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ADAMS-TYPE METHODS FOR THE NUMERICAL SOLUTION OF STOCHASTIC ORDINARY DIFFERENTIAL EQUATIONS *†

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Abstract.

The modelling of many real life phenomena for which either the parameter estimation is difficult, or which are subject to random noisy perturbations, is often carried out by using stochastic ordinary differential equations (SODEs). For this reason, in recent years much attention has been devoted to deriving numerical methods for approximating their solution. In particular, in this paper we consider the use of linear multistep formulae (LMF). Strong order convergence conditions up to order 1 are stated, for both commutative and non-commutative problems. The case of additive noise is further investigated, in order to obtain order improvements. The implementation of the methods is also considered, leading to a predictor-corrector approach. Some numerical tests on problems taken from the literature are also included.

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1 Introduction.

Many real world phenomena are (or appear to be) liable to random noisy perturbation. This is the case, for example, in investment finance, turbulent diffusion, chemical kinetics, VLSI circuit design, etc. (see, e.g., [8, 9, 11]). The mathematical modelling of such phenomena is, therefore, not well matched by deterministic equations, and stochastic equations are preferable instead. When the evolution of such phenomena has to be studied, then one often must handle a system of stochastic ordinary differential equations (SODEs). As in the case of deterministic ODEs, only a few, very simple SODEs can be solved analytically. As a consequence, there is the need for numerical methods for approximating their solutions.

However, this is a relatively new field of investigation and, for this reason, there are not yet general purpose codes for handling SODEs.

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In [1–3, 5–7] the numerical solution of SODEs by means of suitably modified Runge–Kutta methods has been considered. We are here concerned with the use of linear multistep formulae (LMF) for approximating a SODE in the form

$$dy(t) = f(y(t))dt + \sum_{j=1}^{d} g_j(y(t))dW_j(t), \quad t \in [0,T], \qquad (1.1)$$
$$y(0) = y_0 \in \mathbb{R}^m,$$

which, without loss of generality, we have assumed to be autonomous, in order to simplify the notation.

In the formulation (1.1), the $W_j(t)$, $j = 1,\ldots,d$, are independent Wiener processes, modelling independent Brownian motions, which satisfy the initial condition $W_j(0) = 0$ with probability 1 [9]. The deterministic term $f(y)$ is sometimes called the *drift*. The Wiener processes are known to be Gaussian processes satisfying

$$E(W_j(t)) = 0, \quad E(W_j(t)W_j(s)) = \min\{t,s\},$$

whose increments

$$\int_t^s dW_j$$

are, if not overlapping, independent and $N(0, |t-s|)$ distributed.

The solution of (1.1) can be formally written as

$$y(t) = y_0 + \int_0^t f(y(s))ds + \sum_{j=1}^{d} \int_0^t g_j(y(s))dW_j(s),$$

where the integrals

$$\int_0^t g_j(y(s))dW_j(s), \quad j = 1,\ldots,d,$$

are *stochastic integrals* (see, for example, [9]). They are defined as the limit (in the mean square sense), as $n \to \infty$, of the approximating sums

$$\sum_{i=1}^{n} g_j(y(\xi_i))(W_j(t_i) - W_j(t_{i-1})),$$

where $\xi_i = \theta t_i + (1-\theta)t_{i-1}$, for a fixed $\theta \in [0,1]$ and, for simplicity, $t_i = it/n$, $i = 0,\ldots,n$. For stochastic integrals, different choices of θ usually result in different values for the integral. The most common choices for the parameter θ are

- $\theta = 0$, which gives an Itô integral, and
- $\theta = \frac{1}{2}$, which gives a Stratonovich integral.

The Itô formulation has the advantage of preserving the *Martingale* property of the Wiener process, so that

$$E\Big(\int_a^b q(t)dW_j(t)\Big) = 0, \qquad E\Big(\Big\|\int_a^b q(t)dW_j(t)\Big\|^2\Big) = \int_a^b E\left(\|q(t)\|^2\right) dt.$$

On the other hand, the Stratonovich integrals formally satisfy the usual rules of calculus. For example,

$$\begin{aligned}\int_a^b W_j(t)dW_j(t) &= \frac{1}{2}\left(W_j^2(b) - W_j^2(a)\right) + \left(\theta - \frac{1}{2}\right)(b-a)\\ &\equiv \frac{1}{2}\left(W_j^2(b) - W_j^2(a)\right),\end{aligned}$$

because $\theta = \frac{1}{2}$ for Stratonovich integrals. The latter are usually denoted by

$$\int_a^b q(t) \circ dW_j(t),$$

whereas the usual notation is referred to as Itô integrals.

As a consequence, we may reformulate equation (1.1) in its equivalent Stratonovich form. Considering that in general (see, for example, [9]) one has

$$\int_0^t q(W_j) \circ dW_j = \int_0^t q(W_j)dW_j + \frac{1}{2}\int_0^t \frac{d}{dW_j} q(W_j(s))ds,$$

it follows that the Stratonovich formulation of (1.1) is given by

$$dy(t) = g_0(y(t))dt + \sum_{j=1}^{d} g_j(y(t)) \circ dW_j(t), \tag{1.2}$$

where

$$g_0(y(t)) = f(y(t)) - \frac{1}{2}\sum_{j=1}^{d} g_j'(y(t))g_j(y(t)),$$

with g_j' denoting the Jacobian matrix of g_j.

In the following, we shall always use Stratonovich calculus. For this reason, we shall assume that the problems have been recast in the corresponding Stratonovich formulations.

We are now concerned with the numerical approximation of (1.2), by means of a numerical method in the form

$$\begin{aligned}&\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=0}^{d}\sum_{i=0}^{k} \beta_{in}^j g_{j,n+i}, \qquad n = 0,1,\ldots,\\ &y_0,\ldots,y_{k-1} \ \text{fixed},\end{aligned} \tag{1.3}$$

where, as usual, if $y(t)$ is the continuous solution to (1.2), y_{n+i} is the numerical approximation to $y(t_{n+i})$ and $g_{j,n+i} = g_j(y_{n+i})$. We shall only consider here the case of a uniform partition of the integration interval, $t_n = nh$, $n = 0, \ldots, N$, $h = T/N$. The coefficients $\{\alpha_i\}$ are assumed to be independent of n (moreover, we shall fix the usual scaling $\alpha_k = 1$), whereas the remaining coefficients $\{\beta_{in}^j\}$ are in general stochastic variables.

In the next section, we shall obtain conditions for the coefficients, in order to meet suitable accuracy requirements.

2 Strong order conditions.

When speaking about the accuracy of numerical methods for SODEs, we distinguish between two kind of convergence:

Weak convergence: this case concerns the situations where one is interested in the moments. One then requires that there exist constants $C, \delta, p > 0$ such that

$$\max_n \|E\left(q(y_n) - q(y(t_n))\right)\| \le Ch^p,$$

for all stepsizes $h < \delta$ and polynomials q. In such a case, it is said that the method has *weak order* p.

Strong convergence: in this case, one is interested in the mean square convergence of the trajectories, which means that

$$\max_n E\left(\|y_n - y(t_n)\|\right) \le Ch^p,$$

for all stepsizes $h < \delta$, for methods having *strong order* p.

The second requirement is more critical, and will be our matter of investigation, for methods in the form (1.3).

In the case of deterministic ODEs there is a well established theory which relates the local order of a numerical method to the global order. Essentially if the local order behaves as $O(h^{p+1})$ then the global order is $O(h^p)$. Unfortunately, the situation in the stochastic setting is considerably more complicated and has only been treated in a sufficiently general way in [4], where the authors have extended the use of B-series to study the order conditions of stochastic Runge–Kutta methods. Without describing too many of the details we quote the main result of that paper.

THEOREM 2.1. *Let the g_j possess all necessary partial derivatives for all $y \in \mathbb{R}^m$, and let l_n and ϵ_N denote the local error and global error at step n and N, respectively. Then for any stochastic Runge–Kutta (SRK) method if, for all $n = 1, \ldots, N$,*

$$\left(E\left(\|l_n\|^2\right)\right)^{1/2} = O(h^{p+1/2}), \tag{2.1}$$

$$E(l_n) = O(h^{p+1}), \tag{2.2}$$

then

$$\left(E\left(\|\epsilon_N\|^2\right)\right)^{1/2} = O(h^p). \tag{2.3}$$

Remark 2.1. We observe that

- condition (2.1) is (in [1]) called *strong local order p*, (2.2) is called *mean local order p*, and (2.3) is called *strong global order p*;
- the essential result is that a factor $h^{1/2}$ is removed from the local error when moving to the strong error but only if the additional condition (2.2) is satisfied. While this result has only been proved for SRKs, it can easily be extended to other classes of stochastic methods through the same formulation as in the deterministic case—namely general linear (also known as multivalue) methods.

We now examine the local truncation error in light of the above comments by analyzing the truncation error of the class of stochastic linear multistep methods, obtained by inserting the continuous solution evaluated at the grid-points in the discrete equation, to give

$$\tau_n = \sum_{i=0}^{k} \alpha_i y(t_{n+i}) - \sum_{j=0}^{d} \sum_{i=0}^{k} \beta_{in}^{j} g_j(y(t_{n+i})). \tag{2.4}$$

We need to introduce some preliminary results and notations. First of all, we recall the following stochastic Taylor expansions for the solution $y(t)$ of (1.2) (see, e.g., [9]):

$$\begin{aligned} y(t+h) = y(t) &+ \sum_{j=0}^{d} g_j(y(t))J_j(t) + \sum_{\ell,j=0}^{d} g_j'(y(t))g_\ell(y(t))J_{\ell j}(t) \\ &+ \sum_{r,\ell,j=0}^{d} \left(g_j''(y(t))\left(g_\ell(y(t)), g_r(y(t))\right) + g_j'(y(t))g_\ell'(y(t))g_r(y(t))\right) J_{r\ell j}(t) \\ &+ \cdots \end{aligned}$$

where, by setting $W_0(t) = t$,

$$J_j(t) = \int_t^{t+h} \circ dW_j, \qquad J_{\ell j}(t) = \int_t^{t+h} \int_t^{s} \circ dW_\ell(s_1) \circ dW_j(s),$$

$$J_{r\ell j}(t) = \int_t^{t+h} \int_t^{s} \int_t^{s_1} \circ dW_r(s_2) \circ dW_\ell(s_1) \circ dW_j(s).$$

More generally, for a suitably smooth function $g(y)$, one obtains that

$$\begin{aligned} g(y(t+h)) = g(y(t)) &+ \sum_{\ell=0}^{d} g'(y(t))g_\ell(y(t))J_\ell(t) \\ &+ \sum_{r,\ell=0}^{d} \left(g''(y(t))\left(g_\ell(y(t)), g_r(y(t))\right) + g'(y(t))g_\ell'(y(t))g_r(y(t))\right) J_{r\ell}(t) \\ &+ \cdots . \end{aligned}$$

By using the above expansions, we can evaluate the truncation error (2.4) as follows (all functions are evaluated at $y(t_n)$):

$$\tau_n = \sum_{i=0}^{k} \alpha_i \Bigg[y(t_n) + \sum_{j=0}^{d} g_j J_j^{ni} + \sum_{\ell,j=0}^{d} g_j' g_\ell J_{\ell j}^{ni} + \sum_{r,\ell,j=0}^{d} \left(g_j''(g_\ell, g_r) + g_j' g_\ell' g_r \right) J_{r\ell j}^{ni} + \cdots \Bigg] - \sum_{i=0}^{k} \sum_{j=0}^{d} \beta_{in}^{j} \Bigg[g_j + \sum_{\ell=0}^{d} g_j' g_\ell J_\ell^{ni} + \sum_{r,\ell=0}^{d} \left(g_j''(g_\ell, g_r) + g_j' g_\ell' g_r \right) J_{r\ell}^{ni} + \cdots \Bigg], \tag{2.5}$$

where, for all $r, \ell, j = 0, \ldots, d$, and $i = 0, \ldots, k$,

$$J_j^{ni} = \int_{t_n}^{t_{n+i}} \circ dW_j,$$

$$J_{\ell j}^{ni} = \int_{t_n}^{t_{n+i}} \int_{t_n}^{s} \circ dW_\ell(s_1) \circ dW_j(s) \equiv \int_{t_n}^{t_{n+i}} J_\ell^{ns} \circ dW_j(s), \tag{2.6}$$

$$J_{r\ell j}^{ni} = \int_{t_n}^{t_{n+i}} \int_{t_n}^{s} \int_{t_n}^{s_1} \circ dW_r(s_2) \circ dW_\ell(s_1) \circ dW_j(s) \equiv \int_{t_n}^{t_{n+i}} J_{r\ell}^{ns} \circ dW_j(s).$$

In order to derive the conditions on the coefficients of the method to satisfy Theorem 2.1, let us denote, for any string $\{j_1, \ldots, j_\nu\}$ with $j_i \in \{0, \ldots, d\}$, by $z(j_1, \ldots, j_\nu)$ the number of zeros in the string. Then, by considering (see, e.g., P. M. Burrage [6]) that

$$E(|J_j^{ni}|) = O(h^{(1+z(j))/2}),$$
$$E(|J_{\ell j}^{ni}|) = O(h^{(2+z(\ell,j))/2}),$$
$$E(|J_{r\ell j}^{ni}|) = O(h^{(3+z(r,\ell,j))/2}),$$

the following strong local order conditions are derived from (2.5):

• **deterministic consistency and strong local order** 1/2:

$$\sum_{i=0}^{k} \alpha_i = 0, \qquad (\alpha_k = 1) \tag{2.7}$$

$$\sum_{i=0}^{k} \left(\alpha_i J_j^{ni} - \beta_{in}^{j} \right) = 0, \quad j = 0, \ldots, d; \tag{2.8}$$

• **strong local order** 1: all the previous ones and, moreover,

$$\sum_{\ell,j=1}^{d} g_j' g_\ell \sum_{i=1}^{k} \left(\alpha_i J_{\ell j}^{ni} - \beta_{in}^{j} J_\ell^{ni} \right) = 0; \tag{2.9}$$

• **strong local order** 3/2: all the previous ones and, moreover,

$$\sum_{j=1}^{d} g_0' g_j \sum_{i=1}^{k} \left(\alpha_i J_{j0}^{ni} - \beta_{in}^{0} J_j^{ni}\right) + \sum_{j=1}^{d} g_j' g_0 \sum_{i=1}^{k} \left(\alpha_i J_{0j}^{ni} - \beta_{in}^{j} J_0^{ni}\right) = 0, \tag{2.10}$$

$$\sum_{r,\ell,j=1}^{d} \left(g_j''(g_\ell, g_r) + g_j' g_\ell' g_r\right) \sum_{i=1}^{k} \left(\alpha_i J_{r\ell j}^{ni} - \beta_{in}^{j} J_{r\ell}^{ni}\right) = 0. \tag{2.11}$$

In order to ensure that the methods have strong global order the mean order conditions (2.2) given in Theorem 2.1 need to be also satisfied. Thus, to attain strong global order 1/2, the conditions

$$E\left(\sum_{i=1}^{k} \left(\alpha_i J_{\ell j}^{ni} - \beta_{in}^{j} J_{\ell}^{ni}\right) \right) = 0, \quad \ell, j = 1, \ldots, d,$$

i.e. (see (2.12) below)

$$\frac{h}{2} \sum_{i=1}^{k} i\alpha_i - \sum_{i=1}^{k} E\left(\beta_{in}^{j} J_j^{ni}\right) = 0, \quad j = 1, \ldots, d,$$

obtained from zeroing the expectation of (2.9), are needed in addition to (2.7) and (2.8).

In the case of strong global order 1, we need to zero the expectations of (2.10) and (2.11), in addition to (2.7), (2.8) and (2.9). However, since the expectation of the product of stochastic integrals is zero when the number of nonzero indices is odd, it can be shown (see (2.12) below and the properties of Stratonovich integrals listed after (3.2)) that strong local order 1 automatically implies strong global order 1.

We now look for methods (1.3) such that

$$\beta_{in}^{0} = h\beta_i, \qquad \beta_{in}^{j} = \sum_{r=1}^{k} J_j^{nr} d_{ir}, \quad i = 0, \ldots, k, \quad j = 1, \ldots, d, \tag{2.12}$$

where the scalars $\{\beta_i\}$ and $\{d_{ir}\}$ are to be determined. Consequently, condition (2.8) becomes

$$\begin{aligned} 0 &= \sum_{i=1}^{k} \alpha_i J_j^{ni} - \sum_{i=0}^{k} \sum_{r=1}^{k} J_j^{nr} d_{ir} \\ &= \sum_{i=1}^{k} \alpha_i J_j^{ni} - \sum_{r=1}^{k} J_j^{nr} \sum_{i=0}^{k} d_{ir} \;\equiv\; (\hat{\alpha}^T - e^T D) \hat{J}_j^n, \end{aligned}$$

where

$$\hat{\alpha} = (\ \alpha_1, \ldots, \alpha_k\ )^T, \qquad \hat{J}_j^n = (\ J_j^{n1}, \ldots, J_j^{nk}\ )^T, \qquad e = (1, \ldots, 1)^T \in \mathbb{R}^{k+1},$$

and

$$D = [\hat{d}_0 \ \hat{D}]^T, \qquad \hat{D} = [\hat{d}_1, \dots, \hat{d}_k], \qquad \hat{d}_i = (\ d_{i1}, \dots, d_{ik}\ )^T, \quad i = 0, \dots, k.$$

The above condition then reads as $\hat{\alpha} = D^T e$, that is,

$$\alpha_r = \sum_{i=0}^{k} d_{ir}, \quad r = 1, \dots, k. \tag{2.13}$$

Let us now examine condition (2.9). For this, we shall distinguish between the following two cases:

- **the commutative case**, for which

$$[g_j, g_\ell](y) \equiv (g_j'(y) g_\ell(y) - g_\ell'(y) g_j(y)) = 0,$$

for all $j, \ell = 1, \dots, d$, and $y \in \mathbb{R}$;

- **the non-commutative case**, for which

$$[g_j, g_\ell](y) \neq 0,$$

for at least one pair of different indices $j, \ell = 1, \dots, d$.

Let us now consider the first case, from which (2.9) results in

$$\begin{aligned} 0 &= \sum_{\ell,j=1}^{d} g_j' g_\ell \sum_{i=1}^{k} \left(\alpha_i J_{\ell j}^{ni} - \beta_{in}^{j} J_\ell^{ni} \right) \\ &= \sum_{j=1}^{d} g_j' g_j \sum_{i=1}^{k} \left(\alpha_i J_{jj}^{ni} - \beta_{in}^{j} J_j^{ni} \right) \\ &\quad + \sum_{1 \le j < \ell \le d}^{d} g_j' g_\ell \sum_{i=1}^{k} \left(\alpha_i (J_{\ell j}^{ni} + J_{j\ell}^{ni}) - \beta_{in}^{j} J_\ell^{ni} - \beta_{in}^{\ell} J_j^{ni} \right). \end{aligned}$$

By considering that the following relations hold true for double Stratonovich integrals (see, for example, [9]),

$$J_{\ell j}^{ni} + J_{j\ell}^{ni} = J_j^{ni} J_\ell^{ni}, \quad j, \ell = 0, \dots, d, \tag{2.14}$$

it follows that the previous requirement is equivalent to (see (2.12))

$$\begin{aligned} 0 &= \sum_{j=1}^{d} g_j' g_j \sum_{i=1}^{k} \left(\frac{\alpha_i}{2} (J_j^{ni})^2 - J_j^{ni} \sum_{r=1}^{k} J_j^{nr} d_{ir} \right) \\ &\quad + \sum_{1 \le j < \ell \le d}^{d} g_j' g_\ell \sum_{i=1}^{k} \left(\alpha_i (J_\ell^{ni} J_j^{ni}) - J_\ell^{ni} \sum_{r=1}^{k} J_j^{nr} d_{ir} - J_j^{ni} \sum_{r=1}^{k} J_\ell^{nr} d_{ir} \right) \\ &= \sum_{j=1}^{d} g_j' g_j \frac{1}{2} (\hat{J}_j^n)^T \left(\mathrm{diag}(\hat{\alpha}) - (\hat{D} + \hat{D}^T) \right) \hat{J}_j^n \\ &\quad + \sum_{1 \le j < \ell \le d}^{d} g_j' g_\ell (\hat{J}_j^n)^T \left(\mathrm{diag}(\hat{\alpha}) - (\hat{D} + \hat{D}^T) \right) \hat{J}_\ell^n, \end{aligned}$$

which holds true for $\mathrm{diag}(\hat{\alpha}) = \hat{D} + \hat{D}^T$, that is,

$$\begin{aligned} \alpha_r &= 2d_{rr}, \quad r = 1, \ldots, k, \\ d_{ir} &= -d_{ri}, \quad i, r = 1, \ldots, k, \quad i \neq r. \end{aligned} \tag{2.15}$$

From the above conditions, the following result follows:

THEOREM 2.2. *There are no explicit methods, in the form* (1.3)–(2.12)*, with strong order* 1.

PROOF. In order for the method to be explicit, we should have, for all $j = 1, \ldots, d$,

$$0 = \beta_{kn}^j = \sum_{r=1}^k J_j^{nr} d_{kr}.$$

As a consequence, one obtains $d_{kr} = 0$, for all $r = 1, \ldots, k$. However, from (2.15), it follows that

$$1 = \alpha_k = 2d_{kk},$$

which contradicts such an assumption. □

Theorem 2.2 has important consequences, concerning the properties of the discrete solutions, which we shall analyze in Section 3.1.

We now study the non-commutative case, for which, by means of arguments similar to those previously used, the following condition must be satisfied, in order to get strong local order 1 (conversely, the order collapses to 1/2 (see also [1])):

$$\begin{aligned} 0 &= \sum_{\ell,j=1}^d g_j' g_\ell \sum_{i=1}^k \left(\alpha_i J_{\ell j}^{ni} - \beta_{in}^j J_\ell^{ni} \right) \\ &= \sum_{j=1}^d g_j' g_j \sum_{i=1}^k \left(\alpha_i J_{jj}^{ni} - \beta_{in}^j J_j^{ni} \right) \\ &\quad + \sum_{1 \le j < \ell \le d} g_j' g_\ell \sum_{i=1}^k \left(\alpha_i (J_{j\ell}^{ni} + J_{\ell j}^{ni}) - \beta_{in}^j J_\ell^{ni} - \beta_{in}^\ell J_j^{ni} \right) \\ &\quad - \sum_{1 \le j < \ell \le d} [g_j, g_\ell] \sum_{i=1}^k \left(\alpha_i J_{j\ell}^{ni} - \beta_{in}^\ell J_j^{ni} \right). \end{aligned} \tag{2.16}$$

Consequently, by setting

$$\gamma_{j\ell}^n = \sum_{i=1}^k \left(\alpha_i J_{j\ell}^{ni} - \beta_{in}^\ell J_j^{ni} \right), \tag{2.17}$$

one has that the modified scheme

$$\sum_{i=0}^k \alpha_i y_{n+i} = \sum_{j=0}^d \sum_{i=0}^k \beta_{in}^j g_{j,n+i} + \sum_{1 \le j < \ell \le d} \gamma_{j\ell}^n [g_j, g_\ell] \tag{2.18}$$

has still strong global order 1, provided that (2.12)–(2.15) hold true.

REMARK 2.2. We observe that in (2.18) the commutators $[g_j, g_\ell]$ may be evaluated at any point $y(t_{n+s})$, with $s \in \{0, \dots, k-1\}$, in order that (2.16) be satisfied, provided that the functions g_j are suitably smooth. In fact, in such a case one has that

$$E\left(\|[g_j, g_\ell](y(t_n)) - [g_j, g_\ell](y(t_{n+s}))\|\right) = O(\sqrt{sh}).$$

Moreover,

$$E\left([g_j, g_\ell](y(t_n)) - [g_j, g_\ell](y(t_{n+s}))\right) = O(sh),$$

which is needed to satisfy the corresponding mean local order conditions (2.2). Consequently, the commutators can be recomputed every, say, k consecutive steps.

We now study the problem of getting methods of strong order higher than 1. The following negative result holds true.

THEOREM 2.3. *There are no methods, in the form* (1.3)–(2.12)*, of strong order* 3/2*.*

PROOF. We consider the simpler case of only one Wiener process. That is, $d = 1$ in (1.2). Consequently, from (2.11) one has that the following equation needs to be satisfied:

$$\sum_{i=1}^{k} \left(\alpha_i J_{111}^{ni} - \beta_{in}^{1} J_{11}^{ni}\right) = 0.$$

Considering that $J_{11}^{ni} = (J_1^{ni})^2/2$, and $J_{111}^{ni} = (J_1^{ni})^3/6$ (see, for example, [9]), and taking into account (2.12), it then follows that

$$\begin{aligned}(2.19)\qquad &\sum_{i=1}^{k} \Big(\alpha_i (J_1^{ni})^3 - 3(J_1^{ni})^2 \sum_{r=1}^{k} J_1^{nr} d_{ir}\Big)\\ &= \sum_{i=1}^{k} \Big(\alpha_i \Big(\sum_{r=0}^{i-1} J_1^{n+r,1}\Big)^3 - 3\Big(\sum_{r=0}^{i-1} J_1^{n+r,1}\Big)^2 \sum_{r=1}^{k} d_{ir} \sum_{s=0}^{r-1} J_1^{n+s,1}\Big)\\ &\equiv p_3(J_1^{n1}, \dots, J_1^{n+k-1,1}) \;=\; 0,\end{aligned}$$

where p_3 is a polynomial in k variables of degree 3 (indeed, it contains only monomials of degree 3). Since the number of the distinct monomials of degree 3 made up with the k integrals $J_1^{n+r,1}$, $r = 0, \dots, k-1$, is given by

$$\binom{(k-1)+3}{3} = \frac{(k+2)(k+1)k}{6},$$

we obtain a corresponding number of linear equations to zero the coefficients of all monomials. However, by taking into account (2.7), (2.13) and (2.15), one obtains that the number of free parameters is $\frac{1}{2}(k+1)k-1$. Consequently, there is no solution, for all $k \geq 1$, provided that all such equations are independent. Instead of proving this fact, however, we only prove that there is no solution satisfying (2.19) and the previous order conditions. In fact, the coefficient of the monomial $(J_1^{n+k-1,1})^3$ is easily seen to be $\alpha_k - 3d_{kk}$. It then follows that $d_{kk} = \alpha_k/3 = 1/3$, whereas from (2.15) one has $d_{kk} = \alpha_k/2 = 1/2$. □

We then conclude that we cannot have methods of strong order greater than 1 in the given assumed form. Nevertheless, this can be achieved for particular problems, as shown in the next subsection.

2.1 The case of additive noise.

An important instance of problem (1.2) is that of *additive noise*, namely,

$$g_j(y) \equiv g_j, \quad j = 1, \dots, d.$$

In such a case, in fact, some simplification occurs. First of all, the problem is commutative, since

$$g_j' = 0, \quad j = 1, \dots, d. \tag{2.20}$$

Moreover, the truncation error is given by (see (2.10)–(2.11))

$$\tau_n = \sum_{j=1}^{d} g_0' g_j \sum_{i=1}^{k} \left(\alpha_i J_{j0}^{ni} - \beta_{in}^0 J_j^{ni}\right) + R_n \equiv \tau_n^{(1)} + R_n, \tag{2.21}$$

where $E(\|R_n\|) = O(h^2)$, because from (2.20) it follows that R_n does not contain terms involving the multiple integrals $J_{r\ell j}^{ni}$, $r, \ell, j \geq 1$. Consequently, we can obtain strong order 3/2 by adding to (1.3) an extra term zeroing $\tau_n^{(1)}$, namely

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=0}^{d} \sum_{i=0}^{k} \beta_{in}^j g_{j,n+i} - g_0' \sum_{j=1}^{d} \gamma_{j0}^n g_j, \tag{2.22}$$

where

$$\gamma_{j0}^n = \sum_{i=1}^{k} \left(\alpha_i J_{j0}^{ni} - \beta_{in}^0 J_j^{ni}\right). \tag{2.23}$$

Obviously, we need to satisfy also the corresponding mean local order condition (2.2). It can be easily seen that the latter reduces to require

$$E(R_n) = O(h^{\frac{5}{2}}),$$

which, in turn, is equivalent to have (see also (2.12)), for all $r, \ell = 1, \dots, d$,

$$\begin{aligned} 0 &= E\bigg(\sum_{i=0}^{k} \alpha_i (J_{r\ell 0}^{ni} + J_{\ell r 0}^{ni}) - \beta_{in}^0 (J_{r\ell}^{ni} + J_{\ell r}^{ni}) \bigg) \\ &= hE\bigg(\sum_{i=0}^{k} \int_0^i J_r^{ns} J_\ell^{ns} ds \, - \beta_i J_r^{ni} J_\ell^{ni} \bigg), \end{aligned}$$

where the last equality follows from (2.6) and (2.14). Since

$$E(J_r^{ns} J_\ell^{ns}) = \begin{cases} 0, & \text{for } r \neq \ell, \\ sh, & \text{for } r = \ell, \end{cases}$$

we obtain that the previous equation holds iff

$$\sum_{i=0}^{k} i^2 \alpha_i - 2i\beta_i = 0,$$

namely if the method has order $p \geq 2$, when applied to a deterministic ODE.

In addition to this, in the case where the drift is linear, namely

$$g_0(y) = Ay + b,$$

with A and b constant, then the term R_n in (2.21) contains only the triple integrals J_{j00}^{ni}. Consequently, $E(\|R_n\|) = O(h^{5/2})$, and (2.22) gives a strong order 2 method, since also in this case it can be verified that the mean local order condition, which reduces to require

$$E(R^n) = O(h^3),$$

is satisfied, provided that the deterministic order of the method is at least 2.

3 Adams-type methods for SODEs.

We now consider the following family of strong order 1 methods for SODEs:

$$y_{n+k} = y_{n+k-1} + h \sum_{i=0}^{k} \beta_i g_{0,n+i} + \frac{1}{2} \sum_{j=1}^{d} J_j^{n+k-1,1} \left(g_{j,n+k} + g_{j,n+k-1} \right),$$

obtained by setting (see (2.12))

$$d_{ir} = 0, \quad \text{for} \quad \min\{i, r\} \leq k-2,$$

$$d_{k-1,k} = d_{kk} = -d_{k-1,k-1} = -d_{k,k-1} = \tfrac{1}{2}.$$

The coefficients $\{\beta_i\}$ are those of the Adams–Moulton method of order $k+1$. The above scheme may be conveniently rewritten as

$$y_{n+1} = y_n + h \sum_{i=0}^{k} \beta_{k-i} g_{0,n+1-i} + \frac{1}{2} \sum_{j=1}^{d} J_j^{n1} \left(g_{j,n+1} + g_{jn} \right). \tag{3.1}$$

Moreover, by considering that the following properties hold true for Stratonovich integrals, which allow us to handle combinations of the basic Stratonovich integrals

$$J_j^{n1}, \quad J_{\ell j}^{n1}, \quad j, \ell = 0, \ldots, d: \tag{3.2}$$

1. $J_j^{ni} = J_j^{n,i-1} + J_j^{n+i-1,1}$,
2. $J_{\ell j}^{ni} = J_{\ell j}^{n,i-1} + J_{\ell j}^{n+i-1,1} + J_\ell^{n,i-1} J_j^{n+i-1,1}, \quad i = 2, \dots, k$,
3. $J_j^{n+1,i} = J_j^{ni} + J_j^{n+i,1} - J_j^{n1}$,
4. $J_{\ell j}^{n+1,i} = J_{\ell j}^{ni} + J_{\ell j}^{n+i,1} - J_{\ell j}^{n1} + J_\ell^{ni} J_j^{n+i,1} - J_\ell^{n1} J_j^{n+1,i}, \quad n = 0, 1, \dots,$

the corresponding schemes (2.17)–(2.18) and (2.22)–(2.23) become, respectively,

$$y_{n+1} = y_n + h \sum_{i=0}^{k} \beta_{k-i} g_{0,n+1-i} + \frac{1}{2} \sum_{j=1}^{d} J_j^{n1} \left(g_{j,n+1} + g_{jn} \right) + \sum_{1 \le j < \ell \le d} \left(J_{j\ell}^{n1} - \tfrac{1}{2} J_j^{n1} J_\ell^{n1} \right) [g_j, g_\ell] \tag{3.3}$$

and

$$y_{n+1} = y_n + h \sum_{i=0}^{k} \beta_{k-i} g_{0,n+1-i} + \frac{1}{2} \sum_{j=1}^{d} J_j^{n1} (g_{j,n+1} + g_{jn}) - g_0' \Bigg(\sum_{j=1}^{d} \Bigg(J_{j0}^{n1} + h \Bigg(J_j^{n-k+1,k-1} - \sum_{i=1}^{k} \beta_i J_j^{n-k+1,i} \Bigg) \Bigg) g_j \Bigg). \tag{3.4}$$

3.1 Predictor-corrector implementation.

The formulae (3.1) (or (3.3)–(3.4), depending on the problem to be solved) have some advantages and drawbacks.

Amongst the former, we can mention the possibility of obtaining high deterministic order for the method, which may be useful in the case where the noise terms in (1.2) are very small.

On the other hand, the fact of having implicitness in the stochastic terms of the method may be a severe drawback. As an example, let us apply the scheme (3.1) to the scalar equation

$$dy = \lambda y \, dt + \mu y \circ dW, \quad \lambda, \mu \in \mathbb{C},$$

thus obtaining the scheme

$$\left(1 - h\lambda\beta_k - J^{n1}\mu/2\right) y_{n+1} = \left(1 + h\lambda\beta_{k-1} + J^{n1}\mu/2\right) y_n + h\lambda \sum_{i=2}^{k} \beta_{k-i} y_{n+1-i},$$

whose solution may be unbounded, when the real part of μ is nonzero, even though $Re(\lambda) < 0$. As matter of fact (see, for example, [9]) $E(|y_n|)$ does not exist.

However, we can recover this drawback by using a *predictor-corrector* implementation of the method. In more detail, consider the following explicit scheme:

$$y_{n+1} = y_n + h\sum_{i=1}^{k}\hat\beta_{k-i}g_{0,n+1-i} + \sum_{j=1}^{d}J_j^{n1}g_{jn}, \tag{3.5}$$

where the coefficients $\{\hat\beta_i\}$ are the coefficients of the Adams–Bashforth method of order k. It is easily verified that the scheme (3.5) has strong order 1/2. In particular, for $k = 1$ one obtains the well known Euler–Maruyama scheme.

We can use (3.5) as a predictor for the scheme (3.1):

$$\begin{aligned} \hat y_{n+1} &= y_n + h\sum_{i=1}^{k}\hat\beta_{k-i}g_{0,n+1-i} + \sum_{j=1}^{d}J_j^{n1}g_{jn}, \\ y_{n+1} &= y_n + h\left(\beta_k\hat g_{0,n+1} + \sum_{i=1}^{k}\beta_{k-i}g_{0,n+1-i}\right) + \frac{1}{2}\sum_{j=1}^{d}J_j^{n1}\left(\hat g_{j,n+1} + g_{jn}\right), \end{aligned} \tag{3.6}$$

where $\hat g_{j,n+1} = g_j(\hat y_{n+1})$. In this implementation, the previously mentioned problems are overcome, because the new scheme is essentially an explicit one. Moreover, the following result holds true.

THEOREM 3.1. *For commutative problems, the predictor-corrector scheme* (3.6) *has strong order* 1.

PROOF. By considering that

$$\hat y(t_{n+1}) \equiv y(t_n) + h\sum_{i=1}^{k}\hat\beta_{k-i}g_0(y(t_{n+1-i})) + \sum_{j=1}^{d}J_j^{n1}g_j(y(t_n)) = y(t_{n+1}) + S_n,$$

where $E(\|S_n\|) = O(h)$, one obtains

$$\begin{aligned} \hat\tau_n &\equiv y(t_{n+1}) - y(t_n) - h\left(\beta_k g_0(\hat y(t_{n+1})) + \sum_{i=1}^{k}\beta_{k-i}g_0(y(t_{n+1-i}))\right) \\ &\quad + \frac{1}{2}\sum_{j=1}^{d}J_j^{n1}\left(g_j(\hat y(t_{n+1})) + g_j(y(t_n))\right) \\ &= y(t_{n+1}) - y(t_n) - h\left(\sum_{i=0}^{k}\beta_{k-i}g_0(y(t_{n+1-i})) + R_n^0\right) \\ &\quad + \frac{1}{2}\sum_{j=1}^{d}J_j^{n1}\left(g_j(y(t_{n+1})) + g_j(y(t_n)) + R_n^j\right) \equiv \tau_n + R_n, \end{aligned}$$

where τ_n is the truncation error of formula (3.1), and $E(\|R_n\|) = O(h^{3/2})$, since $E(\|R_n^j\|) = O(h)$, $j = 0, \ldots, d$. Consequently, we obtain strong order 1. □

Similar arguments can be used to prove the following corollaries.

COROLLARY 3.2. *For non-commutative problems, the predictor-corrector scheme*

$$
\begin{aligned}
\hat{y}_{n+1} &= y_n + h\sum_{i=1}^{k}\hat{\beta}_{k-i}g_{0,n+1-i} + \sum_{j=1}^{d} J_j^{n1} g_{jn},\\
y_{n+1} &= y_n + h\Big(\beta_k \hat{g}_{0,n+1} + \sum_{i=1}^{k}\beta_{k-i}g_{0,n+1-i}\Big)\\
&\quad + \frac{1}{2}\sum_{j=1}^{d} J_j^{n1}\left(\hat{g}_{j,n+1} + g_{jn}\right) + \sum_{1\le j<\ell\le d}\Big(J_{j\ell}^{n1} - \frac{1}{2}J_j^{n1}J_\ell^{n1}\Big)[g_j, g_\ell],
\end{aligned}
\tag{3.7}
$$

has strong order 1.

COROLLARY 3.3. *In the case of additive noise, the predictor-corrector scheme*

$$
\begin{aligned}
\hat{y}_{n+1} &= y_n + h\sum_{i=1}^{k}\hat{\beta}_{k-i}g_{0,n+1-i} + \sum_{j=1}^{d} J_j^{n1} g_{jn},\\
y_{n+1} &= y_n + h\Big(\beta_k \hat{g}_{0,n+1} + \sum_{i=1}^{k}\beta_{k-i}g_{0,n+1-i}\Big) + \frac{1}{2}\sum_{j=1}^{d} J_j^{n1}\left(\hat{g}_{j,n+1} + g_{jn}\right)\\
&\quad - g_0'\Big(\sum_{j=1}^{d}\Big(J_{j0}^{n1} + h\Big(J_j^{n-k+1,k-1} - \sum_{i=1}^{k}\beta_i J_j^{n-k+1,i}\Big)\Big)g_j\Big).
\end{aligned}
\tag{3.8}
$$

has strong order 3/2. *If, moreover, the drift* g_0 *is linear, then the scheme has strong order* 2.

In the case of scheme (3.7), the commutator $[g_j, g_\ell]$ may be evaluated at any point y_{n-s}, $s = 0, \ldots, k$. Moreover, it can be implemented in a matrix-free fashion, by approximating the Jacobian-times-vectors involved as follows:

$$
g_j'(y_n)g_\ell(y_n) = \frac{g_j(y_n + hg_\ell(y_n)) - g_j(y_n)}{h} + O(h),
$$

where h is the stepsize used. In such a way, since the quantities $\gamma_{j\ell}^n$ have $O(h)$ mean square expectation, we are able to preserve order 1 for scheme (3.7) and order 3/2 for scheme (3.8). Obviously, for the latter scheme, the approximation is exact when g_0 is linear, so that, in such a case, we still get a strong order 2 method.

4 Numerical tests.

To carry out the numerical tests, we need to generate the basic Stratonovich integrals (3.2). They can be computed, or approximated, as shown in [9, pp. 202–205] (see also [3]). We here report the basic facts for completeness. The starting point is that a Wiener process $W_j(t)$, such that $W_j(0) = 0$ and for which the increment at $t = h$ is known, can be expanded in Fourier series as

$$W_j(t) = \frac{t}{h}W_j(h) + \frac{a_{j0}}{2} + \sum_{r=1}^{\infty}\Big(a_{jr}\cos\Big(\frac{2r\pi t}{h}\Big) + b_{jr}\sin\Big(\frac{2r\pi t}{h}\Big)\Big), \tag{4.1}$$

where

$$a_{jr} = \frac{2}{h}\int_0^h \Big(W_j(s) - \frac{s}{h}W_j(h)\Big)\cos\Big(\frac{2r\pi s}{h}\Big)ds,$$
$$b_{jr} = \frac{2}{h}\int_0^h \Big(W_j(s) - \frac{s}{h}W_j(h)\Big)\sin\Big(\frac{2r\pi s}{h}\Big)ds,$$

$r = 1, 2, \ldots,$ are known to be $N\big(0, (2\pi^2 r^2)^{-1}h\big)$ random variables, and (see [9, p. 203]) a_{j0} is $N(0, h/3)$. Then, one considers a partial sum with p terms, approximating the series in (4.1). The following procedure, which is a slight modification of that presented in [9, pp. 202–203] (see also [3]), is then derived (the upper indices $n1$ of the integrals are omitted, for brevity):

- for $j = 1, \ldots, d$, and $r = 1, \ldots, p$, let $\{\xi_j\}$, $\{\nu_j\}$, $\{\zeta_{jr}\}$ and $\{\eta_{jr}\}$ be independent standard Gaussian random variables;
- set:

$$\begin{aligned}
J_0 &= h,\\
J_{00} &= \frac{h^2}{2},\\
J_{j0} &= \frac{1}{2}h^{3/2}(\xi_j + \nu_j/\sqrt{3}),\\
J_{0j} &= \frac{1}{2}h^{3/2}(\xi_j - \nu_j/\sqrt{3}),\\
a_{j0} &= \frac{2}{h}J_{j0} - J_j \equiv \nu_j\sqrt{\frac{h}{3}}, \quad j = 1, \ldots, d,\\
J_{\ell j} &= \frac{1}{2}\Big(J_\ell J_j - a_{\ell 0}J_j - a_{j0}J_\ell + \frac{h}{\pi}\sum_{r=1}^{p}\frac{1}{r}\left(\zeta_{\ell r}\eta_{jr} - \eta_{\ell r}\zeta_{jr}\right)\Big),\\
&\qquad j, \ell = 1, \ldots, d.
\end{aligned}$$

According to the analysis in [7], the value $p = 5$ has been considered as being appropriate (as matter of fact, the multiple stochastic integrals are sufficiently well approximated for all $h \le 0.5$, which is the maximum stepsize considered in the numerical tests).

For Problems 1 and 2, we implement the predictor-corrector schemes (3.6) and (3.7) (depending on the problem) with $k = 1, 2$. For Problem 3 we consider the scheme (3.8) with $k = 1$. For the two-step method, the second initial condition is provided by the method with $k = 1$, thus preserving strong order 1 and a deterministic order 3. All methods are implemented with a constant stepsize.

Table 4.1: Mean square global error, Problem 4.1.

	$k=1$			
$1/h$	error for scheme (3.6)	rate	error for scheme (3.7)	rate
2	5.76E-1	–	5.75E-1	–
4	9.97E-2	2.53	9.93E-2	2.53
8	2.68E-2	1.90	2.63E-2	1.92
16	8.92E-3	1.59	8.49E-3	1.63
32	3.84E-3	1.22	3.31E-3	1.36
64	2.05E-3	0.90	1.60E-3	1.05
128	1.22E-3	0.75	8.18E-4	0.97
256	7.92E-4	0.63	4.13E-4	0.98
512	5.23E-4	0.59	2.15E-4	0.94
1024	3.41E-4	0.63	1.04E-4	1.05

	$k=2$			
$1/h$	error for scheme (3.6)	rate	error for scheme (3.7)	rate
2	1.93E-1	–	1.94E-1	–
4	3.71E-2	2.38	3.69E-2	2.39
8	1.39E-2	1.42	1.36E-2	1.44
16	7.12E-3	0.96	6.75E-3	1.01
32	3.65E-3	0.97	3.13E-3	1.11
64	2.06E-3	0.82	1.60E-3	0.97
128	1.22E-3	0.75	8.21E-4	0.97
256	7.92E-4	0.63	4.16E-4	0.98
512	5.28E-4	0.59	2.15E-4	0.95
1024	3.41E-4	0.63	1.04E-4	1.05

PROBLEM 4.1. This problem is a nonlinear and non-commutative problem describing the satellite dynamics in the atmosphere of the earth, whose density fluctuates randomly [10]. We here consider the Stratonovich formulation of the problem,

$$\begin{aligned} dy_1 &= y_2 dt, \quad t \in [0,T], \quad y_1(0) = y_2(0) = 1, \\ dy_2 &= \Big(- b_0 y_1 + c_0 \sin(2y_1) - a_0 y_2 - \tfrac{1}{2} a_1^2 y_2 \Big) dt \\ &\quad - a_1 y_2 \circ dW_1 - b_1 y_1 \circ dW_2. \end{aligned} \tag{4.2}$$

The choice of the parameters is the same as those considered in [3]:

$$a_0 = c_0 = 1/4, \quad b_0 = 1, \quad a_1 = b_1 = 10^{-1}, \quad T = 40.$$

In Table 4.1 we report the numerical results obtained with the schemes (3.6) and (3.7), with $k = 1, 2$, both with a PEC implementation. For each stepsize, we have computed the expected mean square global error on 100 random trajectories. The reference solutions have been computed by using a strong order 3/2 Taylor expansion.

For the methods in the form (3.6), which do not handle the non-commutativity of the problem, the order collapses towards 1/2, whereas the modified schemes (3.7) maintain the strong order 1 convergence. From the results, it is evident that there is no apparent advantage in using the two-step method over the one-step scheme.

PROBLEM 4.2. This is a nonlinear problem, whose Stratonovich form is

$$dy = -\alpha(1-y^2)dt + \beta(1-y^2) \circ dW, \quad t \in [0,T], \quad y(0) = y_0. \tag{4.3}$$

Table 4.2: Mean square global error for method (3.6), Problem 4.2.

	$\beta = 1$				$\beta = 10^{-4}$			
$1/h$	$k=1$	rate	$k=2$	rate	$k=1$	rate	$k=2$	rate
8	1.45E-1	–	1.43E-1	–	3.61E-3	–	4.18E-04	–
16	7.70E-2	0.91	7.74E-2	0.89	9.01E-4	2.00	4.96E-05	3.07
32	3.94E-2	0.97	3.95E-2	0.97	2.24E-4	2.00	6.03E-06	3.04
64	2.14E-2	0.88	2.14E-2	0.89	5.60E-5	2.00	7.51E-07	3.00
128	1.01E-2	1.09	1.00E-2	1.09	1.40E-5	2.00	9.81E-08	2.94
256	4.69E-3	1.10	4.68E-3	1.10	3.49E-6	2.00	1.45E-08	2.75
512	2.60E-3	0.85	2.60E-3	0.85	8.73E-7	2.00	2.74E-09	2.41
1024	1.38E-3	0.91	1.38E-3	0.91	2.18E-7	2.00	6.97E-10	1.97
2048	7.03E-4	0.98	7.03E-4	0.98	5.45E-8	2.00	2.38E-10	1.55
4096	3.41E-4	1.04	3.41E-4	1.04	1.36E-8	2.00	8.61E-11	1.47

For this problem, the solution is known to be [9]

$$y(t) = \frac{(1+y_0)\exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1+y_0)\exp(-2\alpha t + 2\beta W(t)) - y_0 + 1}.$$

We have considered the following two sets of parameters for the numerical tests:

- $T = 2, \quad y_0 = 0, \quad \alpha = \beta = 1;$
- $T = 2, \quad y_0 = 0, \quad \alpha = 1, \quad \beta = 10^{-4}.$

In the first case, one has that the stochastic part is significant, whereas it is much smaller in the second case. As a consequence, for the second set of parameters, the deterministic order of the method plays a significant role, as shown by the numerical results listed in Table 4.2. Also in this case, the mean square expectation of the global error has been computed on 100 trajectories. The PEC implementation of method (3.6), with $k = 1, 2$, has again been considered.

PROBLEM 4.3. This is a linear problem with additive noise [9],

$$dy = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y\,dt + \begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix} \circ dW, \quad t \in [0,T], \quad y(0) = y_0. \tag{4.4}$$

Table 4.3: Mean square global error, Problem 4.3.

	$k = 1$			
$1/h$	error for scheme (3.6)	rate	error for scheme (3.8)	rate
2	4.36E-1	–	4.31E-1	–
4	9.90E-2	2.14	9.96E-2	2.11
8	2.75E-2	1.85	2.53E-2	1.98
16	8.18E-3	1.75	6.40E-3	1.98
32	3.27E-3	1.32	1.54E-3	2.06
64	1.52E-3	1.10	3.93E-4	1.97
128	8.20E-4	0.89	9.91E-5	1.99
256	3.80E-4	1.11	2.43E-5	2.03
512	1.93E-4	0.98	6.17E-6	1.98
1024	9.89E-5	0.96	1.50E-6	2.04

According to the analysis in Section 2.1, the modified scheme (3.8) would have strong order 2 convergence, whereas (3.6) has only strong order 1. This is confirmed by the numerical tests, listed in Table 4.3, where the expected mean square global error has been computed over 100 trajectories using a PECE implementation.

The reference solution has been computed by means of a strong order 3 Taylor expansion. The parameters used for the problem are

$$T = 10, \qquad y_0 = (1\ 0)^T, \qquad \sigma^2 = 10^{-1}.$$

5 Conclusions.

In this paper we have shown how to construct strong order 1 linear multistep methods for both commutative and non-commutative problems, using the technique of prediction-correction. In the case of non-commutative problems this required the additional evaluation of commutator (Lie bracket) terms at some point. In the case of additive problems, methods of strong order 1.5 were constructed. The effectiveness of these methods was demonstrated on three simple test problems.

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