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On the Convergence of LMF-type Methods for SODEs

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Abstract. In a previous paper [3], some numerical methods for stochastic ordinary differential equations (SODEs), based on Linear Multistep Formulae (LMF), were proposed. Nevertheless, a formal proof for the convergence of such methods is still lacking. We here provide such a proof, based on a matrix formulation of the discrete problem, which allows some more insight in the structure of LMF-type methods for SODEs.

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1. Introduction

The evolution of many phenomena in applied sciences such as, for example, finance, biology and physics, is modeled by using stochastic ordinary differential equations. The latter are differential equations in which noise terms, modeling some unpredictable behavior, are present. In more detail, a stochastic ordinary differential equation (SODE) is an equation in the form

$$\begin{aligned} dy(t) &= f(y(t))dt + \sum_{j=1}^{d} g_j(y(t))dW_j(t), \qquad t \in [0,T], \qquad (1.1)\\ y(0) &= y_0 \in \mathbb{R}^m, \end{aligned}$$

which has been taken to be autonomous, without loss of generality, in order not to complicate the arguments. In (1.1) the $W_j(t)$, $j = 1,\dots,d$, are independent Wiener processes, modeling independent Brownian motions: i.e., they are Gaussian processes with independent increments, such that (see, for example, [9]) $W_j(0) = 0$ with probability 1, and

$$E(W_j(t)) = 0, \qquad Var(W_j(t) - W_j(s)) = t - s, \quad 0 \le s \le t.$$

That is, the increments $W_j(t)-W_j(s)$ are random variables $N(0,t-s)$ distributed. The deterministic term $f(y)$ is sometimes called the drift. Equation (1.1) can be cast into integral form as

$$y(t) = y(0) + \int_0^t f(y(s))\,ds + \sum_{j=1}^{d} \int_0^t g_j(y(s))\,dW_j(s). \tag{1.2}$$

In (1.2) the first integral is a Riemann-Stieltjes integral, whereas the remaining ones are stochastic integrals (see, for example, [9]). The latter are defined as the limit (in the mean square sense), as $n \to \infty$, of the approximating sums

$$\sum_{i=1}^{n} g_j(\xi_i)(W_j(t_i) - W_j(t_{i-1})), \qquad j = 1\dots,d,$$

where $\xi_i = \theta t_i + (1-\theta)t_{i-1}$, for a fixed $\theta \in [0,1]$, and, for sake of simplicity, $t_i = i \cdot t/n$, $i = 0,\dots,n$. Unlike Riemann integrals, different choices of the parameter θ generally result in different values of the stochastic integral. The most common choices of such a parameter are:

- $\theta = 0$, resulting in a Itô integral,
- $\theta = \frac{1}{2}$, resulting in a Stratonovich integral.

For the numerical approximation of the solution of equation (1.1), some numerical methods have been considered (see, for example, [5] and the references therein). In this paper we shall consider, in more detail, numerical methods in the form (as recently proposed in [3] when equation (1.1) is in Stratonovich form):

$$\sum_{i=0}^{k} \alpha_i y_{n-k+i} \quad = \quad h \sum_{i=0}^{k} \beta_i f_{n-k+i} + \sum_{j=1}^{d} \sum_{s=0}^{k-1} J_j^{n-s} \sum_{i=0}^{k} \gamma_{is} g_{j,n-k+i}, \tag{1.3}$$
$$n = k,\dots,N,$$

where the coefficients are normalized so that $\alpha_k = 1$, $h = T/N$ is the stepsize, y_n is the numerical approximation to $y(t_n)$, $t_n = nh$, $y_0,\dots,y_{k-1}$ are given, $f_n = f(y_n)$, and $g_{jn} = g_j(y_n)$. Finally,

$$J_j^n = W_j(t_n) - W_j(t_{n-1}) = \int_{t_{n-1}}^{t_n} dW_j(t), \qquad j = 1,\dots,d, \quad n = 1,\dots,N,$$

are the Wiener increments, which are random variables $N(0,h)$ distributed. Obviously, all the variables $\{J_j^n\}$ are mutually independent. In general, for methods in the form (1.3), the previous integral will be a Itô or a Stratonovich integral, depending on the case.

Our aim will be to analyse the order of convergence of the methods as the stepsize h tends to 0. In 1955 Maruyama proved the mean-square convergence of the simplest one-step method [10] and, in 1968, Gikhman and Skorokhod analysed its mean-square order of convergence [8]. Some results of mean-square convergence for one-step methods can also be found in [9, 12, 18]. For the study of higher-order methods we refer, for example, to the papers of Milstein [11] and Wagner and Platen [20].

We shall consider here the case of linear multistep methods, i.e., methods in the form (1.3). The framework will be the matrix formulation of the corresponding discrete problem, as already proposed in [2] in the case of deterministic equations. The main results, along with the needed preliminary ones, are presented in Section 2. An example of application is shown in Section 3. A further generalization is then considered in Section 4, based on a predictor-corrector implementation of methods in the form (1.3), as presented in [3]. Finally, some concluding remarks are reported in Section 5.

2. Convergence results

The truncation error of formula (1.3) is given by

$$\begin{aligned}\tau_n &= \sum_{i=0}^{k}\alpha_i y(t_{n-k+i}) - h\sum_{i=0}^{k}\beta_i f(y(t_{n-k+i})) \\ &\qquad - \sum_{j=1}^{d}\sum_{s=0}^{k-1} J_j^{n-s}\sum_{i=0}^{k}\gamma_{is}g_j(y(t_{n-k+i})),\end{aligned} \tag{2.1}$$

which must be interpreted as the error generated in a single step, when a set of k exact initial conditions is given.

We now want to investigate the connection between local and global error, i.e. the problem of determining the order of convergence of the method as the stepsize h tends to 0.

When speaking about the accuracy of numerical methods for SODEs, it is customary to distinguish between two different definitions of convergence, as specified in the following (hereafter, $\|\cdot\|$ denotes the 2-norm).

Weak convergence: this case concerns the situations where one is interested in the moments. One then requires that there exist suitable constants $C, \delta, p > 0$, independent of h, such that

$$\max_n \|E\left(q(y_n) - q(y(t_n))\right)\| \leq Ch^p,$$

for all stepsizes $h \in (0, \delta)$ and appropriate polynomials q. In such a case, the method is said to have weak order p.

Strong convergence: in this case, one is interested in the mean square convergence of the trajectories, which means that

$$\max_n E\left(\|y_n - y(t_n)\|\right) \leq Ch^p,$$

for all stepsizes $h \in (0, \delta)$, and the method is said to have strong order p.

By defining the global error at t_n as

$$e_n = y(t_n) - y_n, \tag{2.2}$$

in the following, we shall study the behavior of the following quantities:

– $\max_n \|E(e_n)\|$,
– $\max_n \sqrt{E(\|e_n\|^2)}$.

Considering that the variance of e_n is given by

$$E(\|e_n\|^2) - \|E(e_n)\|^2,$$

the above arguments are equivalent to studying the mean and the standard deviation of the error. We also observe that, by using the Lyapunov inequality, we obtain

$$E(\|e_n\|) \le \sqrt{E(\|e_n\|^2)},$$

and, therefore, the study of the standard deviation essentially consists in the study of the (global) strong order of the method.

We will now introduce the matrix formulation for the discrete problem generated by the method (1.3) when applied to problem (1.1). In general, when in (1.1) $d > 1$, the form of a numerical method is more entangled, in order to take into account of the possible non commutativity (see e.g. [3] for the case of method (1.3)). Therefore, though the arguments can be suitably extended to the more general case, it is customary to study the convergence in the simpler case where $d = m = 1$:

$$\begin{aligned} dy(t) &= f(y(t))dt + g(y(t))dW(t), \qquad t \in [0,T], \\ y(0) &= y_0. \end{aligned} \tag{2.3}$$

In such a case, the lower index j for the Wiener processes, as well as for its increments, can be avoided, thus giving the discrete problem

$$\sum_{i=0}^{k} \alpha_i y_{n-k+i} = h\sum_{i=0}^{k} \beta_i f_{n-k+i} + \sum_{s=0}^{k-1} J^{n-s} \sum_{i=0}^{k} \gamma_{is} g_{n-k+i}, \qquad n = k,\dots,N. \tag{2.4}$$

By introducing the vectors

$$\mathbf{y} = \begin{pmatrix} y_k \\ \vdots \\ y_N \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} f_k \\ \vdots \\ f_N \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} g_k \\ \vdots \\ g_N \end{pmatrix}, \tag{2.5}$$

and the following $(N-k+1) \times (N-k+1)$ matrices,

$$A = \begin{pmatrix} \alpha_k & & & & \\ \vdots & \ddots & & & \\ \alpha_0 & & \ddots & & \\ & \ddots & & \ddots & \\ & & \alpha_0 & \dots & \alpha_k \end{pmatrix}, \qquad B = \begin{pmatrix} \beta_k & & & & \\ \vdots & \ddots & & & \\ \beta_0 & & \ddots & & \\ & \ddots & & \ddots & \\ & & \beta_0 & \dots & \beta_k \end{pmatrix}, \tag{2.6}$$

$$C_s = \begin{pmatrix} \gamma_{ks} & & & & \\ \vdots & \ddots & & & \\ \gamma_{0s} & & \ddots & & \\ & \ddots & & \ddots & \\ & & \gamma_{0s} & \cdots & \gamma_{ks} \end{pmatrix}, \qquad J_s = \begin{pmatrix} J^{k-s} & & \\ & \ddots & \\ & & J^{N-s} \end{pmatrix}, \tag{2.7}$$

$s = 0, \ldots, k-1$, one easily realizes that the discrete problem (2.4) can be written as

$$A\mathbf{y} - hB\mathbf{f} - \sum_{s=0}^{k-1} J_s\, C_s \mathbf{g} = \mathbf{v}, \tag{2.8}$$

where the vector $\mathbf{v}$ has only the leading k entries different from zero, which depend on the initial conditions

$$y_0, \ldots, y_{k-1}.$$

By introducing the vectors $\hat{\mathbf{y}}$, $\hat{\mathbf{f}}$, $\hat{\mathbf{g}}$, defined similarly as (2.5), with y_n replaced by $y(t_n)$ everywhere, and the vector $\hat{\mathbf{v}}$ containing the exact initial conditions

$$y(t_0) \equiv y_0,\; y(t_1), \ldots, y(t_{k-1}),$$

we also obtain that

$$A\hat{\mathbf{y}} - hB\hat{\mathbf{f}} - \sum_{s=0}^{k-1} J_s\, C_s \hat{\mathbf{g}} = \hat{\mathbf{v}} + \boldsymbol{\tau}, \tag{2.9}$$

where

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_k, & \ldots, & \tau_N \end{pmatrix}^T$$

is the vector containing the truncation errors. From (2.8) and (2.9) we then obtain

$$A(\hat{\mathbf{y}} - \mathbf{y}) = hB(\hat{\mathbf{f}} - \mathbf{f}) + \sum_{s=0}^{k-1} J_s\, C_s(\hat{\mathbf{g}} - \mathbf{g}) + \hat{\mathbf{v}} - \mathbf{v} + \boldsymbol{\tau},$$

which, assuming for simplicity exact initial conditions[1], reduces to

$$A(\hat{\mathbf{y}} - \mathbf{y}) = hB(\hat{\mathbf{f}} - \mathbf{f}) + \sum_{s=0}^{k-1} J_s\, C_s(\hat{\mathbf{g}} - \mathbf{g}) + \boldsymbol{\tau}.$$

By defining the vectors

$$\mathbf{e} = \hat{\mathbf{y}} - \mathbf{y} \equiv (e_n)_{n=k,\ldots,N}, \qquad \tilde{\boldsymbol{\tau}} = A^{-1}\boldsymbol{\tau} \equiv (\tilde{\tau}_n)_{n=k,\ldots,N},$$
$$\boldsymbol{\delta}\mathbf{f} = \hat{\mathbf{f}} - \mathbf{f} \equiv (\delta f_n)_{n=k,\ldots,N}, \qquad \boldsymbol{\delta}\mathbf{g} = \hat{\mathbf{g}} - \mathbf{g} \equiv (\delta g_n)_{n=k,\ldots,N},$$

and the matrices

$$\begin{array}{lcll} A^{-1} & = & (a_{ij})_{i,j=k,\ldots,N}, & \\ \tilde{B} & = & A^{-1}B & \equiv (b_{ij})_{i,j=k,\ldots,N}, \\ \tilde{C} & = & \sum_{s=0}^{k-1} A^{-1} J_s\, C_s & \equiv (c_{ij})_{i,j=k,\ldots,N}, \end{array} \tag{2.10}$$

[1] In the general case, the obtained results will continue to hold, provided that the leading k entries of the vector $\hat{\mathbf{v}} - \mathbf{v}$ have $O(h^p)$ mean and standard deviation, for a method having strong order p.

we then obtain

$$\mathbf{e} = h\tilde{B}\boldsymbol{\delta}\mathbf{f} + \tilde{C}\boldsymbol{\delta}\mathbf{g} + \tilde{\boldsymbol{\tau}}. \tag{2.11}$$

We now need some preliminary results (hereafter, 0-stability is the usual notion of LMFs for ODEs).

Lemma 2.1. *If the method* (2.4) *is* 0*-stable, then the following properties hold true:*

1. *there exists* $0 < \alpha$ *independent of* N *such that* $|a_{ij}| \le \alpha$,
2. *there exists* $0 < \beta$ *independent of* N *such that* $|b_{ij}| \le \beta$,
3. *if*
$$\gamma_{k-\nu,s} = 0, \qquad s = 0, \ldots, k-1, \quad \nu = 0, \ldots, s, \tag{2.12}$$
then c_{ij} *is a stochastic variable depending on* $J^{j+1}, \ldots, J^{\eta}$*; alternatively*
4. c_{ij} *is a stochastic variable depending on* $J^{j-k+1}, \ldots, J^{\eta}$,

where $\eta = \min\{i, j+k\}$ *and, moreover,*

$$E(c_{ij}) = 0, \qquad E(|c_{ij}|) \le \sqrt{E(c_{ij}^2)} = O(\sqrt{h}). \tag{2.13}$$

Proof. The first property is proved, for example, in [1, 2]. The second property then easily follows from the banded structure of the matrix B.

Now from (2.10) one obtains

$$c_{ij} = \sum_{r=j}^{\eta} \sum_{s=0}^{k-1} a_{ir} J^{r-s} \gamma_{k+j-r,s}, \qquad \eta = \min\{i, j+k\}, \tag{2.14}$$

and properties 3 and 4 then follow depending on whether (2.12) holds true or not. Finally,

$$E(c_{ij}) = \sum_{r=j}^{\eta} \sum_{s=0}^{k-1} a_{ir} \gamma_{k+j-r,s} E(J^{r-s}) = 0,$$

and, from the Lyapunov inequality,

$$\begin{aligned} E(|c_{ij}|)^2 &\le E(c_{ij}^2) = \sum_{r,n=j}^{\eta} a_{ir} a_{in} \sum_{s,p=0}^{k-1} \gamma_{k+j-r,s} \gamma_{k+j-n,p} E(J^{r-s} J^{n-p}) \\ &= \sum_{s,p=0}^{k-1} \sum_{r=j}^{\eta} a_{ir} a_{i,r-s+p} \gamma_{k+j-r,s} \gamma_{k+j-r+s-p,p} E((J^{r-s})^2) \\ &\le \left(k^2 (k+1)\, \alpha^2 \left(\max_{r=0,\ldots,k, s=0,\ldots,k-1} |\gamma_{rs}| \right)^2 \right) h. \end{aligned}$$

□

Other simple, though important, results are the following.

Lemma 2.2. *For each* $n \ge 1$*, the global error at* t_n*, i.e.* e_n*, is independent of* J^r*, for all* $r > n$*.*

Lemma 2.3. *Let the method* (2.4) *be* 0*-stable, and let* $E(\tau_n) = O(h^{p+1})$, $n \geq k$, *then*

$$|E(\tilde{\tau}_n)| \leq O(h^p).$$

Proof. From property 1 in Lemma 2.1

$$|E(\tilde{\tau}_n)| = \left| E\left(\sum_{j=k}^{n} a_{ij}\tau_j\right)\right| \leq \sum_{j=k}^{n} |a_{ij}|\,|E(\tau_j)| \leq N\alpha O(h^{p+1}) = O(h^p).$$

□

Moreover, we list the following result.

Lemma 2.4. *Let*

$$F = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ \vdots & \ddots & \ddots & \\ 1 & \dots & 1 & 0 \end{pmatrix}_{(N-k+1)\times(N-k+1)}, \tag{2.15}$$

then, given a complex number μ *independent of* N, *one has that, for all allowed integers* N, *the matrix*

$$I - \frac{|\mu|}{N}F$$

is an M*-matrix (see, for example,* [2]*), and*

$$\left\|\left(I - \frac{\mu}{N}F\right)^{-1}\right\| \leq \left\|\left(I - \frac{|\mu|}{N}F\right)^{-1}\right\| \leq e^{-|\mu|}.$$

Proof. See [2, Section 4.4.1]. □

Finally, we need the higher dimensional version of the Mean Value Theorem.

Lemma 2.5. *Let* $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$ *be suitably smooth, with* Ω *a convex set. Then, for each couple of points* $x, y \in \Omega$, *there exist* $\xi^{(i)} \in (0,1)$, $\quad i = 1, \dots, m$, *such that*

$$f(x) - f(y) = D(\xi)(x - y),$$

where $\xi = (\xi^{(1)}, \dots, \xi^{(m)})^T$ *and*

$$D(\xi) = \begin{pmatrix} \nabla f_1(\xi^{(1)}x + (1-\xi^{(1)})y) \\ \vdots \\ \nabla f_m(\xi^{(m)}x + (1-\xi^{(m)})y) \end{pmatrix}.$$

Now we can state the main results of this paper.

Theorem 2.6. *Let the method* (2.4) *be* 0*-stable and let the truncation error* (2.1) *be such that* $E(\tau_n) = O(h^{p+1})$. *It follows that, if* (2.12) *holds true and the functions* f *and* g *are suitably smooth with* f *Lipschitz-continuous with bounded Lipschitz constant* ℓ, *then the global error* (2.2) *satisfies*

$$|E(e_n)| \le O(h^p), \qquad n = k, \dots, N,$$

provided that the errors on the initial conditions satisfy

$$|E(e_i)| \le O(h^p), \qquad i = 0, \dots, k-1.$$

Proof. For brevity, we shall suppose $e_0 = \cdots = e_{k-1} = 0$, but the arguments used can be easily generalized.

Let us then define the lower triangular matrix, of dimension $N - k + 1$,

$$L = I + F \equiv \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \dots & 1 \end{pmatrix}, \tag{2.16}$$

where F is defined according to (2.15). From (2.11), and the results of Lemmas 2.1 and 2.3, we then obtain that

$$\begin{aligned} |E(\mathbf{e})| &\le h|E(\tilde{B}\boldsymbol{\delta}\mathbf{f})| + |E(\tilde{C}\boldsymbol{\delta}\mathbf{g})| + |E(\tilde{\boldsymbol{\tau}})| \\ &\le h\beta\ell L|E(\mathbf{e})| + |E(\tilde{C}\boldsymbol{\delta}\mathbf{g})| + O(h^p). \end{aligned}$$

On the other hand, we have that, since (2.12) holds true, from property 3 in Lemma 2.1 and Lemma 2.5 it follows that

$$E\left((\tilde{C}\boldsymbol{\delta}\mathbf{g})_i\right) = \sum_{j=k}^{i} E(c_{ij} D(\xi_j) e_j) = \sum_{j=k}^{i} E(c_{ij}) E(D(\xi_j) e_j) = 0,$$

because c_{ij} depends on J^r, $r > j$, and then it is independent of $D(\xi_j)e_j$, which only depends on y_j and $y(t_j)$. The last equality then follows from (2.13). We then conclude that

$$(I - h\beta\ell L)|E(\mathbf{e})| \le O(h^p).$$

The proof is completed by observing that, for all $h \in \left(0, (2\beta\ell)^{-1}\right]$, then from Lemma 2.4 it follows that the matrix on the left-hand side is an M-matrix, and, considering that $h = T/N$,

$$\|(I - h\beta\ell L)^{-1}\| = (1 - h\beta\ell)^{-1} \left\| \left(I - \frac{h\beta\ell}{1 - h\beta\ell} F \right)^{-1} \right\| \le 2e^{2\beta\ell T}.$$

□

Remark 2.7. We observe that the proof of the above result strongly relies on the requirement (2.12) since, conversely, c_{ij} and $D(\xi_j)e_j$ wouldn't be independent. Consequently, it seems that the numerical methods in the form (2.4) must be explicit in their stochastic part.

Theorem 2.8. *Let the method* (2.4) *be* 0*-stable and let the local truncation error* (2.1) *be such that*

$$\begin{aligned} E(\tau_n) &= O(h^{p+1}), && (2.17)\\ E(\tau_n^2) &= O(h^{2p+1}), && (2.18)\\ E(\tau_n \mid y(t_j)) &= O(h^{p+1}), \qquad \forall\, t_j \le t_{n-k}. && (2.19) \end{aligned}$$

It follows that, if (2.12) *holds, and the functions f and g are Lipschitz-continuous with bounded Lipschitz constants, then the global error* (2.2) *satisfies*

$$E(e_n^2) \le O(h^{2p}), \qquad n = k, \dots, N,$$

provided that the errors on the initial conditions satisfy

$$E(e_i^2) \le O(h^{2p}), \qquad i = 0, \dots, k-1.$$

Remark 2.9. From a probabilistic point of view, the request (2.19) means that the random variable τ_n must be measurable with respect to the σ-tail generated by $y(t_j)$ for all $j \le n-k$. That is, for all subsets A of the the Borelian sets $B(\mathbb{R})$, $\tau_n^{-1}(A) = \{\omega \in \Omega : \tau_n(\omega) = A\}$ is an element of $\sigma(y(t_j)) = \{y(t_j)^{-1}(A), A \in B(\mathbb{R})\}$. Moreover, we observe that the requirement (2.19) implies (2.17), since $E(\tau_n) = E\,(\,E(\tau_n \mid y(t_j))\,)$.

From the point of view of numerical methods for differential equations, (2.19) means that the truncation error τ_n must roughly behave like $O(h^{p+1})$, whatever the local trajectory is. Indeed, the requirement is on the expectation of τ_n. In light of this interpretation, it is then evident that the two requirements (2.17) and (2.19) are essentially equivalent.

Proof. As done in the previous Theorem 2.6, for brevity we shall consider the simpler case where $e_0 = \cdots = e_{k-1} = 0$, but the arguments used can be readily generalized. Let $\circ$ denote the Hadamard (i.e., componentwise) product. Then, from (2.11),

$$\begin{aligned} \mathbf{e} \circ \mathbf{e} &= (h\tilde{B}\boldsymbol{\delta}\mathbf{f} + \tilde{C}\boldsymbol{\delta}\mathbf{g} + \tilde{\boldsymbol{\tau}}) \circ (h\tilde{B}\boldsymbol{\delta}\mathbf{f} + \tilde{C}\boldsymbol{\delta}\mathbf{g} + \tilde{\boldsymbol{\tau}})\\ &= h^2(\tilde{B}\boldsymbol{\delta}\mathbf{f}) \circ (\tilde{B}\boldsymbol{\delta}\mathbf{f}) + (\tilde{C}\boldsymbol{\delta}\mathbf{g}) \circ (\tilde{C}\boldsymbol{\delta}\mathbf{g}) + \tilde{\boldsymbol{\tau}} \circ \tilde{\boldsymbol{\tau}}\\ &\quad + 2h(\tilde{B}\boldsymbol{\delta}\mathbf{f}) \circ (\tilde{C}\boldsymbol{\delta}\mathbf{g}) + 2h(\tilde{B}\boldsymbol{\delta}\mathbf{f}) \circ \tilde{\boldsymbol{\tau}} + 2(\tilde{C}\boldsymbol{\delta}\mathbf{g}) \circ \tilde{\boldsymbol{\tau}}. \end{aligned}$$

It then follows that, for all $n = k, \dots, N$,

$$\begin{aligned} E(e_n^2) &\le 3\,h^2 E\left[\left(\sum_{j=k}^{n} b_{nj}\delta f_j\right)^2\right] && (2.20)\\ &+ 3\,E\left[\left(\sum_{j=k}^{n-1} c_{nj}\delta g_j\right)^2\right] && (2.21)\\ &+ 3\,E(\tilde{\tau}_n^2). && (2.22) \end{aligned}$$

Let us now analyze all the expressions on the right-hand side of the last inequality. Starting from (2.20), we have,

$$\begin{aligned} & h^2 E\left[\left(\sum_{j=k}^{n} b_{nj}\delta f_j\right)^2\right] \\ \le\ & h^2 E\left[\left(\sum_{j=k}^{n} |b_{nj}|\,|\delta f_j|\right)^2\right] \le h^2\ell^2 \sum_{j,r=k}^{n} |b_{nj}|\,|b_{nr}|\,E(|e_j|\,|e_r|) \\ \le\ & \frac{h^2\ell^2}{2}\sum_{j,r=k}^{n} |b_{nj}|\,|b_{nr}|\,E(e_j^2+e_r^2) \le h^2\ell^2\beta^2\sum_{j=k}^{n}(n-k)E(e_j^2) \\ \le\ & hT\ell^2\beta^2\sum_{j=k}^{n} E(e_j^2), \end{aligned}$$

where β is the uniform upper bound for the entries of $\tilde{B}$, as defined in Lemma 2.1, ℓ is the maximum between the Lipschitz constants of f and g, and, moreover,

$$n-k \le N-k \le \frac{T}{h}.$$

For (2.21), we have

$$E\left[\left(\sum_{j=k}^{n-1} c_{nj}\delta g_j\right)^2\right] = E\left(\sum_{j,r=k}^{n-1} c_{nj}c_{nr}\delta g_j\delta g_r\right) = \sum_{j,r=k}^{n-1} E\left(c_{nj}c_{nr}\delta g_j\delta g_r\right).$$

Now, from Lemma 2.1 we have that, if $|j-r|\ge k$, than

$$E\left(c_{nj}c_{nr}\delta g_j\delta g_r\right) = E(c_{nj})\,E(c_{nr}\delta g_j\delta g_r) = 0.$$

Conversely, from Lemma 2.2 we have that

$$E\left(c_{nj}c_{nr}\delta g_j\delta g_r\right) \le \ell^2\,E(c_{nj}c_{nr}e_je_r) \le \ell^2\,O(h)\,E\left(\frac{e_j^2+e_r^2}{2}\right).$$

Consequently, there exists $c>0$, independent of N, such that

$$\begin{aligned} \sum_{j,r=k}^{n-1} E\left(c_{nj}c_{nr}\delta g_j\delta g_r\right) \ \le\ & \frac{ch\ell^2}{2}\sum_{j=k}^{n-1}\ \sum_{r=\max\{k,j-k\}}^{\min\{n-1,j+k\}} \left(E(e_j^2)+E(e_r^2)\right) \\ \le\ & 2(k+1)ch\ell^2\sum_{j=k}^{n-1} E(e_j^2). \end{aligned}$$

Considering now (2.22), by taking into account the results of Lemmas 2.1 (property 1) and 2.3 and the hypotheses (2.18) and (2.19), we have

$$E(\tilde{\tau}_n^2) = E\left[\left(\sum_{j=k}^{n} a_{nj}\tau_j\right)^2\right] = \sum_{j,r=k}^{n} a_{nj}a_{rj}E(\tau_j\tau_r)$$

$$\le \alpha^2 \left(\sum_{\substack{j,r=k \\ |j-r|<k}}^{n} E\left(\frac{\tau_j^2+\tau_r^2}{2}\right) + \sum_{\substack{j,r=k \\ |j-r|\ge k}}^{n} \left| E(\tau_{\max\{j,r\}}|y(t_{\min\{j,r\}}))\, E(\tau_{\min\{j,r\}})\right| \right)$$

$$\le \alpha^2 \left(k\, n\, O(h^{2p+1}) + n^2\, O(h^{2p+2}) \right).$$

By taking into account all the above intermediate results, we finally obtain

$$E(\mathbf{e}\circ\mathbf{e}) \le h\,\zeta\, L\, E(\mathbf{e}\circ\mathbf{e}) + O(h^{2p}),$$

where the scalar $\zeta > 0$ is independent of N, and L is the matrix defined in (2.16). Consequently,

$$(I - h\,\zeta\, L)\, E(\mathbf{e}\circ\mathbf{e}) \le O(h^{2p}).$$

From Lemma 2.4, we then obtain that, for $h \in (0, h\zeta/2]$,

$$E(\mathbf{e}\circ\mathbf{e}) \le \left\|(I - h\zeta L)^{-1}\right\| O(h^{2p})$$

$$= (1-h\zeta)^{-1} \left\| \left(I - \frac{h\zeta}{1-h\zeta}F\right)^{-1} \right\| O(h^{2p}) \le 2e^{2\zeta T} O(h^{2p}) \equiv O(h^{2p}).$$

Therefore, we finally obtain

$$E(|e_n|) \le \sqrt{E(e_n^2)} = O(h^p), \qquad n = k, \ldots, N,$$

which implies that the method (2.4) has strong order p. □

Remark 2.10. In addition to what is stated in Remark 2.7, we observe that the above arguments cannot be extended to cover the case where (2.12) does not hold. Consequently, this confirms that the method must be explicit in its stochastic component.

3. An example: the Euler-Maruyama method

As an example, let us apply Theorems 2.6 and 2.8 to the Euler-Maruyama method, which, when applied to the one-dimensional problem (2.3), has the following simple form

$$y_n = y_{n-1} + hf(y_{n-1}) + J^n g(y_{n-1}), \qquad n = 1, \ldots, N. \tag{3.1}$$

If we denote by τ_n and $\widetilde{\tau}_n$ the local truncation errors of method (3.1) when the equation is in Itô and in Stratonovich form, respectively, we obtain

$$E(\tau_n) = O(h^2), \qquad E(\tau_n^2) = O(h^2),$$
$$E(\widetilde{\tau}_n) = O(h), \qquad E(\widetilde{\tau}_n^2) = O(h^2).$$

Consequently, from Theorem 2.8, it follows that the Euler-Maruyama method is strongly convergent only if the problem is in the Itô form, with order $p = \frac{1}{2}$, which is known to be its actual order, in this case. Moreover, it is also well known that the Euler-Maruyama method does not converge if applied to a Stratonovich problem [7]: in such a case, the method converges to the solution of the problem formally written in the Itô form. This can be easily proved by resorting to the matrix formulation of the discrete problem, which is the framework used so far. In more detail, let us assume that the one dimensional problem (2.3) is in Stratonovich form. By introducing the vectors

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix},$$

and the following $N \times N$ matrices,

$$A = \begin{pmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} J^1 & & \\ & \ddots & \\ & & J^N \end{pmatrix},$$

the method (3.1) can be written as

$$A\mathbf{y} - hB\mathbf{f} - J\,B\mathbf{g} = e_1 \otimes \left(y_0 + hf_0 + J^1 g_0\right).$$

After left multiplication by the matrix

$$A^{-1} = \begin{pmatrix} 1 & & & \\ \vdots & 1 & & \\ \vdots & & \ddots & \\ 1 & \dots & \dots & 1 \end{pmatrix},$$

we then obtain that the ith approximation to the solution of (2.3) is given by

$$\begin{aligned} y_i &= y_0 + hf_0 + J^1 g_0 + h\sum_{j=2}^{i-1} f_{j-1} + \sum_{j=2}^{i-1} J^j g_{j-1} \\ &= y_0 + h\sum_{j=1}^{i-1} f_{j-1} + \sum_{j=1}^{i-1} J^j g_{j-1}. \end{aligned}$$

From the definition of Riemann's and Itô's integrals we easily see that the latter converges to the solution of problem (2.3) in the Itô form, as $N \to \infty$.

4. Generalizations

The results of Theorems 2.6 and 2.8 can be readily generalized to the case of block methods (that is, general linear methods), i.e. methods which can be cast in the form (let us consider again, for sake of simplicity, the case $m = d = 1$)

$$\mathcal{A}\begin{pmatrix} y_{rn+1} \\ \vdots \\ y_{r(n+1)} \end{pmatrix} = h\mathcal{B}\begin{pmatrix} f_{rn+1} \\ \vdots \\ f_{r(n+1)} \end{pmatrix} + \sum_{s=0}^{r} J^{n-s}\mathcal{C}_s \begin{pmatrix} g_{rn+1} \\ \vdots \\ g_{r(n+1)} \end{pmatrix} + \mathbf{w}_n,$$

$n = 0, 1, \ldots,$ where the vector $\mathbf{w}_n$ depends on already known quantities. In such a case, in fact, the corresponding matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_s$ (see (2.6) and (2.7)) are block Toeplitz matrices, for which the convergence statements can be easily generalized.

As an example, in [3] a family of Adams-type methods for Stratonivich SODEs, of the form

$$y_n - y_{n-1} = h\sum_{i=0}^{k} \beta_i f_{n-k+i} + \frac{J^n}{2}(g_n + g_{n-1}), \tag{4.1}$$

has been introduced, where the coefficients $\{\beta_i\}$ are those of the corresponding Adams-Moulton method. Such a method, whose local errors satisfy

$$E(\tau_n) = O(h^2), \qquad E(\tau_n^2) = O(h^3), \tag{4.2}$$

would be a candidate for a (strong) order 1 method. According to our analysis, this would be the case if the method would have been explicit in its stochastic component. Nevertheless, this is obviously not the case for (4.1). On the other hand, in [3] the following predictor-corrector implementation for the method was proposed:

$$\begin{aligned} y_n^* &= y_{n-1} + h\sum_{i=0}^{k-1} \beta_i^* f_{n-k+i} + J^n g_{n-1}, \\ y_n &= y_{n-1} + h\sum_{i=0}^{k-1} \beta_i f_{n-k+i} + \frac{J^n}{2} g_{n-1} + h\beta_n f_n^* + \frac{J^n}{2} g_n^*, \end{aligned}$$

where the coefficients $\{\beta_i^*\}$ are those of the corresponding Adams-Bashforth method of order k and, obviously, $f_n^* = f(y_n^*)$, $g_n^* = g(y_n^*)$.

In such a case, the arguments of Theorems 2.6 and 2.8 can be generalized, provided that some conditions hold true. First of all, the vector of the unknowns is now given by

$$\mathbf{y} = \begin{pmatrix} y_k^*, & y_k, & y_{k+1}^*, & y_{k+1}, & \ldots, & y_N^*, & y_N \end{pmatrix}^T,$$

the vectors **f** and **g** are similarly defined and, consequently,

$$A = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & -1 & 1 & & & & \\ & -1 & & 1 & & & \\ & & & -1 & 1 & & \\ & & & -1 & & 1 & \\ & & & & & & \ddots \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & & & & & & \\ \beta_k & 0 & & & & & \\ 0 & \beta^*_{k-1} & 0 & & & & \\ 0 & \beta_{k-1} & \beta_k & 0 & & & \\ 0 & \beta^*_{k-2} & 0 & \beta^*_{k-1} & 0 & & \\ 0 & \beta_{k-2} & 0 & \beta_{k-1} & \beta_k & 0 & \\ & & & & & & \ddots \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 0 & & & & & & \\ \frac{1}{2} & 0 & & & & & \\ & 1 & 0 & & & & \\ & \frac{1}{2} & \frac{1}{2} & 0 & & & \\ & & & 1 & 0 & & \\ & & & \frac{1}{2} & \frac{1}{2} & 0 & \\ & & & & & & \ddots \end{pmatrix},$$

$$J_0 = \begin{pmatrix} J^k & & & & & & \\ & J^k & & & & & \\ & & J^{k+1} & & & & \\ & & & J^{k+1} & & & \\ & & & & \ddots & & \\ & & & & & J^N & \\ & & & & & & J^N \end{pmatrix}.$$

The error equation turns out to be formally still given by (2.11) (again, let us assume for simplicity exact initial conditions), with

$$\boldsymbol{\tau} = \begin{pmatrix} \tau^*_k, & \tau_k, & \tau^*_{k+1}, & \tau_{k+1}, & \dots, & \tau^*_N, & \tau_N \end{pmatrix}^T,$$

where τ^*_n is the local error at step n for the predictor scheme, and τ_n is that of the corrector. Such errors (see [3]) satisfy (4.2) and

$$E(\tau^*_n) = O(h), \qquad E((\tau^*_n)^2) = O(h^2).$$

Moreover, it is not difficult to see, by setting as before $\tilde{\boldsymbol{\tau}} = A^{-1}\boldsymbol{\tau}$, that

$$E(\tilde{\boldsymbol{\tau}}) = O(h), \qquad E(\tilde{\boldsymbol{\tau}} \circ \tilde{\boldsymbol{\tau}}) \le O(h^2).$$

Consequently, the same arguments used in the proof of Theorems 2.6 and 2.8 can be used, to prove (strong) order 1 convergence, since now, in such a formulation, the predictor-corrector method is explicit in its stochastic component.

In light of the above results, it is possible to show that the following semi-implicit scheme has (strong) order 1:

$$\begin{aligned} y_n^* &= y_{n-1} + h \sum_{i=0}^{k-1} \beta_i^* f_{n-k+i} + J^n g_{n-1}, \\ y_n &= y_{n-1} + h \sum_{i=0}^{k} \beta_i f_{n-k+i} + \frac{J^n}{2}(g_{n-1} + g_n^*), \end{aligned}$$

since the corresponding matrices B and C_0 are lower triangular and strictly lower triangular, respectively and, consequently, the results of Theorems 2.6 and 2.8 apply. Such a formulation may be useful, when $k = 1$, for problems where the drift is stiff.

5. Conclusions

In this paper we have discussed the convergence of numerical methods for stochastic ordinary differential equations in the form (1.3). The main results can be summarized as follows. Assuming that:

- the method is 0-stable,
- the local errors of the method are random variables with $O(h^{p+1})$ mean and $O(h^{2p+1})$ variance,
- the explicitness condition (2.12) is satisfied,

then the method has global error which has (at least) $O(h^p)$ mean and $O(h^{2p})$ variance. In such a case, we say that the method has strong global order p.

Concerning the explicitness condition (2.12), it is a sufficient condition for convergence, even though its fulfilment allows us to avoid the drawbacks that, for example, may arise when applying the scheme to the scalar equation

$$dy = \lambda y\, dt + \mu y\, dW,$$

where λ and μ are complex parameters. In fact, in such a case, at the nth step one should solve an equation in the form

$$(\alpha_k - h\lambda\beta_k - J^n \mu \gamma_{k0}) y_n = \phi_n,$$

where ϕ_n is a known quantity. The solution may, in such a case, be unbounded, when the real part of μ is nonzero, even though $Re(\lambda) < 0$.

Equivalently, we may reformulate the explicitness condition (2.12) by saying that, in formula (1.3), each Wiener increment J_j^{n-s} must multiply quantities which are independent of it (i.e., terms involving y_r, $r < n - s$). By using such interpretation, the above explicitness condition is easily generalized to the case where the formula contains multiple stochastic integrals.

Finally, we have seen that the obtained results can be extended to handle the case of block methods, thus providing a proof of convergence for the predictor-corrector formulae introduced in [3].

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