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ON THE CHARACTERIZATION OF STIFFNESS FOR ODES

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Abstract: It is well known that in the last forty years one of most studied phenomenon in Numerical Analysis has been the problem of *stiffness* for ODEs. A precise mathematical characterization of it, however, has not yet been given. There are at the moment many definitions and, far from tending to unifications, the newer of them tend to diverge. In this paper, by using two measures of the conditioning of the continuous problem, a definition which seems to cover the most important facets of the phenomenon will be given.

Keywords. Ordinary differential equations, initial value problems, boundary value problems, stiffness, conditioning.

AMS (MOS) subject classification: 65L05.

1. Introduction

As pointed out by Dahlquist [3], the binary alternative "well posed"-"ill posed" problems, used by pure mathematicians to classify differential problems, is insufficient for the purposes of Numerical Analysis. As matter of fact, such discipline needs a more refined classifications. Even very stable, and therefore well conditioned, problems may originate serious difficulties when attempting to approximate them by discrete models. Such stable but numerically difficult problems are usually called *stiff*, even if the term is often also used to denote a wider class of problems, as shown later.

Such problems arise in an countless amount of applications and their solutions have challenged professional Numerical Analysts as well as Applied Mathematicians, Theoretical Physicists, Electrical Engineers, etc. Their numerical treatment has been studied during the last forty years: a large number of symposiums, mini symposiums, etc., have been dedicated to them; a complete list of papers devoted to their study would certainly fill a book of the dimension of a dictionary.

One would expect that, after such amount of activity, what a *stiff problem* is should be out of question. On the contrary, not only it is still an open problem, but it seems that the opinions of some experts tend to diverge on the characterization of the phenomenon. In the recent edition of his book (1991), Lambert, one of the leading experts, gives five different definitions of stiffness, each of them capturing some important features of it. A recent book [4], the most comprehensive on the subject, gives a negative definition saying that *"stiff equations are problems for which explicit (numerical) methods don't work"*. This is an empirical definition, which certainly is not mathematically acceptable. In a recent paper, Higham and Trefethen propose to rely the stiffness on the non-normality of the coefficient matrix. Even if the raised question is numerically important (see Section 5), such definition cannot be accepted as well. In fact,

1. in the literature there are many examples of stiff problems having symmetric coefficient matrices;

2. even a single scalar equation may be stiff.

In attempting to formulate a unifying definition, let us go back briefly to the historical development of the concept. From the early time of such story, it was clear that stiffness was related to different time scales present in the problems. The term itself derives from such peculiarity. In fact it seems to descend from mechanical models of systems of weights connected with springs having very different rigidity constants (stiff constants). The solutions of the corresponding equations are characterized by fast modes, corresponding to the effect of the stronger springs, and slow modes, corresponding to the effect of the soft ones.

This feature is clearly stressed in the following description of stiff problems given by Liniger in 1972 (translated from French [8]):

"... they represent coupled physical systems having components varying with very different time scales: that is they are systems having some components varying much more rapidly than the others ... ".

The same picture is extendible to models from chemical reactions, where the different time scales come from the different reaction rates of the components, and to many other similar models in many branches of the applications.

The presence of at least two time scales is then the main characteristic of stiffness. This is even true for a single equation such as

$$\begin{aligned} y' &= \lambda y, \quad t \in [t_0, T], \qquad \lambda < 0, \\ y(t_0) &= y_0. \end{aligned} \tag{1}$$

Here the two time scales are $T - t_0$ and $|\lambda|^{-1}$. It is quite obvious that if $T - t_0 \approx |\lambda|^{-1}$, any numerical method would be able to do a good job in few steps. The problem arises when $T - t_0 \gg |\lambda|^{-1}$. In this case an explicit method would require the use of a large amount of steps to give a solution whose qualitative behavior mimics that of the continuous solution $e^{\lambda(t-t_0)}y_0$.

For many years the ratio between the extreme time constants of the problem was considered a measure of stiffness. In the above example such ratio is $(T - t_0)|\lambda|$.

So far, so clear. The extension to more complicated models such as nonautonomous problems, singular perturbation problems, nonlinear problems, semi discretization of PDEs by means of the method of lines, etc., have enormously varied the phenomenon and the original definition has become insufficient, giving rise to the flourishing of the new ones.

In this paper, by imbedding the analysis of stiffness in the more general setting of *conditioning analysis*, we give a definition which encompasses all the aspects of the phenomenon. To avoid unnecessary complications, we shall restrict our initial analysis to linear problems. The extension to nonlinear problems will be briefly sketched in Section 5.

2. Classification of problems

Even though the notion of stiffness originated in the ambit of dissipative initial value problems, there are many instances in the literature where such notion has been extended to wider ambits such as boundary value problems, singular perturbation problems, etc. Therefore, we shall consider the more general setting of boundary value problems, such as,

$$
\begin{aligned}
y' &= L(t)y, \qquad y \in \mathbb{R}^s, \\
B_0 y(t_0) + B_1 y(T) &= \eta,
\end{aligned}
\tag{2}
$$

where B_0 and B_1 are $s \times s$ matrices. In the case of initial value problems B_0 and B_1 will be the identity matrix and the null matrix, respectively. The solution of (2) is

$$y(t) = \Phi(t, t_0) Q^{-1} \eta,$$

where $\Phi(t, t_0)$ is the fundamental matrix and

$$Q = B_0 + B_1 \Phi(T, t_0).$$

A perturbation $\delta\eta$ of the boundary condition will cause a perturbation δy to the solution which is bounded by (see, for example, [1])

$$\|\delta y(t)\| \leq \|\Phi(t, t_0) Q^{-1}\| \|\delta\eta\|.$$

Here $\|\cdot\|$ is any norm in $\mathbb{R}^s$. We define

$$\phi(t) = \|\Phi(t, t_0) Q^{-1}\| \tag{3}$$

and

$$
\begin{aligned}
\kappa &= \max_{t \in [t_0, T]} \phi(t), \\
\gamma &= \frac{1}{T - t_0} \int_{t_0}^{T} \phi(t) dt.
\end{aligned}
\tag{4}
$$

Of course, κ and γ are, respectively, the uniform norm and the L_1 norm in the space $C[t_0, T]$.

Consider the following three cases, which will represent three typical classes of problems:

1. both κ and γ are of moderate size and $\kappa \approx \gamma$,
2. γ is of moderate size and $\kappa \gg \gamma$,
3. both parameters are large.

We shall classify the problems according to their belonging to one of the previous classes. In the first case, the problems are *well conditioned* in both norms, while in the second case they are well conditioned at least in the L_1 norm. Finally, in the third case the problems are *ill conditioned* in both norms.

Definition. *Stiff problems* are those having a large ratio

$$\sigma = \frac{\kappa}{\gamma}$$

(at least for one of the modes of the problem).

The quantity σ will be called *stiffness ratio.* The part of the definition enclosed in parentheses will be clarified later. Let us first comment the cases where σ is large for the entire problem. It is obvious that the scalar problem (1) may belong either to the first class or to the second one, according to the value of $|\lambda|(T-t_0)$. In fact, it is easily checked that for such problem $\kappa = 1$ and

$$\gamma = \frac{1-e^{\lambda(T-t_0)}}{|\lambda|(T-t_0)}, \qquad \sigma \approx |\lambda|(T-t_0).$$

More generally, for λ complex with negative real part, one obtains:

$$\gamma = \frac{1-e^{Re(\lambda)(T-t_0)}}{|Re(\lambda)|(T-t_0)}, \qquad \sigma \approx |Re(\lambda)|(T-t_0).$$

The given definition then agrees with the analysis made in the Section 1. It is also worth to note that our definition does not exclude the possibility that $\lambda > 0$ (or $Re(\lambda) > 0$). Some people would not agree to consider the problems with positive λ as stiff. However, recently some authors do (see e.g. [9, p. 402].) In such case one easily obtains,

$$\kappa = e^{Re(\lambda)(T-t_0)}, \qquad \gamma = \frac{e^{Re(\lambda)(T-t_0)}-1}{Re(\lambda)(T-t_0)}, \qquad \sigma \approx Re(\lambda)(T-t_0).$$

Then, the stiffness ratio continues to depend on the quantity $|Re(\lambda)|(T-t_0)$. The only difference with the previous case is that now the problem is also *ill conditioned* (since both κ and γ are large, when T is not very small).

Note that the given definition is independent of a time rescaling. That is, a change of variable $t = \alpha\tau$ leaves κ, γ and, therefore, σ unchanged.

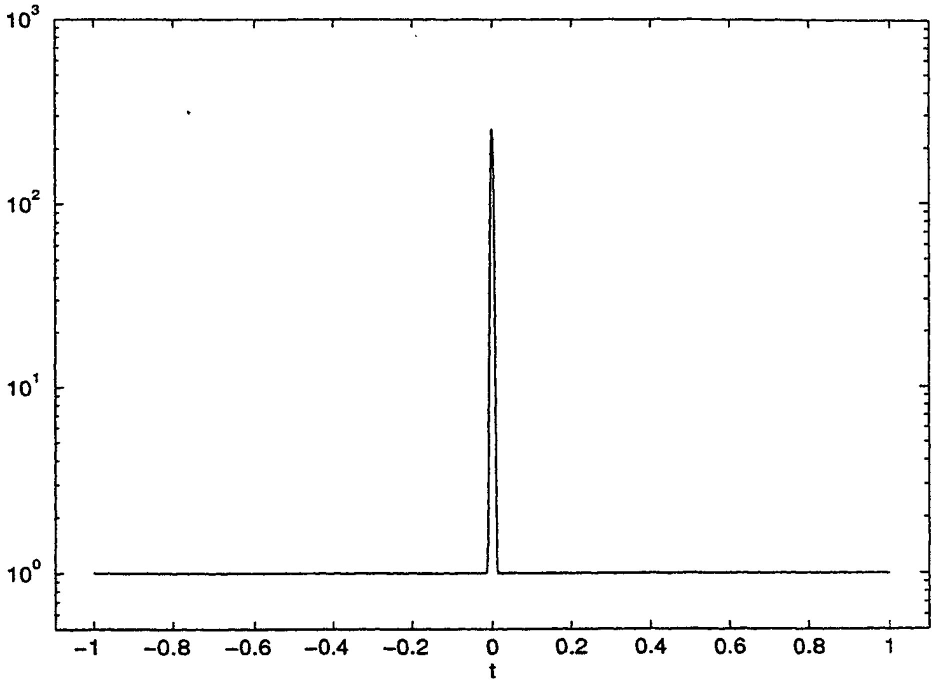

Figure 1: Function $\phi(t)$ for a stiff problem.

The above example permits to give a simple geometric description of our definition. In fact, it only states that the average value of $\phi(t)$ is, for stiff problems, much smaller than its maximum value (see Figure 1).

How many of the problems quoted as stiff in the literature are embraced by our definition? It is quite obvious that all problems for which $\|\Phi(t,t_0)Q^{-1}\|$ assumes huge values in short subintervals have γ much smaller than κ, and then they are stiff. This is the case, for example, of singular perturbation problems. Let us see this with an example.

Consider the following singular perturbation problem:

$$\begin{aligned} \varepsilon y'' + ty' &= 0, \\ y(-1) = 0, \qquad & y(1) = 1. \end{aligned} \tag{5}$$

It can be stated as a first order problem by posing:

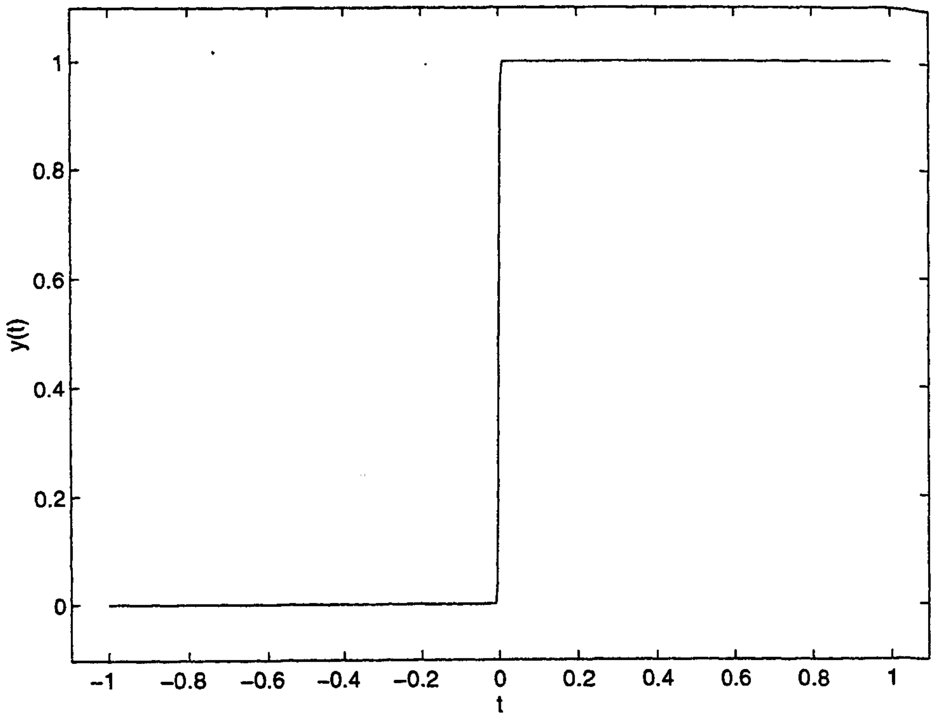

Figure 2: Solution of problem (5), $\varepsilon = 1e-5$.

$$y' = z, \qquad z' = -\frac{t}{\varepsilon} z.$$

One verifies that

$$\Phi(t,-1) = \begin{pmatrix} 1 & \phi_1(t,-1) \\ 0 & \phi_2(t) \end{pmatrix},$$

where

$$\phi_2(t) = e^{-\frac{t^2-1}{2\varepsilon}}, \qquad \phi_1(t,t_0) = \int_{t_0}^{t} \phi_2(s)ds.$$

Moreover, the matrix Q is given by:

$$Q = \begin{pmatrix} 1 & 0 \\ 1 & \phi_1(1,-1) \end{pmatrix},$$

so that one obtains:

$$\Phi(t,-1)Q^{-1} = \frac{1}{\phi_1(1,-1)} \begin{pmatrix} \phi_1(1,t) & \phi_1(t,-1) \\ -\phi_2(t) & \phi_2(t) \end{pmatrix}.$$

Finally, by using the infinity norm, and supposing $\varepsilon < 0.1$, one has,

$$\kappa(\varepsilon) = \frac{1}{\int_0^1 e^{-\frac{s^2}{2\varepsilon}} ds}, \qquad 1 \le \gamma(\varepsilon) < \frac{3}{2}.$$

In the following Table 1 we report the values of $\kappa(\varepsilon)$ and of the stiffness ratio $\sigma(\varepsilon)$ for different values of ε, from which one deduces that the problem becomes *stiff* as ε tends to zero. In fact, the problem falls in the second class of our classification.

Table 1.

ε	10^{-5}	10^{-10}	10^{-15}	10^{-20}
$\kappa(\varepsilon)$	2.52e2	7.98e4	2.52e7	7.98e9
$\sigma(\varepsilon)$	>1.6e2	>5.3e4	>1.6e7	>5.3e9

As an example, the function $\phi(t)$ plotted in Figure 1 is that of problem (5), for $\varepsilon = 1e-5$. The maximum value $\kappa(\varepsilon)$ is reached at $t = 0$. The function is almost everywhere equal to 1, except for a small neighborhood of $t = 0$. This facts reflects in the solution of the problem, which has a layer at $t = 0$ (see Figure 2).

Concerning boundary value problems, it seems to us that our definition embraces all the possibilities, since in this case the interval of integration is fixed in advance. More subtle is the case of initial value problems.

3. The case of Initial Value Problems

Consider now the initial value problem. For sake of clarity we assume the matrix L in (2) to be diagonal and independent of time. Suppose that

$$\lambda_{max} = \lambda_1 < \lambda_2 < \ldots < \lambda_s = \lambda_{min}$$

are its eigenvalues, which we suppose to be real, for simplicity. Moreover, as almost usual in the theory of stiff equations, they are supposed to be negative. Consider now the single uncoupled equations and let $\gamma_1, \gamma_2, \ldots, \gamma_s$ be the corresponding values defined by (4). It is easily checked that $\kappa_1 = \ldots = \kappa_s = 1$. According to the previous definition, some scalar equations will be stiff, some others not. Suppose, for example, that the equation corresponding to λ_1 is stiff, while the one corresponding to λ_s is not stiff.

By looking at the values κ and γ for the whole problem, the information that a single equation is stiff is lost, since in this case one obtains $\kappa = 1$ and $\gamma \approx \gamma_s$. In conclusion, considered as a whole the problem lies in the first class of our classification, while considered as a set of decoupled problems, it contains problems in the second class.

According to the previous definition, problems either belonging to the second class, or with at least one of its decoupled problems in the second class, are stiff.

Observe that our definition embraces the usual one, for the present class of problems. In fact one usually integrates until a complete information about the behavior of the solution has been reached, that is until the solution has reached the steady state. This requires to assume $T - t_0 \approx |\lambda_s|^{-1}$. Consequently $\gamma_1 \approx |\lambda_s/\lambda_1|$ and then

$$\sigma \approx \left|\frac{\lambda_{max}}{\lambda_{min}}\right|,$$

thus giving the usual stiffness ratio which, for a long time, has been used to define stiffness.

When L is not diagonal, but diagonalizable by means of a constant matrix M, then both κ and γ may grow by a factor $k(M)$, the condition number of the matrix M. As consequence, a problem which is stiff because it has a *stiff mode*, may also become ill conditioned, since κ and γ may become large (note that σ may not vary sensibly).

The case where M is not constant is more interesting since it permits to show that our definition is able to give the right classification, not well evidenced by the usual definitions. For example, let us consider the following example taken from [9, p. 409]:

$$y' = A_\nu(t)y + f(t), \quad t \in [0, 10\pi], \qquad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{6}$$

where

$$f(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} - A_\nu(t) \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix},$$

ν is a nonnegative parameter, and

$$A_\nu(t) = M(t) \begin{pmatrix} -1001 & 0 \\ 0 & -1 \end{pmatrix} M^T(t), \qquad M(t) = \begin{pmatrix} \cos(\nu t) & \sin(\nu t) \\ -\sin(\nu t) & \cos(\nu t) \end{pmatrix}.$$

The solution of the problem is given by $y(t) = (\cos(t)\ \sin(t))^T$, independently of the value of the parameter ν. Despite the fact that the solution is very smooth, the problem is stiff. In fact, the eigenvalues of the matrix $A_\nu(t)$ are $\lambda_1 = -1$ and $\lambda_2 = -1001$ for all ν and t. However, the problem becomes more difficult as the parameter ν grows, but this fact can not be recovered by the usual analysis based on the eigenvalues of the "frozen" Jacobian. Nevertheless, by using the transformation

$$z(t) = M^T(t)y(t),$$

one obtains the equation

$$z' = \hat{A}z + M^T(t)f(t),$$

where

$$\hat{A} = \begin{pmatrix} -1001 & -\nu \\ \nu & -1 \end{pmatrix}.$$

The eigenvalues of the matrix $\hat{A}$ are

$$\hat{\lambda}_{1/2} = -501 \pm \sqrt{500^2 - \nu^2}.$$

As consequence, one has that

1. when $\nu = 0$ we get the results obtained by the usual analysis based on the eigenvalues of the "frozen" Jacobian;

2. for $\nu > 0$, the problem becomes increasingly more stiff for the given interval $[0;10\pi]$.

By looking at the two parameters κ and σ for this problem, one obtains the results in Table 2, where the $\|\cdot\|_2$ norm is used.

Table 2.

ν	0	10	100	500	1000	2000
κ	1	1	1	1	1	1
σ	3.1e1	3.4e1	3.4e2	6.3e3	1.1e4	1.3e4

As one can see, the problem fits in the second class of our classification, since it is not ill conditioned and the stiffness ratio is much larger than 1. Moreover, this ratio increases as ν increases, thus reflecting the increasing difficulty in the integration of the continuous problem.

The above example permits to outline another important question. In fact, problem (6) is a stiff one, even if its solution is very smooth. This is due to the fact that the initial condition $(1\ 0)^T$ does not activate the fast modes of the equation.

Such situation is also evidenced in the following example, taken from ([7, p. 217]),

$$y' = \begin{pmatrix} -2 & 1 \\ -1.999 & 0.999 \end{pmatrix} y + \begin{pmatrix} 2\sin(t) \\ 0.999(\sin(t) - \cos(t)) \end{pmatrix},$$

whose solution is given by

$$y(t) = \alpha e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{-0.001t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

To the initial condition $(2\ 3)^T$ it corresponds $\beta = 0$ so that the slowest mode is not activated, and in a short interval of time the solution will decay to $(\sin(t)\ \ \cos(t))^T$. However, a small perturbation of the initial condition will give $\beta \neq 0$ and, then, in order to get a complete information, a much longer interval of time would be needed.

One could then define stiffness by considering only the activated modes. We prefer, however, to maintain the definition given above, since it is independent of the initial conditions. In fact, small numerical errors in the initial condition may activate all the modes. This means that it is not correct to look at the solution to decide whether a problem is stiff or not. In fact, neglecting the non activated modes may be potentially more dangerous than one would infer from the previous example. This is stressed in the following problem [9, p. 388],

$$y' = \begin{pmatrix} 0 & 1 \\ 100 & 0 \end{pmatrix} y, \quad 0 \le t \le T, \qquad y(0) = \begin{pmatrix} 1 \\ -10 \end{pmatrix},$$

whose solution is given by

$$y(t) = e^{-10t} \begin{pmatrix} 1 \\ -10 \end{pmatrix},$$

so that the steady state solution is quickly reached and the problem is stiff if T is large enough. However, the general solution of the problem is

$$\alpha_1 e^{-10t} \begin{pmatrix} 1 \\ -10 \end{pmatrix} + \alpha_2 e^{10t} \begin{pmatrix} 1 \\ 10 \end{pmatrix},$$

so that a slightly different initial condition may activate the unstable mode. Obviously, finite precision computation *always* activates the increasing mode, so that the problem becomes unstable. This is promptly recovered by our definition, since one obtains:

$$\kappa = e^{10T}, \qquad \gamma \approx \frac{\kappa}{10T}, \qquad \sigma \approx 10T,$$

so that the problem fits in the third class of our classification (ill conditioned problems) and it is also stiff for large values of T.

Anyway, a problem is not only defined by the equation but also by the interval of integration and by the initial (or boundary) conditions. Once the problem is fixed, our definition defines uniquely if it is stiff or not.

4. Stiffness and stability

In literature it is very frequent the statement *"stiff problems are very stable."* This is certainly true for dissipative linear autonomous problems for which stiffness was first studied, since for such problems the steady state solution is asymptotically stable. Our definition, as far as such problems are concerned, agrees with the mentioned statement. However, the statement is no longer valid for boundary value problems or for nonautonomous initial value problems. In such cases the concept of conditioning is more appropriate, with respect to that of stability. Our definition of stiffness applies to ill conditioned problems as well. In fact a situation where both κ and γ are large and moreover $\sigma \gg 1$ is certainly possible. Such problems are both ill conditioned and stiff. Of course, they must be handled with care, since they are very sensitive to perturbations.

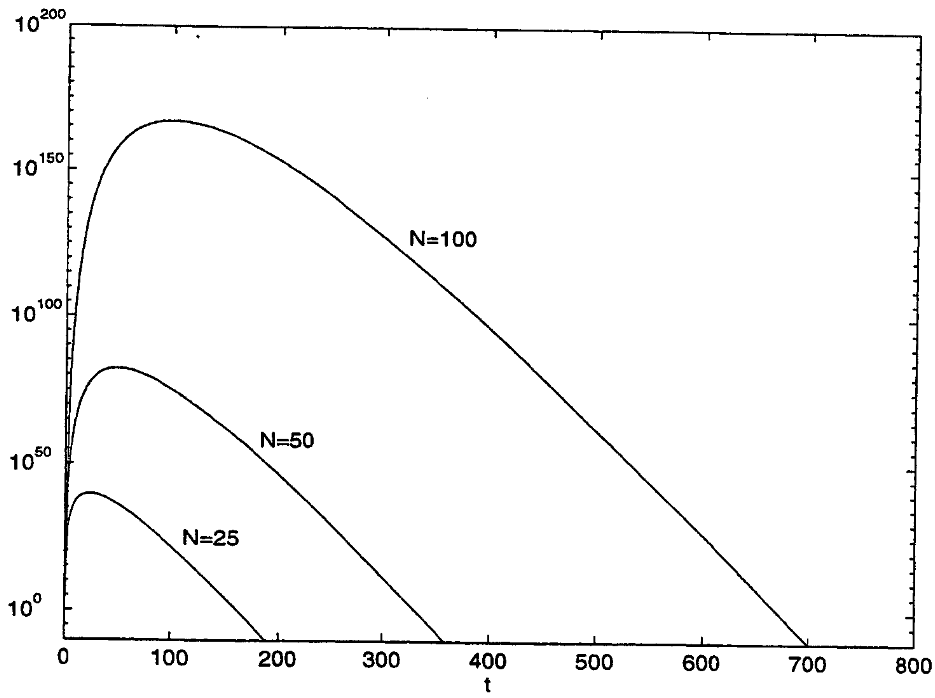

Figure 3: Function $\phi(t)$ for problem (7), $\eta = 50$ and $N = 25, 50, 100$.

In order to illustrate the question, we shall provide with two examples: the former chosen among initial value problems and the latter among boundary value problems.

Example 1. Consider the following problem,

$$y' = \begin{pmatrix} -1 & \eta & & \\ & \ddots & \ddots & \\ & & -1 & \eta \\ & & & -1 \end{pmatrix}_{N\times N} y, \qquad 0 \le t \le 1000, \tag{7}$$

where $\eta > 1$ is a parameter, and the initial condition $y(0)$ is given. It may seem that the length of the interval of integration is artfully large: observe, however, that as N increases the solutions need increasingly larger intervals to approach the steady state solution. This is shown in Figure 3 where we

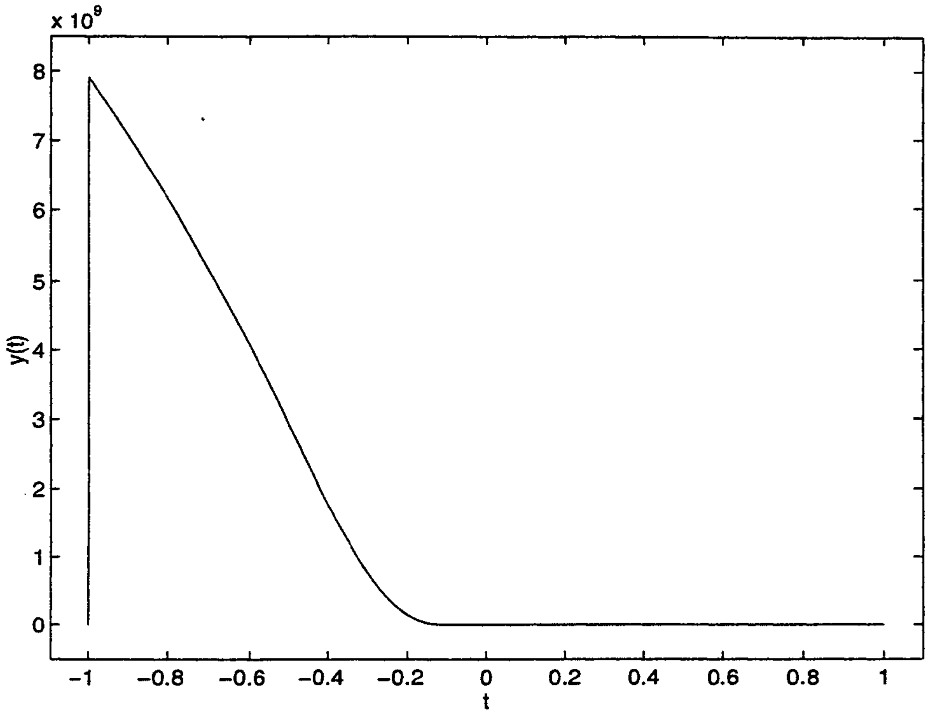

Figure 4: Solution of problem (8).

report the function $\phi(t)$ (see (3)) for problem (7), $\eta = 50$ and $N = 25, 50, 100$. In Table 3 we report the parameters κ, γ and the stiffness ratio σ for $N = 10$, for different values of the parameter η.

Table 3.

η	1	10	50	100
κ	1	1.5e8	2.6e14	1.3e17
γ	1.0e-2	1.1e6	2.0e12	1.0e15
σ	9.5e1	1.3e2	1.3e2	1.3e2

As one can see, the stiffness ratio of the problem remains almost constant and the problem can be classified as moderately stiff. However, it migrates from well-conditioning to ill-conditioning as η increases. It is evident that ill-conditioning is not due to the spectrum of the matrix, but to its non-normality (see also [5]).

Example 2. Consider the following problem [6],

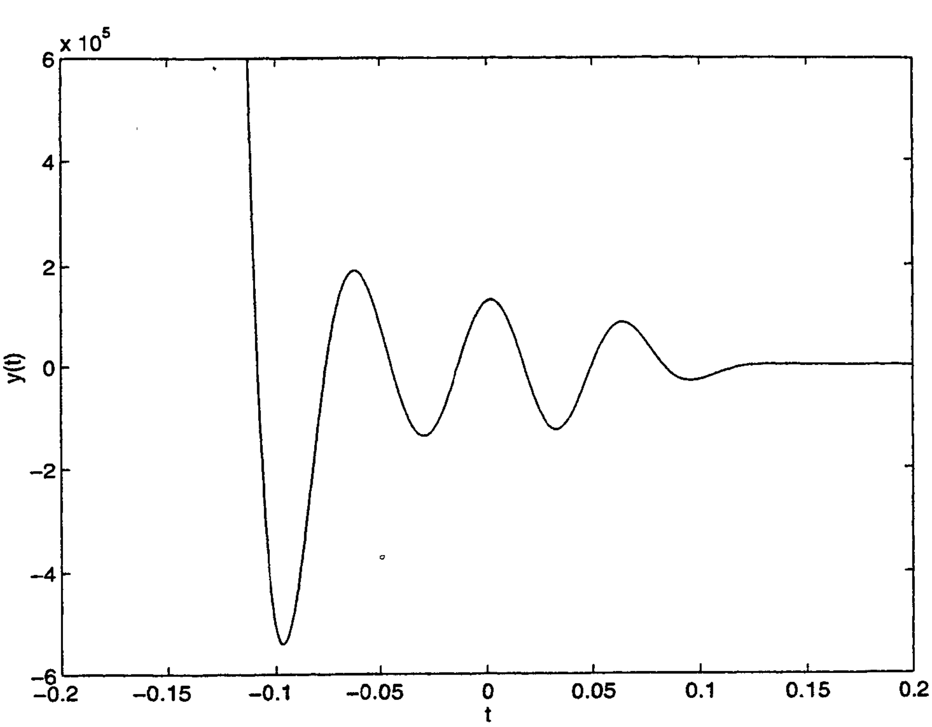

Figure 5: Solution of problem (8), zoom at $t = 0$.

$$\varepsilon y'' + t^2 y' + y = 0, \qquad y(-1) = 1, \quad y(1) = 2, \tag{8}$$

where the parameter $\varepsilon = 1e - 4$. This is a very hard to solve singular perturbation problem. In fact, its solution has a layer at $t = -1$, so that for $t = -1 + O(\varepsilon)$ it reaches a value $\approx 7.9e9$ (see Figure 4). Moreover, the solution heavily oscillates near $t = 0$ (see Figure 5). The (estimated) values of κ, γ and σ are:

$$\kappa \approx 4e13, \qquad \gamma \approx 5e9, \qquad \sigma \approx 8e3.$$

One then concludes that the problem is both very ill conditioned and stiff.

5. Nonlinear problems

The currently used definitions of stiffness for nonlinear problems suffer lim-

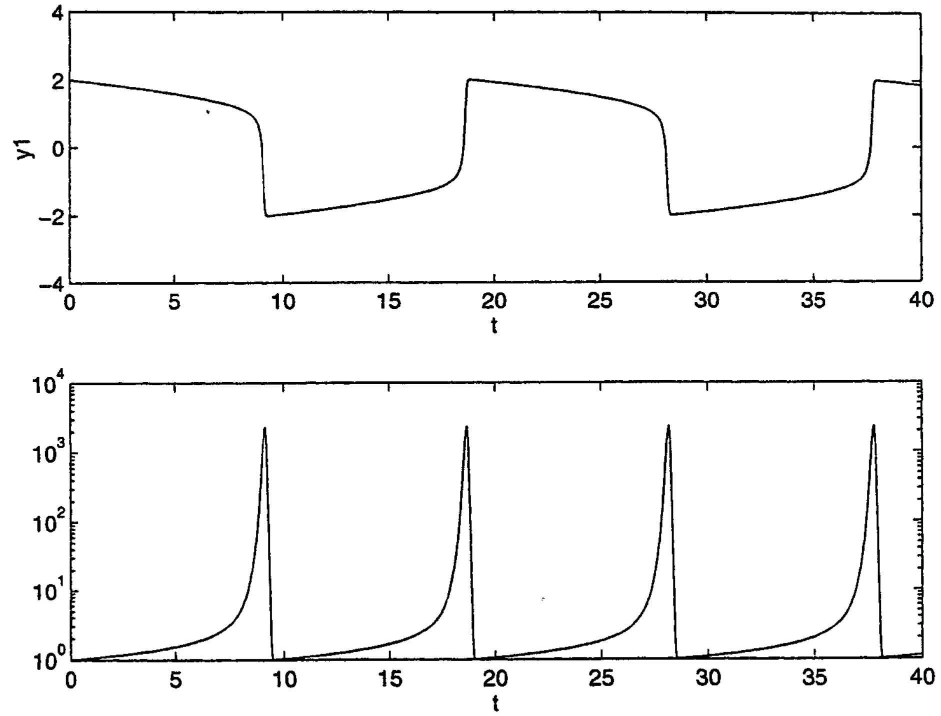

Figure 6: First component of the solution and function $\phi(t)$, for problem (9).

itations similar to those mentioned when discussing about problem (6). In that occasion it was in fact shown that the eigenvalues of the matrix were not able to describe the increasing difficulty of the problem in correspondence of the growth of the parameter ν. Similarly, for nonlinear problems the eigenvalues of the Jacobian matrix may not be sufficient to precisely evidence (and measure) stiffness. Moreover, even problems having asymptotically stable solution sets, may show up Jacobians having eigenvalues with positive real parts. A famous example is given by the Van der Pol equations,

$$\begin{array}{rcl} y_1' &=& y_2, \\ y_2' &=& -y_1+\mu y_2(1-y_1^2), \end{array} \qquad \begin{array}{l} y_1(0)=2, \\ y_2(0)=0, \end{array} \qquad 0\le t\le T, \tag{9}$$

where μ is a positive parameter. The initial condition lies near an asymptotically stable limit cycle. It is well known that this problem becomes numerically difficult as μ increases. Usually, it is classified as a stiff one, even for

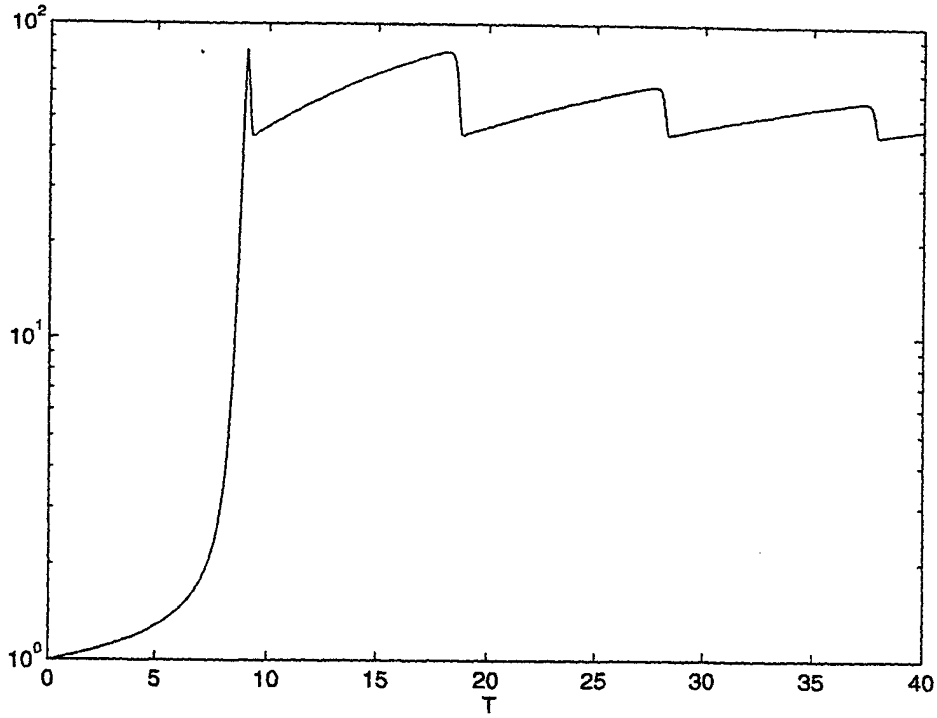

Figure 7: Stiffness ratio for problem (9), with respect to the length T of the interval.

moderate values of μ, although the eigenvalues of the Jacobian (on the limit circle) have positive real part for some time.

The generalization of the definition given in Section 2 to the non linear case is naturally made thru the linearization of the problem. That is, if

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{10}$$

is the given nonlinear problem, one considers the variational problem

$$z' = f_y(t, y(t))z, \qquad z(t_0) \text{ given}, \tag{11}$$

where f_y denotes the Jacobian matrix of f. Problem (11) is linear, so that the function $\phi(t)$ and the parameters κ, γ and σ can be defined as in Section 2.

Usually a non linear problem is solved by means of a sequence of linear ones (for example, those defined by the Newton method), whose solutions $y^{(j)}(t)$ are better approximations, as j increases, of the solution of problem (10). One then defines a sequence of values $\kappa^{(j)}$, $\gamma^{(j)}$ and $\sigma^{(j)}$, converging to κ, γ and σ, respectively, which describe the conditioning and the stiffness of the intermediate linear problems. Such intermediate quantities turn out to be very useful to design an efficient stepsize variation strategy (see next section).

As an example, in Figure 6 we report the (estimated) function $\phi(t)$ for the Van der Pol equation, with $\mu = 10$, along with the first component of the solution, y_1. As one can see, the "spikes" in the function $\phi(t)$ are in correspondence of the rapid variations of $y_1(t)$. In Figure 7 the stiffness ratio $\sigma(T)$ is plotted as function of the length of the interval of integration. It is clear that the problem is stiff for T greater than half period.

Note that the function $\phi(t)$ is able to provide, as by-product, also the information about the periodicity of the solution.

6. Concluding remarks

We have introduced a mathematically acceptable definition of stiffness, which is able to cover all the known aspects of this phenomenon, including its extension to the wider class of continuous boundary value problems.

Strictly related to stiffness, is the problem of designing an efficient strategy for the variation of stepsizes. Actually, such questions may be considered as the two faces of the same medal, since, without the use of variable stepsizes, stiff problems could not be solved accurately, even with methods giving bounded solutions (stable methods). On the contrary, it is well known that non stiff problems are well integrated by using constant stepsizes.

The use of the parameters κ and γ permits to design an innovative strategy for the stepsize variation [2]. The usual approaches to the stepsize variation are based on the local estimates of the errors. Our approach, for the first time, uses explicitly the definition of stiffness. The details of such procedure will be discussed in a forthcoming paper. Its effectiveness may, however, be inferred by the figures and the tables of this paper, obtained by using a package based on it.

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