}$$

$$q := \frac{r\{(1-\alpha)[d(1-\alpha-\delta)+\alpha]+(d+\alpha)\varepsilon\}}{(1-\alpha)[d(1-\alpha-\delta)+\alpha]}. \tag{12}$$

Then,

(i) if $p > 0$, $(\tilde{K}, \tilde{\lambda})$ is a saddle point of system (3)–(4);
(ii) if $p < 0$ and $q > 0$, $(\tilde{K}, \tilde{\lambda})$ is a source of system (3)–(4);
(iii) if $p < 0$ and $q < 0$, $(\tilde{K}, \tilde{\lambda})$ is a sink of system (3)–(4).

In particular, case (ii) corresponds to

$$\alpha < \varepsilon, \quad \delta < 1 - \alpha + \frac{\alpha}{d} \tag{13}$$

or

$$\alpha > \varepsilon, \quad \delta > 1 - \alpha + \frac{\alpha}{d} + \frac{(\alpha + d)\varepsilon}{(1-\alpha)d}$$

while case (iii) corresponds to

$$\alpha > \varepsilon, \quad 1 - \alpha + \frac{\alpha}{d} < \delta < 1 - \alpha + \frac{\alpha}{d} + \frac{(\alpha + d)\varepsilon}{(1-\alpha)d}. \tag{14}$$

The stability properties of a fixed point in system (3)–(5) are obtained by combining the above results for system (3)–(4) with those concerning the sign of the eigenvalue in the direction of E-axis.

If case (i) holds, then the equilibrium $(\tilde{K}, \tilde{\lambda}, \tilde{E})$ has a two-dimensional stable manifold; so, given the initial values of K and E, K^0 and E^0, there exists (at least locally) a unique initial value of λ, λ^0 (λ^0 is the "shadow" price determined by the representative agent), from which the economy approaches the fixed point under dynamics (3)–(5).

If case (ii) holds, then the above stable manifold is one dimensional; consequently, the economy cannot (generically) reach the fixed point.

In case (iii) the stable manifold has full dimension; so there exists a continuum of initial values of λ leading to the fixed point. In such a case, an indeterminacy problem arises (see e.g. [3]). Precisely there exists a continuum of (Nash) equilibrium orbits that the economy can follow, given K^0 and E^0; each equilibrium orbit is associated to a particular initial value of λ, which is chosen by economic agents among infinitely many (a continuum).

Indeterminacy implies that, although orbits have the same limit point, there is, however, a continuum of (observable) transient paths in (K, E) coordinates leading to it. Along some of these, E may reach very low values; this makes the environmental good particularly vulnerable to exogenous shocks negatively affecting it.

6. A Hopf bifurcation

As a consequence of formulae (13), (14), when $\alpha > \varepsilon$ and δ crosses the value

$$\bar{\delta} = 1 - \alpha + \frac{\alpha}{d} + \frac{(\alpha + d)\varepsilon}{1-\alpha}, \tag{15}$$

system (3)–(4) generically undergoes a Hopf bifurcation (observe that $(\partial q/\partial\delta)(\bar{\delta}) > 0$) and a limit cycle $\Gamma(t)$ arises for δ belonging either to a right or a left neighborhood of $\bar{\delta}$.

Checking the attractiveness, or repellingness, of $\Gamma(t)$ requires computing the sign of an expression involving derivatives up to the third order of the vector field $(\dot{K}, \dot{\lambda})$ (see e.g. [5]). In fact, posed $\mu := \delta - \bar{\delta}$, one can show that there exists a smooth change of variables such that, passing, subsequently, to polar coordinates, system (3)–(4) becomes

$$\dot{\rho} = (\chi\mu + B\rho^2)\rho + h.o.t., \tag{16}$$

$$\dot{\vartheta} = \omega + \eta\mu + C\rho^2 + h.o.t.,$$

where $\chi q'(\bar{\delta}) > 0$ and $\omega := \sqrt{-p(\bar{\delta})(1-\alpha)/\tilde{K}\tilde{\lambda}}$. It follows that the limit cycle is attractive if $B < 0$. Recall, for example, the change of coordinates

$$u := \ln(\lambda K^{1-\alpha}), \quad v := \ln(\lambda K),$$

giving place to system (10). Posing

$$x := v - \tilde{v}, \quad y := \sqrt{-\frac{p(\bar{\delta})\tilde{K}\tilde{\lambda}}{1-\alpha}}(u - \tilde{u}),$$

system (10) can be re-written, at $\delta = \bar{\delta}$, as

$$\dot{x} = -\omega y + \varphi(x, y), \tag{17}$$

$$\dot{y} = \omega x + \psi(x).$$

Then, one can show (see, e.g., [5]) that

$$\begin{aligned} \operatorname{sign} B = \operatorname{sign} \Bigg[& \omega \left(\frac{\partial^3 \varphi}{\partial x^3}(0,0) + \frac{\partial^3 \varphi}{\partial x \partial y^2}(0,0) \right) \\ & + \frac{\partial^2 \varphi}{\partial x \partial y}(0,0) \left(\frac{\partial^2 \varphi}{\partial y^2}(0,0) + \frac{\partial^2 \varphi}{\partial x^2}(0,0) \right) - \psi''(0) \frac{\partial^2 \varphi}{\partial x^2}(0,0) \Bigg]. \end{aligned} \tag{18}$$

7. Supercriticality of the Hopf bifurcation

We want to prove that the Hopf bifurcation of system (3)–(4), occurring, for $\alpha > \varepsilon$, as δ crosses the value $\bar{\delta}$ given in (15), is always supercritical, so giving place to an attractive limit cycle when $\delta > \bar{\delta}$.

To this end, let us utilize coordinates (u, v), defined above, yielding system (10). Let

$$w = \frac{dl}{1-l} \tag{19}$$

and

$$F(l, u, v) = -w + \alpha \left(\frac{w}{d+w} \right)^{\alpha+\delta} e^{[(\alpha-\varepsilon)/\alpha]u} e^{(\varepsilon/\alpha)v} = 0, \tag{20}$$

which is equivalent to condition (6). It easily follows

$$\dot{w} = -\frac{\partial F/\partial u}{\partial F/\partial w}\dot{u} - \frac{\partial F/\partial v}{\partial F/\partial w}\dot{v} = \frac{(d+w)w}{w-d(\alpha+\delta-1)}\left[\frac{\alpha-\varepsilon}{\alpha}\dot{u} + \frac{\varepsilon}{\alpha}\dot{v}\right].$$

Thus we obtain, in coordinates (v, w), the system

$$\dot{v} = r + (w-1)e^{-v}, \tag{21}$$

$$\dot{w} = \frac{(d+w)w}{w-d(\alpha+\delta-1)}\left[r - (1-\alpha)e^{-v} + \frac{\varepsilon}{\alpha}(w-\alpha)e^{-v}\right],$$

whose unique fixed point is $(v_0, w_0) = (\ln(1-\alpha)/r, \alpha)$. Consider a new variable z defined, in a neighborhood of $w_0 = \alpha$, by

$$z'(w) = \frac{w-d(\alpha+\delta-1)}{(d+w)w}. \tag{22}$$

Renaming v as x, pose

$$z - \frac{\varepsilon}{\alpha}x = hy \tag{23}$$

with

$$h = \frac{\sqrt{\varepsilon(\alpha-\varepsilon)}}{\alpha}. \tag{24}$$

Multiplying the resulting vector field by e^x, we get the system

$$\dot{x} = re^x + w\left(\frac{\varepsilon}{\alpha}x + hy\right) - 1, \tag{25}$$

$$\dot{y} = \frac{\sqrt{\alpha-\varepsilon}}{\varepsilon}[re^x - (1-\alpha)]$$

which is equivalent to (10). If $\delta = \bar{\delta}$, system (25) can be reconduced to form (17) by a translation, with

$$\omega = (1-\alpha)\frac{\sqrt{\alpha-\varepsilon}}{\varepsilon}. \tag{26}$$

Hence, by utilizing the inverse function derivative formula, straightforward calculations lead to

$$\begin{aligned} \operatorname{sign} B &= \operatorname{sign}\left[(1-\alpha)w'''(z_0) + \frac{\varepsilon}{\alpha}(w''(z_0))^2\right] \\ &= \operatorname{sign}\left[\left(\frac{z''(\alpha)}{z'(\alpha)}\right)^2\left(3(1-\alpha)z'(\alpha) + \frac{\varepsilon}{\alpha}\right) - (1-\alpha)z'''(\alpha)\right]. \end{aligned} \tag{27}$$

It follows from (15) that

$$d(\alpha+\bar{\delta}-1) = \frac{\varepsilon(d+\alpha)}{1-\alpha} \tag{28}$$

and

$$z'(\alpha) = -\frac{\varepsilon}{\alpha(1-\alpha)}. \tag{29}$$

Through (28) and (29) we finally get

$$\operatorname{sign} B = \operatorname{sign}\left[-\frac{1-\alpha}{\varepsilon}\left(\alpha + \frac{\varepsilon(2\alpha+d)}{1-\alpha}\right) + 2\alpha - \sigma\right] \tag{30}$$

with $\sigma > 0$. Therefore $B < 0$, implying the Hopf bifurcation is supercritical, as claimed.

8. Periodic orbits

We have pointed out the existence of a Hopf bifurcation generating a locally attracting periodic orbit of system (3)–(4). In this section we analyze the dynamics of system (3)–(5) when a periodic orbit of (3)–(4) exists.

Remark 1. Let $p < 0$ and Δ a compact not pointwise limit set of system (3)–(5) contained in the positive orthant. Then Γ, the projection of Δ on the (K, λ) plane, is, by the Poincaré– Bendixson theorem, a periodic orbit of system (3)–(4).

Theorem 2. *Assume system* (3)–(4) *admits a periodic orbit* $\Gamma(t) = (K^*(t), \lambda^*(t))$, $K^*(t) > 0$, $\lambda^*(t) > 0$, *of period* $T > 0$. *Let*

$$a_0 := \frac{1}{T}\int_0^T \frac{1}{\lambda^*(t)}\,\mathrm{d}t \tag{31}$$

and C *be the* (*invariant*) *cylinder*, *contained in the positive orthant*

$$\{K > 0, \lambda > 0, E > 0\}$$

constituted by orbits of (3)–(5) *whose projections on the* (K, λ) *plane coincide with* Γ. *Then*,

(a) *if* $\beta\bar{E} - \gamma a_0 \leqslant 0$, *along any orbit lying on* $C \lim_{t\to+\infty} E(t) = 0$;
(b) *if* $\beta\bar{E} - \gamma a_0 > 0$, *the orbits lying on* C *are bounded for* $t > 0$, *but* $E(t)$ *does not tend to a constant value when* $t \to +\infty$.

Proof. Note that Eq. (5) can be written as a non-autonomous linear differential equation,

$$\dot{x} + \left(\beta\bar{E} - \frac{\gamma}{\lambda(t)}\right)x = \beta, \tag{32}$$

where $x := 1/E$. By assumption, $K^*(t)$ and $\lambda^*(t)$ are smooth periodic functions, so that we can consider the Fourier expansion of $\sigma(t) := 1/\lambda^*(t)$, that is

$$\sigma(t) = a_0 + \sum_{n=1}^{+\infty}\left(a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}\right). \tag{33}$$

Replacing $\lambda^*(t)$ in Eq. (32), we obtain

$$x(t) = \left[x_0 + \beta\int_0^t \mathrm{e}^{\int_0^\tau(\beta\bar{E}-\gamma\sigma(u))\,\mathrm{d}u}\,\mathrm{d}\tau\right]\mathrm{e}^{-\int_0^t(\beta\bar{E}-\gamma\sigma(\tau))\,\mathrm{d}\tau} \tag{34}$$

with $x_0 > 0$ and $x(t) = 1/E(t)$.

Assume, firstly, $\beta\bar{E} - \gamma a_0 < 0$. The function

$$\psi(t) := \mathrm{e}^{\gamma \int_0^t (\sigma(\tau) - a_0)\,\mathrm{d}\tau} \tag{35}$$

is periodic (of period T) and positive. Let

$$\psi_1 := \min \psi(t).$$

Hence, for $t > 0$,

$$x(t) > x_0 \mathrm{e}^{-(\beta\bar{E} - \gamma a_0)t} \psi_1$$

which implies $x(t) \to +\infty$ [and thus $E(t) \to 0$] as $t \to +\infty$.

Secondly, if $\beta\bar{E} - \gamma a_0 = 0$, then

$$\int_0^t \mathrm{e}^{\int_0^\tau (\beta\bar{E} - \gamma\sigma(u))\,\mathrm{d}u}\,\mathrm{d}\tau = \int_0^t \varphi(\tau)\,\mathrm{d}\tau,$$

where $\varphi(\tau) := (\psi(\tau))^{-1}$ is periodic and positive. Therefore,

$$\varphi(\tau) = \varphi_0 + \sum_{n=1}^{+\infty} \left(c_n \cos \frac{2\pi n\tau}{T} + d_n \sin \frac{2\pi n\tau}{T} \right)$$

with $\varphi_0 > 0$. It follows that, for a suitable $\zeta > 0$,

$$x(t) + \zeta > \beta\varphi_0\psi_1 t$$

when $t > 0$, and $\lim_{t\to+\infty} x(t) = +\infty$.

Finally, assume $\beta\bar{E} - \gamma a_0 > 0$. Let φ_1 and φ_2 be, respectively, the minimum and maximum value of $\varphi(t)$. Then,

$$\varphi_1 \frac{\mathrm{e}^{(\beta\bar{E} - \gamma a_0)t} - 1}{\beta\bar{E} - \gamma a_0} < \int_0^t \mathrm{e}^{\int_0^\tau (\beta\bar{E} - \gamma\sigma(u))\,\mathrm{d}u}\,\mathrm{d}\tau < \varphi_2 \frac{\mathrm{e}^{(\beta\bar{E} - \gamma a_0)t} - 1}{\beta\bar{E} - \gamma a_0}. \tag{36}$$

It follows that

$$\begin{aligned} 0 &< \frac{\varphi_1}{\varphi_2}\beta \frac{\mathrm{e}^{(\beta\bar{E} - \gamma a_0)t} - 1}{(\beta\bar{E} - \gamma a_0)\mathrm{e}^{(\beta\bar{E} - \gamma a_0)t}} < x(t) - x_0\psi(t)\mathrm{e}^{-(\beta\bar{E} - \gamma a_0)t} \\ &< \frac{\varphi_2}{\varphi_1}\beta \frac{\mathrm{e}^{(\beta\bar{E} - \gamma a_0)t} - 1}{(\beta\bar{E} - \gamma a_0)\mathrm{e}^{(\beta\bar{E} - \gamma a_0)t}} \end{aligned}$$

and, since

$$\lim_{t\to+\infty} x_0\psi(t)\mathrm{e}^{(\beta\bar{E} - \gamma a_0)t} = 0,$$

$x(t)$ is bounded and bounded away from zero, when $t > 0$. Suppose, by absurd, that in this case

$$\lim_{t\to+\infty} x(t) = \bar{x}. \tag{37}$$

Then $x(t) = \bar{x} + \phi(t)$, with $\lim_{t\to+\infty} \phi(t) = 0$. Let $t \in [nT, (n+1)T]$, n large enough. Write

$$\dot{x}(t) = -(\beta\bar{E} - \gamma\sigma(t))\bar{x} + \eta(t) \tag{38}$$

with $\lim_{t\to+\infty} \eta(t) = 0$. For (37) to hold, it is easily seen to be necessary that $\bar{x} = \beta/(\beta\bar{E} - \gamma a_0)$. Then,

$$\dot{x}(t) = \rho(t) + \eta(t), \tag{39}$$

where $\rho(t) := \gamma\bar{x}(\sigma(t) - a_0)$ is a periodic function with time average 0. Hence, $\xi(t) := \int_{nT}^{t} \rho(\tau)\,d\tau$ is also periodic. Let ξ_1 and ξ_2 denote its minimum and maximum value. If n is large enough,

$$|\eta(t)| < (\xi_2 - \xi_1)/2,$$

when $t > nT$, so that $x(t)$ has an oscillation of amplitude $> (\xi_2 - \xi_1)/2$ in the interval $[nT, (n+1)T]$, yielding a contradiction.

Theorem 3. *Assume the conditions of Theorem* 2 *and let* $\beta\bar{E} - \gamma a_0 > 0$. *Then, there exists a unique periodic orbit* $\Omega(t)$ *of system* (3)–(5) *lying on the cylinder* C, *which is the* ω*-limit set of the orbits of* C.

Proof. If $\beta\bar{E} - \gamma a_0 > 0$, Theorem 2 shows that orbits lying on C have compact ω-limit sets still lying on C. Then each of these limit sets is a periodic orbit $\Omega(t)$. Consider such an orbit and pose $\Omega(0) = (K_0, \lambda_0, E_0)$. Naming $x = 1/E$, $x_0 = 1/E_0$, it is clear that for $\Omega(t)$ to be periodic

$$\lim_{t\to-\infty} x_0 + \beta \int_0^t e^{\int_0^\tau (\beta\bar{E} - \gamma\sigma(u))\,du}\,d\tau = 0. \tag{40}$$

Consider two orbits lying on C starting, respectively, at (K_0, λ_0, x_0') and (K_0, λ_0, x_0''), with $x_0' = x_0 + \mu$, $x_0'' = x_0 - \mu$, $\mu > 0$ and sufficiently small. Then, by (40), it is easily calculated that

$$\lim_{t\to-\infty} x'(t) = +\infty \quad \text{(i.e., } \lim_{t\to-\infty} E'(t) = 0), \tag{41}$$

while $x''(\bar{t}) = 0$ for some $\bar{t} < 0$, implying $\lim_{t\to\bar{t}^+} E''(t) = +\infty$. This proves the uniqueness of the periodic orbit Ω lying on C. □

Notice that the periodic orbit $\Gamma(t)$, arising in system (3)–(4) through the Hopf bifurcation and lying on the plane $E = 0$, is also a periodic orbit under system (3)–(5). However, its stability properties may be different in the two systems.

In fact, although $\Gamma(t)$ is always attractive in the plane $E = 0$ (i.e., in system (3)–(4)), it may be not attractive in the space (K, λ, E) (i.e., in system (3)–(5)). The above analysis tells that, when $\Gamma(t)$ exists, two scenarios are possible.

If $\beta\bar{E} - \gamma a_0 > 0$ (i.e., if condition (b) in Theorem 2 holds), then there exists a unique periodic orbit $\Omega(t)$, along which $E > 0$, whose projection in the plane (K, λ) is $\Gamma(t)$. In this case $\Omega(t)$ is attractive, while the periodic orbit $\Gamma(t)$ is not, for system (3)–(5).

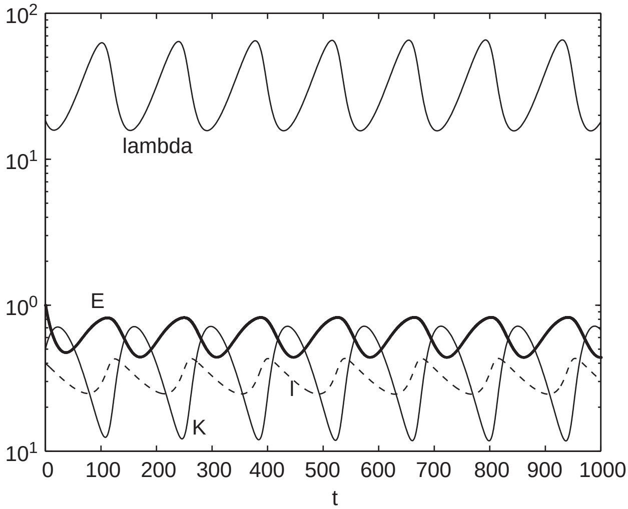

Fig. 1. $\delta = 1.8$, $\bar{E} = 1$.

If $\beta\bar{E} - \gamma a_0 \leqslant 0$ (i.e., if condition (a) in Theorem 2 holds), then there is no periodic orbit with $E > 0$ whose projection is $\Gamma(t)$, and $\Gamma(t)$ becomes attractive for system (3)–(5).

Since the average consumption $\bar{c}$ along the periodic orbit $\Gamma(t)$ coincides with $1/\lambda^*(t)$ (see (7)), the value a_0 defined in (31) represents the time-average of $\bar{c}$ along the orbit $\Gamma(t)$. Consequently, γa_0 expresses the time-average impact on the environmental asset of consumption along $\Gamma(t)$. Therefore, condition $\beta\bar{E} - \gamma a_0 \leqslant 0$ says that, if the "endowment" $\bar{E}$ of the environmental asset and the rate of regeneration β of E are not high enough with respect to γa_0, the growth dynamics represented by orbits lying on C is not sustainable; that is, the orbits approach $\Gamma(t)$, where $E = 0$.

Observe that, being the periodic orbit $\Omega(t)$ (or $\Gamma(t)$) locally attracting, an indeterminacy problem arises analogous to the one concerning an attractive fixed point. Given the initial values K^0, E^0, agents can choose the initial λ among a continuum of values, in order to approach $\Omega(t)$ (respectively, $\Gamma(t)$). Indeterminacy makes transient dynamics unpredictable, because it depends on the initial value of the *price* λ. Economies with the same production technology, starting from the same initial values of K and E, may have very different transient evolution patterns. This feature can make some economies more exposed than others to exogenous shocks, as suggested above.

Fig. 1 shows a numerical simulation of the time evolution of K, λ and E along the limit cycle $\Omega(t)$.

Fig. 2 represents an orbit approaching it in the space (K, λ, E). Notice that on the left side of Fig. 2, along $\Omega(t)$, the stock E of the environmental good reaches its minimum level; such a level is associated to a relatively low value of the capital stock K. This is a rather dangerous scenario, since the economy is not wealthy enough to protect the environmental good. On the contrary, on the right side of Fig. 2 a relatively high value

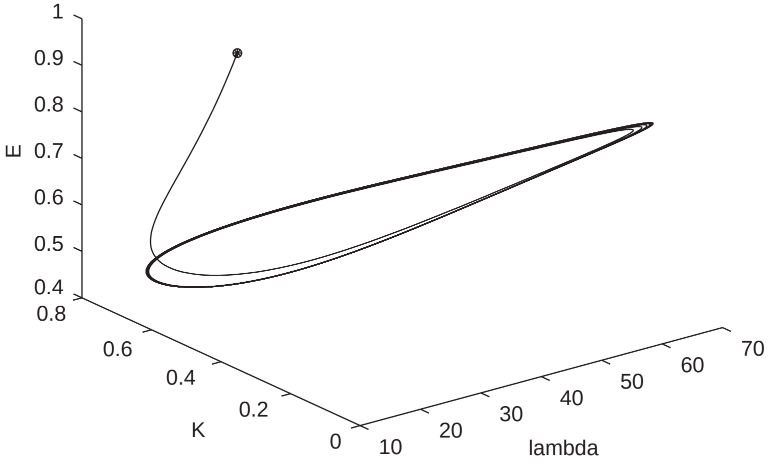

Fig. 2. $\delta = 1.8$, $\bar{E} = 1$.

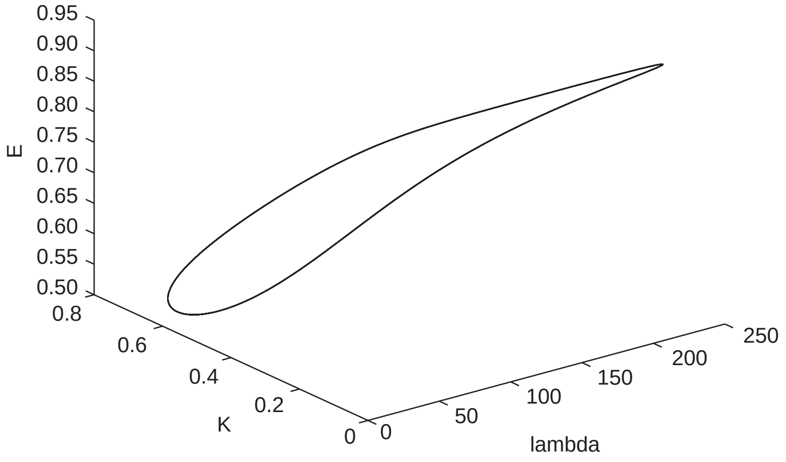

Fig. 3. $\delta = 1.85$, $\bar{E} = 1$.

of E corresponds to a relatively high value of K. In such a case the economy would have the means to protect environment, but there is no apparent need to do so.

Numerical simulations in Figs. 3–5 show how the periodic orbit $\Omega(t)$ modifies when the value of the parameter δ changes. Note that, when δ grows, the orbit stretches and, in fact, it gets more and more stretched in the λ direction, until it disappears. Observe also that the minimum value taken by E, along the periodic orbit, becomes lower when δ increases.

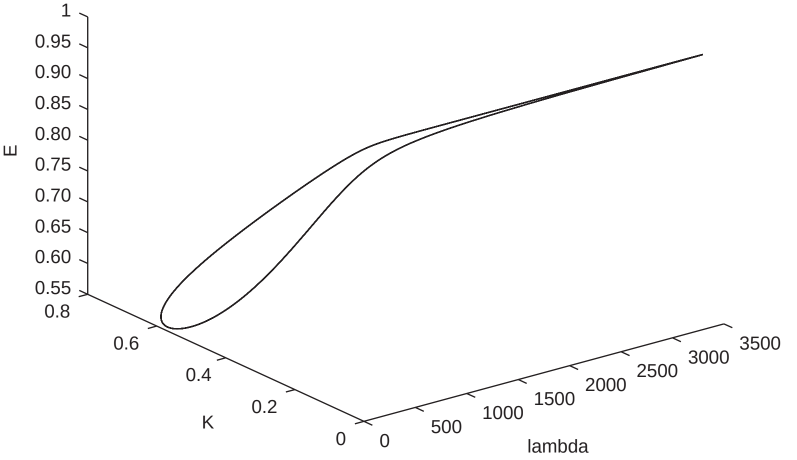

Fig. 4. $\delta = 1.89$, $\bar{E} = 1$.

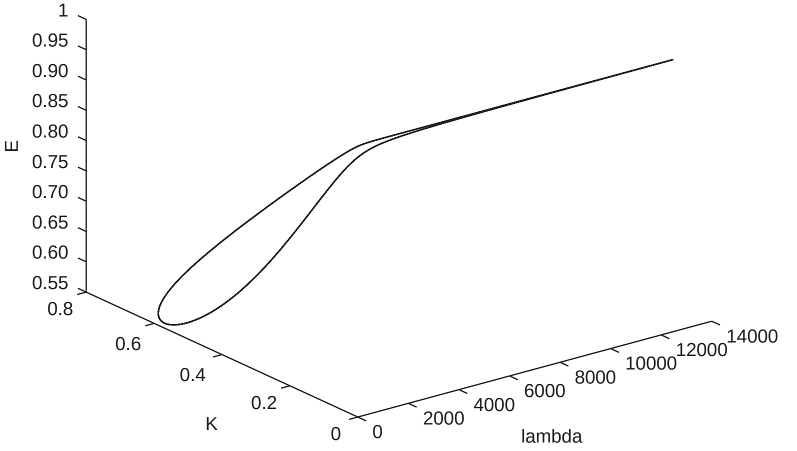

Fig. 5. $\delta = 1.893$, $\bar{E} = 1$.

9. Local attractiveness

We prove the following.

Theorem 4. *An attractor (equilibrium or periodic orbit) of system (3)–(4) is never globally attractive in the quadrant* $\lambda > 0$, $K > 0$.

Proof. Consider coordinates (9) and system (10). In the (u, v) plane,

$$l = \text{constant} = \bar{l}$$

corresponds to a straight line,

$$(\alpha-\varepsilon)u+\varepsilon v=m. \tag{42}$$

An attractor of (10) may exist, as we have shown, only if $\alpha\leqslant\varepsilon$ or $\alpha>\varepsilon$ and $\alpha+\delta>1$. Let

$$\bar{l}>\sup\left(\frac{1}{1+d},\frac{\alpha}{\alpha+\delta},\frac{\alpha+\delta-1}{\alpha+\delta}\right). \tag{43}$$

If $\alpha\leqslant\varepsilon$, consider the angle

$$\mathscr{A}:=\left\{(\alpha-\varepsilon)u+\varepsilon v\geqslant m,\quad v\geqslant\ln\frac{1-\alpha+\varepsilon}{r}\right\}.$$

Since $1-\alpha+\varepsilon\geqslant 1$, it is easily computed that, along the boundary of $\mathscr{A}$, the vector field $(\dot{u},\dot{v})$ points into $\mathscr{A}$, which does not contain the unique equilibrium of (10). Hence, no orbit starting in $\mathscr{A}$ can converge to some periodic orbit, which must surround the equilibrium.

If, instead, $\alpha>\varepsilon$ and $\alpha+\delta>1$, we can take the angle

$$\tilde{\mathscr{A}}:=\left\{(\alpha-\varepsilon)u+\varepsilon v\geqslant m,\quad v\geqslant\ln\frac{1-\alpha}{r}\right\}.$$

It is easily checked that, along the boundary of $\tilde{\mathscr{A}}$, $(\dot{u},\dot{v})$ points into $\tilde{\mathscr{A}}$, since, in particular, when $l\geqslant\bar{l}$ and $\bar{l}$ satisfies (43), $\dot{l}>0$ if $(\alpha-\varepsilon)\dot{u}+\varepsilon\dot{v}>0$. Therefore the orbits starting in $\tilde{\mathscr{A}}$ stay in $\tilde{\mathscr{A}}$, which does not contain the unique equilibrium of (10), and cannot converge either to the equilibrium or to some periodic orbit surrounding it. □

Theorem 4 implies that there are not global attractors (equilibria or periodic orbits) also in system (3)–(5). Therefore, the existence of a local attractor belonging to the region $\{E>0\}$ is not a sufficient condition for the sustainability of growth dynamics; in fact, if the economy starts from an initial position which does not belong to the attraction basin of the attractor, then the stock of environmental good may approach zero.

The numerical simulation in Fig. 6 concerns an example in which the fixed point of system (3)–(4) is locally attracting; it shows the boundary of its attraction basin in the plane (K,l). The simulation suggests that the two-dimensional attraction basin is compact and bounded by a closed orbit.

For all figures, the parameters not listed in the corresponding captions are the following ones: $r=0.05,\alpha=0.5,\varepsilon=0.25,d=1,\beta=0.1$.

10. Conclusions

The model we have proposed can be extended in several directions. For example, in a collateral work (see [1]) the authors introduce, in the same basic model, the additional hypothesis that private assets, produced in the economy, are also consumed by the agents as substitutes for degraded environmental assets. In this framework, it is shown

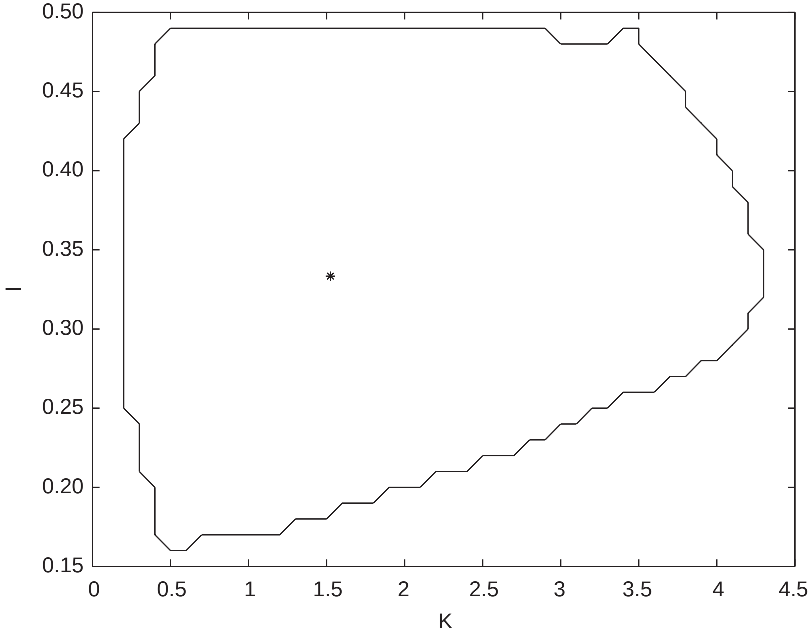

Fig. 6. $\delta = 1.5$, $\bar{E} = 10$.

that increasing the environmental degradation may lead to increasing the production and consumption of private assets for substitutive needs. This increase generates a further increase in environmental degradation, which in turn generates a further increase in the consumption of goods for substitutive ends, and so on. It is proved that such a mechanism leads the economy down to growth paths which, from the point of view of social well-being, are worse than those described in the basic model analyzed in the present work.

Another extension of the model could be that of considering the effects on the economic and environmental dynamics of "red code"—type interventions, which means assuming that the policy-maker intervenes only when environmental degradation reaches certain threshold values. This "adaptive" type of environmental policy is the one usually adopted to manage traffic in major Italian cities and, although responding to organizational requirements, certainly it is not optimal. Therefore its impact on environmental dynamics cannot be predicted by traditional growth models.

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