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Time Reversal Symmetry and Energy Drift in Conservative Systems

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Abstract. According to some authors (see, e.g., [1]) Time Reversal Symmetry (TRS) is a murky problem. By considering that usually the TRS, to which they refer to, regards continuous variation of time, it is easy to estrapolate that TRS in the discrete domain is even more murky. In Numerical Analysis we come across to this problem, usually confined in the Physics domain, when trying to approximate the solutions of Hamiltonian problems (or, more generally, conservative problems). Often, for such problems an energy drift is observed even when using the most accurate numerical methods. It was believed that Hamiltonian systems satisfyng TRS (in the form established by the currently used definition) would prevent the drift, but recent counterexamples have shown that this is not always the case (see, e.g., [2]). In this paper we intend to analyze the problem, starting from the most current definition of TRS.

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1. DEFINITIONS OF TRS

The most intuitive definition of TRS, though not the most useful for mathematical needs, is the following one [3]:

Definition 1 *If for a motion picture of a mechanical system one cannot decide whether it is shown in forward or reverse direction, the system is said to have time-reversal symmetry.*

On the other hand, for continuous flows, the dynamics of a physical system, in the phase space D, can often be described by a family of maps $U(t): D \to D$ which associate the initial state of the system, say y_0, to the state at time t, i.e. $y(t) = U(t)y_0$. The operator $U(t)$ has the structure of a one-parameter group, since $U(t+s) = U(t)U(s)$. Under such hypothesis, the TRS is usually defined as follows [1]:

Definition 2 *The system has TRS if there exist a map $S: D \to D$ such that $U(-t) = S^{-1}U(t)S$.*

Usually, for Hamiltonian systems of dimension $2m$, S is the matrix

$$S = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}.$$

Because of the use of the one parameter family $U(t)$, this definition requires continuity of time. In the current literature, one often relates the TRS to the property of the hamiltonian function $H(y)$ to be invariant under the map S, i.e. $H(y) = H(Sy)$ (*S-symmetry property*), which implies that $y(t)$ and $Sy(-t)$ satisfy the same equation. It is not clear, however, to what extent the S-symmetry property is equivalent to the TRS as defined in Definition 1. Certainly it does for linear problems. It remains doubtful whether it can be extended to non linear systems, especially in the case of discrete time where the one parameter map cannot be used. In our opinion, the definition which eliminates most of the doubts and can also be extended to discrete systems, is a variation of the one given in [4, p. 234]:

Definition 3 *A continuous system, satisfying the S-symmetry property, has TRS when in his phase space there exist* bounded simply connected *orbits which contain both $y(t)$ and $Sy(-t)$, for all $t \in (t_0, +\infty)$. In other words, if $y_0 \in \mathrm{Ker}(I - S)$, such orbits contain both the past and the future solutions.*

For two dimensional systems, this implies that there exist periodic solutions (closed orbits). In the case of the nonlinear pendulum, the phase plane contains both closed and open orbits (see Figure 1): only for the former orbits Definition 1 is clearly applicable, for, the motion on an open orbit implies the growth (or the decrease) of one variable and this

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makes recognizable the past from the future. The above definition of TRS has the advantage to be easily extendible to discrete systems. However, when passing from continuous to discrete time, there are many differences, despite the similarity in the definition of TRS. In fact, in the discrete case closed orbits are sets of discrete points and no more closed and smooth curves (as in the continuous case) even though, with the naked eye, this seems still to be the case for orbits corresponding to quasi-periodic trajectories, or periodic trajectories with a very long period. Moreover, solutions may interlace, i.e. different solutions may look as if they intersect, or they seem to coincide whereas they have only the starting point in common. In Figure 1 the circles denote the future solution, while the stars denote the past solution. For the internal orbit, since the solution is not periodic, they do not have common points, except for the initial one. On the contrary, all the points of the two solutions coincide when they are periodic.

2. APPLICATIONS TO NUMERICAL ANALYSIS

How the previous considerations may help in eliminating the energy drift, which is sometimes observed when integrating Hamiltonian systems with periodic solutions? Suppose to have methods which are able to generate, for some values of the stepsize h, discrete periodic solutions of period $N(h)$. Then, of course, no drift may appear even when integrating over very large time intervals. The problem is then reduced to use such methods to get periodic solutions. Methods able to reproduce periodic solutions, at least for linear problems, are known and they are called *symmetric methods*, which can be derived either by local arguments or by more global ones, the latter ones strictly related to TRS (see, for example, [5]). We wish to outline, at this point, that the drift is not a numerical effect, since it appears every time one tries to approximate a periodic solution, even with continuous functions. This point will be clarified in the full paper. The problem is now to find the values of h which provide periodic orbits, if any. In order to make the arguments as simple as possible, we shall refer to the trapezoidal rule. The answer looks simple for problems satisfying the S-symmetry property $H(y) = H(Sy)$ such as, for example, the nonlinear pendulum. In such a case, in fact, there are periodic solutions of any period, as h tends to 0: for any N, in fact, there are many values of h providing periodic solutions of period N. Moreover, as h tends to 0, whatever neighborhood of the stepsize h is considered, it will contain values of the stepsize corresponding to almost all periods. In other words, this means that it is likely to always approximate a periodic solution. What about the counterexample given in [2]? In that paper the hamiltonian

$$H(y) \equiv H(q,p) = T(p) + U(q), \quad T(p) = \frac{p^3}{3} - \frac{p}{2}, \quad U(q) = \frac{q^6}{30} + \frac{q^4}{4} - \frac{q^3}{3} + \frac{1}{6}, \tag{1}$$

was used as example of a non TRS system (thus showing a drift in the numerical hamiltonian) while the hamiltonian

$$H_1(y) \equiv H_1(q,p) = (T(p) + U(q))(T(-p) + U(q)) \tag{2}$$

was used as example of TRS system showing the drift as well. In Figure 2 it is shown that when a periodic solution is approximated (see left figure), no drift arise for the hamiltonian $H(y)$ (see right figure). By the way, we observe that this is a fortunate situation where the periodic solution is attractive for the dynamical system defined by the trapezoidal rule, but this opens a different question. Concerning $H_1(y)$, it nicely evidentiates the difference between our Definition 3 of TRS and the currently used S-symmetry property on which Definition 2 relies. As matter of fact, the orbit is made of *two separate branches* (see Figure 3) and then when $y(t)$ moves on one of them, $Sy(-t)$ moves on the other branch, even if they satisfy the same equation. Note that the system defined by $H_1(y)$, while satisfying Definition 2, does not satisfy Definitions 3 and 1, because it is evident that it would be recognizable on which branch the motion is occurring.

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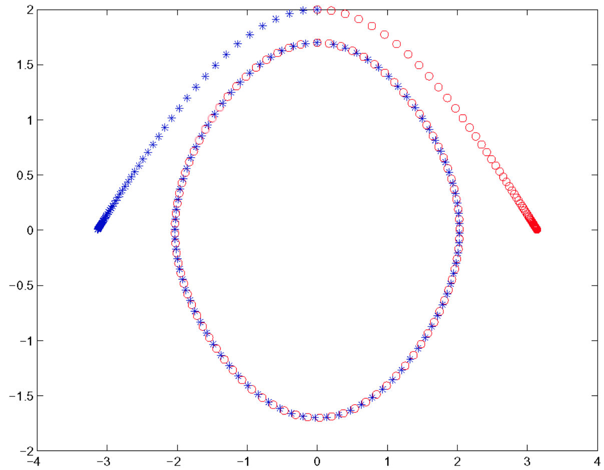

FIGURE 1. Orbits for the nonlinear pundulum: red circles future solutions; blue stars past solutions.

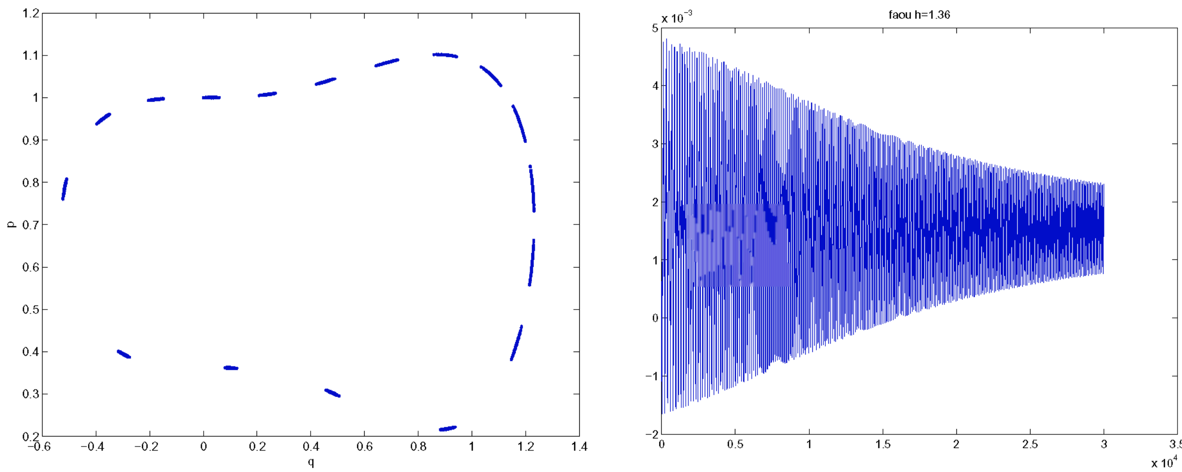

FIGURE 2. Almost periodic orbit for (1) (left picture) generating no drift in the numerical hamiltonian (right picture).

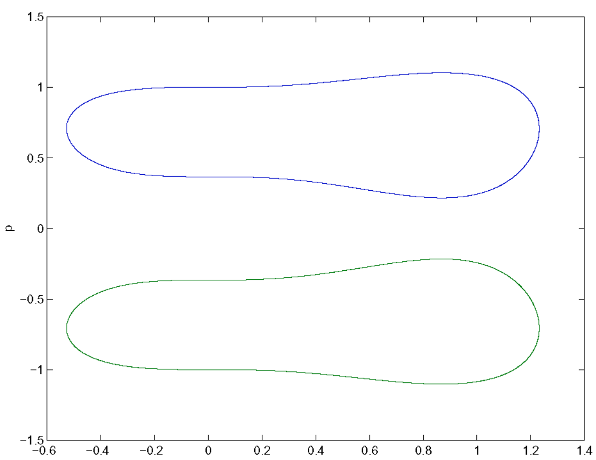

FIGURE 3. Different branches for the orbit corresponding to $H_1(y) = 0$ (see (1)-(2)).