BIT Numer Math
DOI 10.1007/s10543-009-0229-9

The BS class of Hermite spline quasi-interpolants on nonuniform knot distributions

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Received: 6 February 2009 / Accepted: 3 April 2009

Abstract The BS Hermite spline quasi-interpolation scheme is presented. It is related to the continuous extension of the BS linear multistep methods, a class of Boundary Value Methods for the solution of Ordinary Differential Equations. In the ODE context, using the numerical solution and the associated numerical derivative produced by the BS methods, it is possible to compute, with a local approach, a suitable spline with knots at the mesh points collocating the differential equation at the knots and having the same convergence order as the numerical solution. Starting from this spline, here we derive a new quasi-interpolation scheme having the function and the derivative values at the knots as input data. When the knot distribution is uniform or the degree is low, explicit formulas can be given for the coefficients of the new quasi-interpolant in the B-spline basis. In the general case these coefficients are obtained as solution of suitable local linear systems of size $2d \times 2d$, where d is the degree of the spline. The approximation order of the presented scheme is optimal and the numerical results prove that its performances can be very good, in particular when suitable knot distributions are used.

Keywords Splines · B-splines · Quasi-interpolation · Linear multistep methods

Mathematics Subject Classification (2000) 65D07 · 65D15 · 65L06

Communicated by Tom Lyche.

Work developed within the project "Numerical methods and software for differential equations".

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Published online: 29 May 2009

1 Introduction

Univariate spline Quasi-Interpolants (QIs) are function approximations with the following general form,

$$Q_d y = \sum_{j \in J} \mu_j(y) B_j,$$

where $\{B_j, j \in J\}$ is the B-spline basis of some space of splines of degree d on a bounded interval I defined by some partition π of I. The $\mu_j, j \in J$, are local linear functionals for the definition of which several different approaches have been considered in the literature, producing functionals of *Differential type* (DQIs) (see e.g. [2, 4]), of *integral type* (iQIs) (see e.g. [15, 18]), or of *discrete type* (dQIs) (see e.g. [7, 17]). We refer to [16] and to references mentioned therein for recent developments on univariate quasi-interpolation and for its interesting extension to the multivariate setting.

Even if many different types of spline quasi-interpolation schemes exist, they all share *locality* as common denominator and this is the reason why quasi-interpolation is usually preferred to interpolation or to least-squares approximation whenever the computational cost is fundamental. This is for instance the case when real-time processing of large streams of data is required and also when it is assumed that the input information on the function y can be dynamically updated (and consequently also the spline approximation is). Our specific interest for spline quasi-interpolation was in particular motivated by the necessity of associating with the numerical solution produced by a Boundary Value Method (BVM) for Ordinary Differential Equations (see [1] for a general introduction to BVMs) an easy to compute continuous approximation having the same convergence order [12]. Observe that, when dealing with numerical methods for ODEs, the values at some mesh points of (an approximation of) y are available together with the associated values of (an approximation of) y'. This is the reason why here we focus our attention on DQIs and in particular on a scheme requiring the knowledge of y and y' only at the knot set. Now, for one class of BVMs, namely the BS methods, it is possible to compute a spline Hermite interpolation scheme locally, that is to use a local procedure for associating the numerical solution and the corresponding numerical derivative produced by such methods with a spline with knots at the mesh points and there collocating the differential equation [10, 11, 13]. In this paper we prove that the local approach introduced in [13] for the definition of such spline defines a general Hermite spline quasi-interpolation scheme when the input data are not necessarily produced by the BS methods for ODEs. In particular, we prove that our quasi-interpolation operator is a projector in the space of C^{d-1} splines of degree d and that it has optimal approximation order $p = d + 1$ when $y \in C^{d+1}(I)$, provided that two positive (lower and upper) bounds exist for the ratios between successive mesh sizes. Considering that the quasi-interpolation approach here introduced is strictly related to BS methods, we call the new scheme BS Hermite quasi-interpolant. A preliminary application to differential equations of the strategy here proposed has been considered in [12], where we dealt with another important class of BVMs, the Top Order Methods. On the other hand, the goal of this paper is to present our approach in the general setting of quasi-interpolation because we believe that it can be useful not only for the specific application to differential equations but also for different applications.

The paper is organized as follows. In Sect. 2 we give a general introduction to BS methods and then, in Sect. 3, after some necessary preliminaries on the adopted notation, the constructive approach used for defining the new quasi-interpolation operator $Q_d^{(BS)}$ is introduced in the general case of a nonuniform partition. Section 4 is devoted to the analysis of its convergence behaviour. Finally, in Sect. 5 the results of some numerical experiments are reported in order to check the performances of the new quasi-interpolation scheme.

2 The BS methods

The BS methods are a recently studied class [10, 11] of Boundary Value Methods for ODEs which have been efficiently implemented for the numerical solution of Boundary Value Problems [13]. BVMs are Linear Multistep Methods which are combined with a specific number of initial and final additional methods in order to equip the full scheme with good stability features which can be absent or poor when it is used as a traditional Initial Value Method (see [1] for a general introduction to Boundary Value Methods). In particular, the k-step BS method is correctly used as BVM if it is combined with $k_1 - 1$ left and k_2 right additional methods, where $k_1 := \lceil \frac{k}{2} \rceil$ and $k_2 := \lfloor \frac{k}{2} \rfloor$, [10]. Even if clearly a suitable choice of the additional methods is necessary for a good behaviour of the numerical scheme, in the following we concentrate only on the *main* BS method because the additional methods are not of specific interest for this paper (the interested reader can refer to [11]). If $y'(x) = f(x, y(x)), x \in [a, b]$, is the considered differential equation (associated with suitable boundary conditions) and $\pi := \{a = x_0 < x_1 < \cdots < x_N = b\}$ denotes any fixed mesh in the integration interval, the numerical solution $\{y_i, i = 0, \ldots, N\}$ computed using the k-step BS method satisfies the following main equations:

$$\sum_{j=-k_1}^{k_2} \alpha_{j+k_1}^{(i)} y_{i+j} = h_i \sum_{j=-k_1}^{k_2} \beta_{j+k_1}^{(i)} f_{i+j}, \quad i = k_1, \ldots, N - k_2, \tag{1}$$

where $h_i := x_i - x_{i-1}$, $f_l := f(x_l, y_l)$ and $\boldsymbol{\alpha}^{(i)} := (\alpha_0^{(i)}, \ldots, \alpha_k^{(i)})^T$ and $\boldsymbol{\beta}^{(i)} := (\beta_0^{(i)}, \ldots, \beta_k^{(i)})^T$, $i = k_1, \ldots, N - k_2$, are the coefficient vectors characterizing the linear multistep BS method, used with variable mesh size.

Only in the special case of uniform meshes the vectors $\boldsymbol{\alpha}^{(i)}$ and $\boldsymbol{\beta}^{(i)}$ do not depend on i; in this case their components are a priori known and defined as follows,

$$\alpha_j := B'(k - j + 1), \qquad \beta_j := B(k - j + 1), \quad j = 0, \ldots, k, \tag{2}$$

where $B(x)$ here denotes the $k + 1$ degree B-spline with integer knots $0, \ldots, k + 2$. In Table 1 the α and β coefficients of the BS methods with $k = 1, \ldots, 5$ are reported. For $k = 1$ the method corresponds to the trapezoidal rule, for $k = 2$ to the Simpson rule.

Table 1 The α and β coefficients of the BS methods with $k = 1, \ldots, 5$

k	α						β					
1	-1	1					$\frac{1}{2}$	$\frac{1}{2}$				
2	$-\frac{1}{2}$	0	$\frac{1}{2}$				$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$			
3	$-\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$			$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$		
4	$-\frac{1}{24}$	$-\frac{5}{12}$	0	$\frac{5}{12}$	$\frac{1}{24}$		$\frac{1}{120}$	$\frac{13}{60}$	$\frac{11}{20}$	$\frac{13}{60}$	$\frac{1}{120}$	
5	$-\frac{1}{120}$	$-\frac{5}{24}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{24}$	$\frac{1}{120}$	$\frac{1}{720}$	$\frac{19}{240}$	$\frac{151}{360}$	$\frac{151}{360}$	$\frac{19}{240}$	$\frac{1}{720}$

In the general nonuniform case, each couple of vectors $\boldsymbol{\alpha}^{(i)}$ and $\boldsymbol{\beta}^{(i)}$ has to be computed solving the following linear system of size $(2k+2) \times (2k+2)$,

$$G^{(i)}(\boldsymbol{\alpha}^{(i)T}, \boldsymbol{\beta}^{(i)T})^T = \mathbf{e}_{2k+2}, \tag{3}$$

where $\mathbf{e}_{2k+2} = (0, \ldots, 0, 1)^T \in \mathbb{R}^{2k+2}$ and

$$G^{(i)} := \begin{bmatrix} A_1^{(i-k_1)T} & -h_i A_2^{(i-k_1)T} \\ \mathbf{0}^T & \mathbf{e}^T \end{bmatrix}, \tag{4}$$

with $\mathbf{e} := (1, \ldots, 1)^T \in \mathbb{R}^{k+1}$, and $A_1^{(j)}$, $A_2^{(j)}$, $j \in \mathbb{N}$, defined as,

$$\begin{aligned} A_1^{(j)} &:= \begin{bmatrix} B_{j-k-1}(x_j), & \ldots, & B_{j+k-1}(x_j) \\ \vdots & \vdots & \vdots \\ B_{j-k-1}(x_{j+k}), & \ldots, & B_{j+k-1}(x_{j+k}) \end{bmatrix}_{(k+1)\times(2k+1)}, \\ A_2^{(j)} &:= \begin{bmatrix} B'_{j-k-1}(x_j), & \ldots, & B'_{j+k-1}(x_j) \\ \vdots & \vdots & \vdots \\ B'_{j-k-1}(x_{j+k}), & \ldots, & B'_{j+k-1}(x_{j+k}) \end{bmatrix}_{(k+1)\times(2k+1)}, \end{aligned} \tag{5}$$

where $B_j(x)$, $j = -(1+k), \ldots, N-1$, denote the B-spline basis of degree $k+1$ with extended knot vector $\{x_{-1-k}, \ldots, x_{-1}, x_0, \ldots, x_N, x_{N+1}, \ldots, x_{N+k+1}\}$. In [11] it has been proved that the local matrix $G^{(i)}$ is always non singular[1] and an efficient algorithm for the solution of (3) is given.

Observe that the local truncation error $\tau^{(i)}$ of the main k-step BS method is defined as follows,

$$\tau^{(i)} := \sum_{j=-k_1}^{k_2} \alpha_{j+k_1}^{(i)} y(x_{i+j}) - h_i \sum_{j=-k_1}^{k_2} \beta_{j+k_1}^{(i)} f(x_{i+j}, y(x_{i+j})), \quad i = k_1, \ldots, N-k_2, \tag{6}$$

[1] In [11] distinct auxiliary knots are assumed. However the proof of the nonsingularity of $G^{(j)}$ reported in Corollary 1 in Appendix of that paper does not depend on such assumption and can be repeated also when the auxiliary knots are coincident.

where $y(x)$ is the exact solution of the differential problem. In [10, 11] it has been proved that $\tau^{(i)}$ for the k-step BS method is $O(h^{k+2})$, where h is the maximum mesh size; thus, if a suitable selection of the additional methods is done, we can deduce that the full scheme in (1) has approximation order $p = k + 1$. The stability features of the BS methods are optimal, as proved in [10], and this is a fundamental point for a robust numerical solution of differential problems; however, the characterizing feature of such class of methods is that it is possible and easy to determine a spline of degree $k + 1$ and with knots at the mesh points, $s = \sum_{i=-1-k}^{N-1} c_i B_i$, verifying all the following Hermite interpolation conditions (and, as a consequence, collocating the differential equation at the knots),

$$s(x_i) = y_i, \qquad s'(x_i) = f_i, \quad i = 0 \dots, N. \tag{7}$$

We refer to [13] for the proof of this result, but we recall here the local approach which can be used for the spline computation because this will be useful to derive the new quasi-interpolation scheme. If we put $d := k + 1$[2] and we denote as $s^{(j)}$, $j = -1, \dots, N - d$, the restriction of s to the interval $[x_{j+1}, x_{j+d}]$, (i.e. $s^{(j)} = \sum_{i=j+1-d}^{j-1+d} c_i B_i$), (7) implies that

$$\begin{cases} s^{(j)}(x_i) = y_i, & j+1 \le i \le j+d, \\ \frac{ds^{(j)}}{dx}(x_i) = f_i, & j+1 \le i \le j+d, \end{cases} \tag{8}$$

which can be recast in matrix form deducing that

$$G^{(j+k_1+1)T} \hat{\mathbf{c}}^{(j)} = (y_{j+1}, \dots, y_{j+d}, -h_{j+k_1+1} f_{j+1}, \dots, -h_{j+k_1+1} f_{j+d})^T, \tag{9}$$

where $\hat{\mathbf{c}}^{(j)} = (c_{j+1-d}, \dots, c_{j-1+d}, 0)^T \in \mathbb{R}^{2d}$.

Denoting as $\hat{\boldsymbol{\alpha}}^{(j,r)} = (\hat{\alpha}_1^{(j,r)}, \dots, \hat{\alpha}_{k+1}^{(j,r)})^T$ and $\hat{\boldsymbol{\beta}}^{(j,r)} = (\hat{\beta}_1^{(j,r)}, \dots, \hat{\beta}_{k+1}^{(j,r)})^T$ two vectors belonging to $\mathbb{R}^{k+1}$ such that,

$$G^{(j+k_1+1)} (\hat{\boldsymbol{\alpha}}^{(j,r)T}, \hat{\boldsymbol{\beta}}^{(j,r)T})^T = \mathbf{e}_r, \quad 1 \le r \le 2d, \tag{10}$$

(thus, in particular, it is $\hat{\boldsymbol{\alpha}}^{(j,2d)} = \boldsymbol{\alpha}^{(j+k_1+1)}$ and $\hat{\boldsymbol{\beta}}^{(j,2d)} = \boldsymbol{\beta}^{(j+k_1+1)}$), we can observe that,

$$(\hat{\boldsymbol{\alpha}}^{(j,r)T}, \hat{\boldsymbol{\beta}}^{(j,r)T}) G^{(j+k_1+1)T} \hat{\mathbf{c}}^{(j)} = \begin{cases} c_{j-d+r}^{(j)}, & \text{if } r < 2d, \\ 0, & \text{if } r = 2d. \end{cases}$$

[2] Observe that, throughout the paper, we use not only the integer d denoting the spline degree but also the superfluous integer $k = d - 1$ denoting the spline smoothness. This use is done in the following sections mainly for brevity reasons; on the other hand, at the beginning of this section, k is preferred to d because it can be interpreted as step number in the BVM setting.

Thus, the following local approach for the computation of the spline coefficients $c_j, j=-d,\ldots,N-1$, is suggested in [13],

$$\begin{cases} c_j = \sum_{i=1}^d \hat{\alpha}_i^{(-1,j+d+1)} y_{i-1} - h_{k_1} \sum_{i=1}^d \hat{\beta}_i^{(-1,j+d+1)} f_{i-1}, \\ \quad j=-d,\ldots,-2, \\ c_j = \sum_{i=1}^d \hat{\alpha}_i^{(j,d)} y_{i+j} - h_{j+k_1+1} \sum_{i=1}^d \hat{\beta}_i^{(j,d)} f_{i+j}, \\ \quad j=-1,\ldots,\hat{N}, \\ c_j = \sum_{i=1}^d \hat{\alpha}_i^{(\hat{N},j-\hat{N}+d)} y_{\hat{N}+i} - h_{N-k_2} \sum_{i=1}^d \hat{\beta}_i^{(\hat{N},j-\hat{N}+d)} f_{\hat{N}+i}, \\ \quad j=\hat{N}+1,\ldots,N-1, \end{cases} \tag{11}$$

where the symbol $\hat{N}=N-d$ has been used for brevity reasons. Under the assumption of quasi-uniform meshes (i.e. meshes such that for all i it is $m \le h_i/h_{i+1} \le M$, where m and M are two assigned positive constants) in [13] it has been proved that, when the solution of the differential problem is C^{d+1} smooth, the spline s extending the numerical solution produced by the k-step BS method has convergence order $O(h^d)$.

The quasi-interpolation approach introduced in this paper takes the local formulas in (11) as a tool for defining a spline of degree d approximating a function $y(x)$, assuming as input data its values and derivative values on the mesh π. Observe that, in the general case, we can not determine the spline by requiring all the conditions in (7) because the dimension of the spline space is only $N+d$; this fact can be also interpreted locally observing that, in general it is not true that the solution of the local system in (9) has a vanishing last component. On the other hand, the formulas given in (11) can still be used for defining the coefficients of a d degree good quasi-interpolant approximation of $y(x)$, as it will be shown in the next section.

3 The BS Hermite spline quasi-interpolant

In this section we introduce our BS Hermite spline quasi-interpolation scheme which approximates on an interval $[a,b]$ a function y which is known together with its derivative at $N+1$ mesh points, $\pi=\{x_0,\ldots,x_N\}$ with $a=x_0<\cdots<x_N=b$. The data, that is the function and derivative values at the mesh points, are denoted as $y_0,\ldots,y_N$ and $f_0,\ldots,f_N$, respectively. The spline here defined belongs to $S_{d,\pi}$, i.e. to the linear space of all polynomial splines of degree d defined in the interval $[a,b]$ with knots in $\pi=\{x_0,\ldots,x_N\}$, and with smoothness $C^k[a,b]$, with $k=d-1$. As in the previous section, the B-spline basis of $S_{d,\pi}$ is denoted as $B_j(x)$, $j=-d,\ldots,N-1$, and the associated extended knot vector is $\{x_{-d},\ldots,x_{-1},x_0,\ldots,x_N,x_{N+1},\ldots,x_{N+d}\}$. By using this notation, our quasi-interpolant spline is represented as follows,

$$Q_d^{(BS)}(y) = \sum_{j=-d}^{N-1} \mu_j^{(BS)}(y) B_j, \tag{12}$$

where $\mu_j^{(BS)}(y)$ are the local linear combinations of function and derivative values already introduced in (11) which we report here for clarity reasons,

$$\begin{cases} \mu_j^{(BS)}(y) = \sum_{i=1}^d \hat{\alpha}_i^{(-1,j+d+1)} y_{i-1} - h_{k_1} \sum_{i=1}^d \hat{\beta}_i^{(-1,j+d+1)} f_{i-1}, \\ \quad j = -d, \ldots, -2, \\ \mu_j^{(BS)}(y) = \sum_{i=1}^d \hat{\alpha}_i^{(j,d)} y_{i+j} - h_{j+k_1+1} \sum_{i=1}^d \hat{\beta}_i^{(j,d)} f_{i+j}, \\ \quad j = -1, \ldots, \hat{N}, \\ \mu_j^{(BS)}(y) = \sum_{i=1}^d \hat{\alpha}_i^{(\hat{N},j-\hat{N}+d)} y_{\hat{N}+i} - h_{N-k_2} \sum_{i=1}^d \hat{\beta}_i^{(\hat{N},j-\hat{N}+d)} f_{\hat{N}+i}, \\ \quad j = \hat{N}+1, \ldots, N-1, \end{cases} \tag{13}$$

where $\hat{N} = N - d$ and the vectors $\hat{\boldsymbol{\alpha}}^{(j,r)} = (\hat{\alpha}_1^{(j,r)}, \ldots, \hat{\alpha}_d^{(j,r)})^T$ and $\hat{\boldsymbol{\beta}}^{(j,r)} = (\hat{\beta}_1^{(j,r)}, \ldots, \hat{\beta}_d^{(j,r)})^T$ are defined as solution of the linear system in (10).

In the next subsection we give an interpretation of the above formulas within the general context of quasi-interpolation and in the following one we report their explicit analytic expression for low degree cases. The study of the convergence behaviour of the associated quasi-interpolant $Q_d^{(BS)}(y)$ is developed in Sect. 4.

3.1 Interpretation of the functional definition

In this subsection we show how the scheme introduced in (12) and (13) can be interpreted and explained by using a common strategy in the quasi-interpolation setting (see for example [6, 8]) which is based on the use of local projectors. In particular we outline that each $\mu_j^{(BS)}(y)$ can be associated with a suitable local function which approximates $y|_{I_j}$, where $I_j \subset [a,b]$ is fixed as follows,

$$I_j = \begin{cases} [x_{j+1,}, x_{j+d}], & \text{if } -1 \le j \le N-d, \\ [x_0, x_{d-1}], & \text{if } -d \le j \le -2, \\ [x_{N-d+1}, x_N], & \text{if } N-d+1 \le j \le N-1. \end{cases}$$

The local function defined in I_j is a spline denoted as $BSQ^{(j)}$ belonging to the space S_{d,N_j} of the (local) splines defined in I_j with degree d and knots at the d inner active knots of B_j, $j = -1, \ldots, N-d$. Consequently, such local spline can be represented as follows,

$$BSQ^{(j)}(x) = \sum_{s=j-d+1}^{j+d-1} c_s^{(j)} B_s(x), \quad \text{if } -1 \le j \le N-d, \tag{14}$$

and it is assumed that

$$\begin{aligned} BSQ^{(j)}(x) &= BSQ^{(-1)}(x), \quad \text{if } -d \le j \le -2, \\ BSQ^{(j)}(x) &= BSQ^{(N-d)}(x), \quad \text{if } N-d+1 \le j \le N-1. \end{aligned}$$

The $(2d-1)$ coefficients defining in (14) each local approximation $BSQ^{(j)}(x)$ in the corresponding local B-spline basis are obtained by requiring the following conditions,

$$\begin{cases} BSQ^{(j)}(x_i) = y_i, & j+1 \le i \le j+d, \\ \frac{dBSQ^{(j)}}{dx}(x_i) = f_i + \frac{\tau^{(j+k_1+1)}}{h_{j+k_1+1}}, & j+1 \le i \le j+d, \end{cases} \tag{15}$$

where we remind that $k_1 := \lceil \frac{d-1}{2} \rceil$ and $h_i := x_i - x_{i-1}$; the symbol $\tau^{(j+k_1+1)}$ denotes an additional necessary unknown; in fact one can observe that in (15) $2d$ conditions are required but the coefficients defining $BSQ^{(j)}(x)$ are only $2d-1$. We can also remark that this system is similar to the local system (8), except for the additional unknown $\tau^{(j+k_1+1)}$ and we will show at the end of this subsection that, if the function $y(x)$ is sufficiently smooth in $[a,b]$, $\tau^{(j+k_1+1)}$ is $O(h^{d+1})$, where h is the maximal mesh size.

The equations in (15) can be recast in matrix form by using the local matrix $G^{(j+k_1+1)}$ introduced in (4), as follows,

$$G^{(j+k_1+1)T}\hat{\mathbf{c}}^{(j)} = (y_{j+1}, \ldots, y_{j+d}, -h_{j+k_1+1} f_{j+1}, \ldots, -h_{j+k_1+1} f_{j+d})^T, \tag{16}$$

where now $\hat{\mathbf{c}}^{(j)} := (c_{j-d+1}^{(j)}, \ldots, c_{j+d-1}^{(j)}, \tau^{(j+k_1+1)})^T$.

By using again the vectors $\hat{\boldsymbol{\alpha}}^{(j,r)}$, $\hat{\boldsymbol{\beta}}^{(j,r)}$ introduced in (10), we can write,

$$(\hat{\boldsymbol{\alpha}}^{(j,r)T}, \hat{\boldsymbol{\beta}}^{(j,r)T}) G^{(j+k_1+1)T}\hat{\mathbf{c}}^{(j)} = \begin{cases} c_{j-d+r}^{(j)}, & \text{if } r < 2d, \\ \tau^{(j+k_1+1)}, & \text{if } r = 2d, \end{cases} \tag{17}$$

and such formula, considering also (16), implies that,

$$c_{j-d+r}^{(j)} = \sum_{i=1}^{d} \hat{\alpha}_i^{(j,r)} y_{j+i} - h_{j+k_1+1} \sum_{i=1}^{d} \hat{\beta}_i^{(j,r)} f_{j+i}, \quad r = 1, \ldots, 2d-1. \tag{18}$$

Then we can conclude that our functional values $\mu_j^{(BS)}(y)$ can be interpreted as follows,

$$\mu_j^{(BS)}(y) = \begin{cases} c_j^{(-1)}, & -d \le j \le -2, \\ c_j^{(j)}, & -1 \le j \le N-d, \\ c_j^{(N-d)}, & N-d+1 \le j \le N-1, \end{cases} \tag{19}$$

that is each inner $\mu_j^{(BS)}(y)$ can be associated with the central coefficient of the corresponding local approximation $BSQ^{(j)}$ and the left (right) $(d-1)$ boundary ones can be associated with the first (last) $(d-1)$ coefficients of $BSQ^{(-1)}$ $(BSQ^{(N-d)})$.

Let us now analyze more deeply the meaning of the additional unknown $\tau^{(j+k_1+1)}$ appearing in (15) and consequently in (16). For this aim, first we observe that (16) and (17) imply also that $\tau^{(j+k_1+1)}$ can be expressed as follows,

$$\tau^{(j+k_1+1)} = \sum_{i=1}^{d} \hat{\alpha}_i^{(j,2d)} y_{j+i} - h_{j+k_1+1} \sum_{i=1}^{d} \hat{\beta}_i^{(j,2d)} f_{j+i}. \tag{20}$$

Then, by using a general result about the approximation order of the BS methods for ODEs proved in [11], we can state the following proposition.

Proposition 1 *The values of* $\tau^{(j+k_1+1)}$, $j = -1, \ldots, N-d$, *are the local truncation errors of the* k*-step BS linear multistep method constructed on the knots* $x_{j+1}, \ldots, x_{j+d}$. *As a consequence, the following properties are satisfied*:

1. *If* $y \in C^{d+1}[a,b]$, *then* $\tau^{(j+k_1+1)} = O(h^{d+1})$, *where* h *denotes the maximal mesh size*;
2. *If* y *is a polynomial of degree* d, *then* $\tau^{(j+k_1+1)} = 0$;
3. *If* $y \in S_{d,\pi}$, *then* $\tau^{(j+k_1+1)} = 0$.

Proof First, observe that (10) and (3) imply that $\boldsymbol{\alpha}^{(j+k_1+1)} = \hat{\boldsymbol{\alpha}}^{(j,2d)}$ and $\boldsymbol{\beta}^{(j+k_1+1)} = \hat{\boldsymbol{\beta}}^{(j,2d)}$. Thus, considering (6) and relating to the differential equation $y' = f$, we can say that $\tau^{(j+k_1+1)}$ in (20) is the local truncation error associated with the corresponding linear k-step BS scheme. In [11] it has been proved that such method satisfies the order conditions with order $p = k+1$ $(= d)$. This immediately implies the first two statements of this proposition. The third one is a consequence of the fact that if $y \in S_{d,\pi}$, its values and its derivative values at the mesh points verify (1). As a consequence, from the analysis reported in the previous section, we can infer that the last component of the solution of (16) vanishes if $y \in S_{d,\pi}$, that is $\tau^{(j+k_1+1)} = 0$. □

Summarizing, we can conclude that the determination of the coefficients of $Q_d^{(BS)}(y)$ in the B-spline basis requires the solution of all the local systems (10) for $j = -1$, $r = 1, \ldots, d$, for $0 \le j \le N-d-1$, $r = d$ and for $j = N-d$, $r = d, \ldots, 2d-1$. Thus, the computational cost is related to the solution of these $N+d$ linear systems of size $2d \times 2d$ which can be solved in an efficient and stable way using the algorithm described in [11] for general nonuniform meshes.

In the following subsection the explicit analytic expression of our quasi-interpolant for some splines of low degrees is reported. Observe that, when a uniform mesh π is used, the inner coefficient vectors $\hat{\boldsymbol{\alpha}}^{(j,d)}$ and $\hat{\boldsymbol{\beta}}^{(j,d)}$ do not depend on j, $j = -1, \ldots, N-d$.

3.2 Low degree cases

When low degree cases are considered, even if general nonuniform knot distribution are assumed, it is possible to compute the analytic expression of all the $\mu_j^{(BS)}(y)$ defining our quasi-interpolant with the help of symbolic computation. Tables 2 and 3 report such expressions for $d = 2$ and $d = 3$ and we can observe that for $d = 2$ they do not depend on the knot distribution while for $d = 3$ each $\mu_j^{(BS)}(y)$ only depends

Table 2 Values of μ_j for $d=2$. General knot distribution

$$\mu_j^{(BS)}(y) = \tfrac{1}{2}(y_{j+1}+y_{j+2}) - \tfrac{1}{4}h_{j+1}(-f_{j+1}+f_{j+2}), \quad j=-1,\ldots,N-2,$$
$$\mu_{-2}^{(BS)}(y) = y_0, \qquad \mu_{N-1}^{(BS)}(y) = y_N$$

Table 3 Values of μ_j for $d=3$. General knot distribution. $R_j = h_{j+3}/h_{j+2}$

$$\begin{aligned}
\mu_j^{(BS)}(y) &= \tfrac{1}{3}\big(-\tfrac{R_j(2+R_j)}{1+R_j}y_{j+1} + \tfrac{R_j^2+4R_j+1}{R_j}y_{j+2} - \tfrac{1+2R_j}{R_j(1+R_j)}y_{j+3}\big) \\
&\quad - \tfrac{1}{9}h_{j+2}\big(\tfrac{R_j(2+R_j)}{1+R_j}f_{j+1} + (1-R_j)f_{j+2} - \tfrac{1+2R_j}{1+R_j}f_{j+3}\big), \quad j=-1,\ldots,N-3, \\
\mu_{-3}^{(BS)}(y) &= y_0, \qquad \mu_{N-1}^{(BS)}(y) = y_N, \\
\mu_{-2}^{(BS)}(y) &= \tfrac{1}{3}\big(\tfrac{3+2R_{-1}}{1+R_{-1}}y_0 + \tfrac{R_{-1}-1}{R_{-1}}y_1 + \tfrac{1}{R_{-1}(1+R_{-1})}y_2\big) \\
&\quad - \tfrac{1}{9}h_1\big(-\tfrac{3+2R_{-1}}{1+R_{-1}}f_0 + 2f_1 + \tfrac{1}{1+R_{-1}}f_2\big), \\
\mu_{N-2}^{(BS)}(y) &= \tfrac{1}{3}\big(\tfrac{R_{N-3}^2}{1+R_{N-3}}y_{N-2} + (1-R_{N-3})y_{N-1} + \tfrac{2+3R_{N-3}}{1+R_{N-3}}y_N\big) \\
&\quad - \tfrac{1}{9}h_{N-1}\big(-\tfrac{R_{N-3}^2}{1+R_{N-3}}f_{N-2} - 2R_{N-3}f_{N-1} + \tfrac{R_{N-3}(2+3R_{N-3})}{1+R_{N-3}}f_N\big)
\end{aligned}$$

Table 4 Values of μ_j for $d=4$. Uniform knot distribution

$$\begin{aligned}
\mu_j^{(BS)}(y) &= \tfrac{1}{12}(5y_{j+1}+y_{j+2}+y_{j+3}+5y_{j+4}) \\
&\quad - \tfrac{1}{48}h(-5f_{j+1}-41f_{j+2}+41f_{j+3}+5f_{j+4}), \quad j=-1,\ldots,N-4, \\
\mu_{-4}^{(BS)}(y) &= y_0, \qquad \mu_{N-1}^{(BS)}(y) = y_N, \\
\mu_{-3}^{(BS)}(y) &= \tfrac{1}{24}(23y_0-3y_1+3y_2+y_3) \\
&\quad - \tfrac{1}{96}h(-23f_0+11f_1+11f_2+f_3), \\
\mu_{-2}^{(BS)}(y) &= \tfrac{1}{24}(-11y_0+47y_1-7y_2-5y_3) \\
&\quad - \tfrac{1}{96}h(11f_0+41f_1-47f_2-5f_3), \\
\mu_{N-3}^{(BS)}(y) &= \tfrac{1}{24}(-5y_{N-3}-7y_{N-2}+47y_{N-1}-11y_N) \\
&\quad - \tfrac{1}{96}h(5f_0+47f_1-41f_2-11f_3), \\
\mu_{N-2}^{(BS)}(y) &= \tfrac{1}{24}(y_{N-3}+3y_{N-2}-3y_{N-1}+23y_N) \\
&\quad - \tfrac{1}{96}h(-f_{N-3}-11f_{N-2}-11f_{N-1}+23f_N)
\end{aligned}$$

on one of the ratios of consecutive mesh sizes. Table 4 reports their form for $d=4$ in the special case of uniform knot distribution. Observe that in all the reported tables we relate to the case of coincident additional knots for the definition of the boundary functionals (see also Remark 2 at the end of the following section).

4 Convergence behaviour of $Q_d^{(BS)}(y)$

Theorem 1 *The quasi-interpolation operator $Q_d^{(BS)}$ is a projector on the spline space $S_{d,\pi}$.*

Proof Keeping in mind the locality of the B-spline basis and considering the representation given in (19) of the functionals $\mu_j^{(BS)}(y)$, it is sufficient to prove that the local approximation $BSQ^{(j)}(y)$ reproduces the local spline space S_{d,N_j}, $\forall -1 \le j \le N-d$, where $N_j := \pi \cap I_j$ (e.g. see Lemma 8.2 in [8]). Thus, let us assume that, $y|_{I_j}(x) = \sum_{s=j-d+1}^{j+d-1} b_s B_s(x)$. Then, from Proposition 1, the right hand side of the local approximation problem (16) can be written as $(G^{(j+k_1+1)})^T \hat{\mathbf{b}}^{(j)}$, where $\hat{\mathbf{b}}^{(j)} := (b_{j-d+1}, \dots, b_{j+d-1}, 0)^T$. Since $G^{(j+k_1+1)}$ is non singular, we have that $\hat{\mathbf{c}}^{(j)} = \hat{\mathbf{b}}^{(j)}$. This immediately implies that $BSQ^{(j)}(y) = y|_{I_j}$. □

As a special case, the previous theorem implies that $Q_d^{(BS)}(y) = y$ when y is a polynomial of degree less or equal to d. Now, in order to derive the approximation order for the BS quasi-interpolation scheme when smooth enough functions are approximated, we need some assumption on the knot distribution. We define:

$$h := \max_{1\le s\le N} h_s, \qquad \hat{h}_i := \max_{\max(i-k_2,1)\le s\le \min(i+k_1+1,N)} h_s, \quad i = 0, \dots, N-1,$$

and the following two scalar quantities related to the coefficients used in (13):

$$\|\hat{\boldsymbol{\alpha}}\| := \max\Big(\max_{-1\le j\le N-d} \|\hat{\boldsymbol{\alpha}}^{(j,d)}\|_1, \max_{1\le r\le k} \|\hat{\boldsymbol{\alpha}}^{(-1,r)}\|_1, \max_{1\le r\le k} \|\hat{\boldsymbol{\alpha}}^{(N-d,2d-r)}\|_1 \Big),$$

$$\|\hat{\boldsymbol{\beta}}\| := \max\Big(\max_{-1\le j\le N-d} \|\hat{\boldsymbol{\beta}}^{(j,d)}\|_1, \max_{1\le r\le k} \|\hat{\boldsymbol{\beta}}^{(-1,r)}\|_1, \max_{1\le r\le k} \|\hat{\boldsymbol{\beta}}^{(N-d,2d-r)}\|_1 \Big).$$

By using this notation, we can first state two preliminary lemmas and then our main theorem concerning the approximation order of our quasi-interpolant.

Lemma 1 *Let $g \in C^1[a,b]$. Then, $\forall 0 \le i \le N-1$ and $\forall 0 \le r \le k$ there holds*

$$\|D^r Q_d^{(BS)}(g)\|_{\infty,[x_i,x_{i+1}]} \le \frac{C_{r,\pi}}{\hat{h}_i^r} \left(\|\hat{\boldsymbol{\alpha}}\| \|g\|_{\infty,[x_{i_1},x_{i_2}]} + \hat{h}_i \|\hat{\boldsymbol{\beta}}\| \|g'\|_{\infty,[x_{i_1},x_{i_2}]} \right),$$

where $i_1 = \max\{i-k, 0\}$, $i_2 = \min\{i+d, N\}$, $C_{r,\pi}$ is a suitable positive constant depending on r and on the ratios between consecutive mesh sizes, with $C_{0,\pi} = 1$.

Proof Considering that the B-splines are nonnegative and that they sum up to one, from the relation $Q_d^{(BS)}(g)|_{[x_i,x_{i+1}]} = \sum_{s=i-d}^{i} \mu_s^{(BS)}(g) B_s(x)$, we get that

$$\|Q_d^{(BS)}(g)\|_{\infty,[x_i,x_{i+1}]} \le \max_{i-d\le s\le i} |\mu_s^{(BS)}(g)|.$$

From (13), we deduce the proof for $r = 0$. For $r > 0$ the result follows by considering the recursive derivative formulas for splines expressed in the B-spline basis (see for example [3]). □

Lemma 2 *Let $x_{-d} < \cdots < x_{-1} < x_0 < \cdots < x_N < x_{N+1} < \cdots < x_{N+d}$. If there exist two positive constants, m and M, with $m \leq 1 \leq M$, such that*

$$m \leq \frac{h_i}{h_{i+1}} \leq M, \quad i = -d+1, \ldots, N+d-1, \tag{21}$$

then there exist two other positive constants $A_{m,M,d}$ and $B_{m,M,d}$ depending only on m, M and on d such that

$$\|\hat{\boldsymbol{\alpha}}\| \leq A_{m,M,d}, \quad \textit{and} \quad \|\hat{\boldsymbol{\beta}}\| \leq B_{m,M,d}.$$

Proof Observe that the recursive definition of the B-spline basis implies that, $\forall s$ with $-1 \leq s \leq N-d$, the entries of the matrix $G^{(s+k_1+1)}$ only depend on the successive ratios h_i/h_{i+1}, $i = s-d+2, \ldots, s+2d-1$. Considering that such matrix is always nonsingular [11], we get that each of the vectors $\hat{\boldsymbol{\alpha}}^{(j,r)}$ and $\hat{\boldsymbol{\beta}}^{(j,r)}$ depends continuously on the ratios between successive mesh sizes. Thus the hypothesis (21) implies the thesis of the Lemma because we can say that $\|\hat{\boldsymbol{\alpha}}\|$ and $\|\hat{\boldsymbol{\beta}}\|$ are continuous functions of all the ratios h_i/h_{i+1}, $i = -d+1, \ldots, N+d-1$. □

Lemma 3 *Let $x_{-d} = \cdots = x_{-1} = x_0 < \cdots < x_N = x_{N+1} = \cdots = x_{N+d}$. If there exist two other positive constants, m and M, with $m \leq 1 \leq M$, such that*

$$m \leq \frac{h_i}{h_{i+1}} \leq M, \quad i = 1, \ldots, N-1, \tag{22}$$

then there exist other two positive constants $A_{m,M,d}$ and $B_{m,M,d}$ depending only on m, M and on d such that

$$\|\hat{\boldsymbol{\alpha}}\| \leq A_{m,M,d}, \quad \textit{and} \quad \|\hat{\boldsymbol{\beta}}\| \leq B_{m,M,d}.$$

Proof Even in this case we can state that each matrix $G^{(j)}$ is nonsingular (see footnote 1). Then, by using arguments analogous to those used in Lemma 2, the thesis can be proved. □

Then we are ready to prove the following main result,

Theorem 2 *Let us assume the hypotheses in Lemmas 2 or 3 for the knot distribution. If $y \in C^{d+1}[a,b]$, then the r-th derivative of the approximation error of the quasi-interpolant $Q_d^{(BS)}(y)$ satisfies the following inequality,*

$$\|D^r(y - Q_d^{(BS)}(y))\|_\infty \leq L h^{d+1-r} \|D^{d+1} y\|_\infty, \quad r = 0, \ldots, k, \tag{23}$$

where L is a suitable positive constant depending on d, r and on the positive quantities m and M introduced in Lemmas 2 or 3.

Proof Let us consider the r-th derivative of the error in the interval $[x_i, x_{i+1}]$, with $0 \leq i \leq N-1$ and for the sake of brevity let us use the notation $\|\cdot\|_i := \|\cdot\|_{\infty,[x_i,x_{i+1}]}$.

We have that, if p belongs to the space Π_d of all polynomials with degree less than or equal to d, we get the following bounds:

$$\begin{aligned}\|D^r(y - Q_d^{(BS)}(y))\|_i &= \|D^r(y-p) - D^r(Q_d^{(BS)}(y) - p)\|_i \\ &\leq \|D^r(y-p)\|_i + \|D^r(Q_d^{(BS)}(y) - p)\|_i \\ &= \|D^r(y-p)\|_i + \|D^r Q_d^{(BS)}(y-p)\|_i.\end{aligned}$$

From Lemma 1 we get the following inequality,

$$\|D^r Q_d^{(BS)}(y-p)\|_i \leq \frac{C_{r,\pi}}{\hat{h}_i^r}\left(\|\hat{\boldsymbol{\alpha}}\|\|y-p\|_{\infty,[x_{i_1},x_{i_2}]} + \hat{h}_i\|\hat{\boldsymbol{\beta}}\|\|y'-p'\|_{\infty,[x_{i_1},x_{i_2}]}\right),$$

where $C_{0,\pi} = 1$ and $C_{r,\pi}$, for each $r > 0$, depends on the positive constants m and M. Now, if p is the Taylor expansion of order d of $y|_{[x_i,x_{i+1}]}$ at the point x_i, we get

$$\|D^r(y-p)\|_i \leq \rho_{d,r} h_{i+1}^{d+1-r}\|D^{d+1}y\|_i,$$

where $\rho_{d,r}$ is a positive constant depending only on d and r, with $r = 0,\ldots,d$.

In conclusion, using the above bound, we obtain the following local error estimate,

$$\|D^r(y - Q_d^{(BS)}(y))\|_i \leq [\rho_{d,r} + C_{r,\pi}(\|\hat{\boldsymbol{\alpha}}\|\rho_{d,0} + \|\hat{\boldsymbol{\beta}}\|\rho_{d,1})]\|D^{d+1}y\|_{\infty,[x_{i_1},x_{i_2}]}\hat{h}_i^{d+1-r}.$$

Taking into account the upper bounds for $\|\hat{\boldsymbol{\alpha}}\|$ and for $\|\hat{\boldsymbol{\beta}}\|$ obtained in Lemmas 2 or 3 under the assumed hypotheses on the knot distribution, this local estimate implies the global estimate in (23), where

$$L := \rho_{d,r} + C_{r,\pi}(A_{m,M,d}\rho_{d,0} + B_{m,M,d}\rho_{d,1}). \qquad \square$$

Remark 1 Even if the statement of the previous theorem is true whatever the fixed positive constants $m \leq 1 \leq M$ are, the value of the constant L in the upper bound on the error (23) deteriorates when m decreases and/or M increases. In our experience it is reasonable to require $m = M^{-1}$, $M = 2$.

Remark 2 The specific selection adopted for fixing the necessary auxiliary left and right knots (respectively $x_{-d},\ldots,x_{-1}$ and $x_{N+1},\ldots,x_{N+d}$) influence only the definition of the boundary functionals, that is only the vector coefficients $\hat{\boldsymbol{\alpha}}^{(-1,j+d+1)}$, $\hat{\boldsymbol{\beta}}^{(-1,j+d+1)}$, $j = -d,\ldots,-2$ and $\hat{\boldsymbol{\alpha}}^{(N-d,j-N+2d)}$, $\hat{\boldsymbol{\beta}}^{(N-d,j-N+2d)}$, $j = N-k,\ldots,N-1$. We observe that the choice of coincident additional knots seems preferable because it ensures that $Q_d^{(BS)}(y)(a) = y(a)$ and $Q_d^{(BS)}(y)(b) = y(b)$. In addition, at least when uniform partitions are used, the corresponding values of $\|\hat{\boldsymbol{\alpha}}\|$ and of $\|\hat{\boldsymbol{\beta}}\|$ increase when a uniform distribution of auxiliary knots with mesh size h is used.

5 Numerical results

In this section we analyze the behavior of $Q_d^{(BS)}$ for approximation by using the following two testing functions,

$$y_1(x) = e^{-x}\sin(5\pi x), \qquad [a,b] = [-1,1],$$
$$y_2(x) = \frac{\exp(-x/\sqrt{\epsilon}) - \exp((x-2)/\sqrt{\epsilon})}{(1-\exp(-2/\sqrt{\epsilon}))}, \quad \epsilon = 0.001, \qquad [a,b] = [0,1].$$

For the first test function we report only results obtained with a uniform knot distribution. In fact in this case we had not a significant gain from using nonuniform distributions. For the second test function, characterized by a boundary layer of width ϵ near $x = 0$, we report results obtained with uniform and nonuniform distributions because in this case the use of a suitable knot sequence can be very profitable. Observe that the nonuniform knot sequences considered for the reported experiments have always geometric distribution (with common ratio α).

The obtained results are summarized in Tables 5–13 where N denotes always the number of mesh steps and ϵ_d^{BS} the infinity norm (which is computed using a uniform mesh with 1000 points) of the absolute error with respect to the BS quasi-interpolant of degree d. When a uniform knot distribution is used, we can compare the accuracy of our approximation with that of the discrete spline quasi-interpolation operator presented in [17] that in the following will be referred to as Q_d (the knot set is unchanged). Thus, in all the tables where a uniform knot distribution is assumed, we report also ϵ_d which is the absolute error with respect to the Q_d quasi-interpolant.

In the general case the comparison is done with the accuracy of the approximant produced by the d-degree spline Hermite interpolation scheme based on the selection

Table 5 Approximation errors for the test function 1, $d = 3$. Uniform knots

N	ϵ_3^{BS}	r_3^{BS}	ϵ_3	r_3	ϵ_3^{HC}	r_3^{HC}	ϵ_3^{HI}	r_3^{HI}
16	2.9e−1		5.8e−1		1.1e−1		2.3e−2	
32	1.2e−2	4.6	1.0e−1	2.5	6.6e−3	4.1	2.5e−3	3.2
64	5.0e−4	4.6	5.1e−3	4.3	4.0e−4	4.0	1.8e−4	3.8
128	2.6e−5	4.2	2.8e−4	4.2	2.4e−5	4.1	1.2e−5	4.0
256	1.5e−6	4.1	1.8e−5	4.0	1.5e−6	4.0	7.2e−7	4.0
512	9.4e−8	4.0	1.1e−6	4.0	9.4e−8	4.0	4.5e−8	4.0

Table 6 Approximation errors for the test function 2, $d = 3$. Uniform knots

N	ϵ_3^{BS}	r_3^{BS}	ϵ_3	r_3	ϵ_3^{HC}	r_3^{HC}	ϵ_3^{HI}	r_3^{HI}
16	1.9e−2		8.4e−2		1.9e−2		7.4e−3	
32	1.7e−3	3.5	1.4e−2	2.6	1.7e−3	3.5	4.5e−4	4.0
64	1.3e−4	3.7	1.4e−3	3.2	1.3e−4	3.7	3.3e−5	3.7
128	8.8e−6	3.9	1.2e−4	3.6	8.8e−6	3.9	2.9e−6	3.5
256	5.8e−7	3.9	7.8e−6	3.9	5.8e−7	3.9	2.2e−7	3.7
512	3.7e−8	4.0	5.2e−7	3.9	3.7e−8	4.0	1.5e−8	3.9

Table 7 Approximation errors for the test function 2 $d=3$. Geometric knot distribution

N	α	ϵ_3^{BS}	r_3^{BS}	ϵ_3^{HC}	r_3^{HC}	ϵ_3^{HI}	r_3^{HI}
8	1.6479	2.9e−3		1.9e−3		1.3e−3	
16	1.3209	1.4e−4	3.8	9.4e−6	3.7	4.7e−6	3.8
64	1.0921	8.5e−7	3.6	8.3e−7	3.5	4.0e−7	3.5
128	1.0504	7.6e−8	3.5	7.6e−8	3.5	3.7e−8	3.5
256	1.0276	6.9e−9	3.5	6.9e−9	3.5	3.3e−9	3.5
512	1.0150	6.1e−10	3.5	6.1e−10	3.5	2.9e−10	3.5

Table 8 Approximation errors for the test function 1, $d=4$. Uniform knots

N	ϵ_4^{BS}	r_4^{BS}	ϵ_4	r_4	ϵ_4^{HI}	r_4^{HI}
16	2.7e−1		4.6e−1		4.3e−3	
32	3.6e−3	6.2	1.4e−2	5.0	6.4e−4	2.7
64	6.0e−5	5.9	3.3e−4	5.5	2.4e−5	4.7
128	1.1e−6	5.8	1.8e−5	4.2	8.0e−7	4.9
256	2.1e−8	5.6	6.3e−7	4.8	2.4e−8	5.1
512	4.5e−10	5.6	2.1e−8	4.9	7.4e−10	5.0

Table 9 Approximation errors for the test function 2, $d=4$. Uniform knots

N	ϵ_4^{BS}	r_4^{BS}	ϵ_4	r_4	ϵ_4^{HI}	r_4^{HI}
16	4.9e−3		2.2e−2		4.9e−3	
32	1.9e−4	4.7	2.3e−3	3.3	2.3e−4	4.4
64	9.3e−6	4.3	1.4e−4	4.1	8.6e−6	4.7
128	2.9e−7	5.0	6.1e−6	4.5	3.0e−7	4.9
256	8.0e−9	5.2	2.3e−7	4.7	9.0e−9	5.0
512	1.5e−10	5.8	7.8e−9	4.9	2.9e−10	5.0

Table 10 Approximation errors for the test function 2, $d=4$. Geometric knot distribution

N	α	ϵ_4^{BS}	r_4^{BS}	ϵ_4^{HI}	r_4^{HI}
8	1.6479	2.6e−3		3.0e−4	
16	1.3209	5.3e−5	5.6	2.6e−6	6.8
32	1.1697	1.6e−6	5.1	8.0e−8	5.0
64	1.0921	5.3e−8	4.9	3.2e−9	4.7
128	1.0504	1.9e−9	4.7	1.4e−10	4.5
256	1.0276	8.1e−11	4.6	5.3e−11	1.4

of an optimal knot distribution, which is available in the Matlab spline package (release 7.2) [19] and here is referred to as H_d (observe that the number of degrees of freedom used by this scheme is $2N+2$); for the details about H_d, the interested reader can refer to [5, 14]. The related absolute error is denoted in the tables as ϵ_d^{HI}. For the case $d=3$ the accuracy of our approach is also compared with that of the classical C^1 cubic spline Hermite interpolant at the knots whose knot sequence is the same

Table 11 Approximation errors for the test function 1 and its derivative, $d = 6$. Uniform knots

N	ϵ_6^{BS}	r_6^{BS}	ϵ_6^{HI}	r_6^{HI}	ϵ'^{BS}_6	r'^{BS}_6	ϵ'^{HI}_6	r'^{HI}_6
16	1.0e−1		9.5e−3		1.5e0		2.9e−1	
32	5.0e−4	7.7	1.7e−5	9.1	8.5e−3	7.5	1.2e−3	7.9
64	1.6e−6	8.3	4.0e−7	5.4	5.7e−5	7.2	5.5e−5	4.5
128	7.0e−9	7.8	3.9e−9	6.7	4.8e−7	6.9	1.0e−6	5.7
256	2.7e−11	8.0	2.7e−11	7.2	5.4e−9	6.5	1.1e−8	6.5
512	1.1e−13	7.9	2.1e−13	7.0	7.2e−11	6.2	1.3e−10	6.5

Table 12 Approximation errors for the test function 2, $d = 6$. Uniform knots

N	ϵ_6^{BS}	r_6^{BS}	ϵ_6^{HI}	r_6^{HI}	ϵ'^{BS}_6	r'^{BS}_6	ϵ'^{HI}_6	r'^{HI}_6
8	1.0e−2		3.0e−2		4.1e−1		9.7e−1	
16	2.8e−4	5.2	9.3e−4	5.0	1.5e−2	4.8	6.1e−2	4.0
32	8.5e−6	5.1	1.5e−5	6.0	9.9e−4	3.9	1.9e−3	5.0
64	1.2e−7	6.2	1.6e−7	6.5	2.1e−5	5.6	4.4e−5	5.5
128	1.1e−9	6.7	1.5e−9	6.7	3.0e−7	6.1	8.2e−7	5.7
256	7.7e−12	7.2	1.1e−11	7.1	4.0e−9	6.3	8.9e−9	6.5
512	3.7e−14	7.7	8.6e−14	7.0	5.9e−11	6.1	1.0e−10	6.4

Table 13 Approximation errors for the test function 2, $d = 6$. Geometric knot distribution

N	α	ϵ_6^{BS}	r_6^{BS}	ϵ_6^{HI}	r_6^{HI}	ϵ'^{BS}_6	r'^{BS}_6	ϵ'^{HI}_6	r'^{HI}_6
8	1.6479	2.4e−3		1.1e−2		3.4e−2		1.1e−1	
16	1.3209	1.8e−5	7.0	2.6e−6	12.0	3.2e−4	6.7	4.1e−5	11.4
32	1.1697	1.6e−7	6.9	2.2e−9	10.0	4.4e−6	6.2	2.1e−7	7.6
64	1.0921	1.5e−9	6.8	1.7e−11	7.0	8.0e−8	5.8	3.5e−9	5.9
128	1.0504	1.4e−11	6.7	6.2e−13	4.8	1.9e−9	5.4	6.0e−9	*
256	1.0276	1.3e−13	6.7	2.9e−11	*	5.0e−11	5.2	1.4e−8	*
512	1.0150	1.3e−15	6.7	2.0e−10	*	8.1e−11	*	8.6e−8	*

used for $Q_d^{(BS)}$. Such local spline approximation will be denoted in the following as $HC_3(y)$ and the associated absolute error as ϵ_3^{HC}. Observe that, we do not compare Q_d^{BS} with such scheme when $d > 3$ because we want to compare approximations with the same approximation order. For facilitating the evaluation of the accuracy of the considered schemes, in all the tables each error column has on the right an associated column reporting the corresponding numerically computed approximation order (denoted as r with the corresponding sub and superscripts).

Tables 5–7 relate to the case $d = 3$ and we can verify that all the schemes have the theoretical convergence order. In particular Table 7 shows that a significant improvement can be obtained by using nonuniform knot distributions for the second test function. The difference between $Q_3^{BS}(y)$ and $HC_3(y)$ is evident only when few knots are used but Q_3^{BS} produces a C^2 approximation while HC_3 a C^1 one. The computational cost of our scheme is linear with respect to N, as well as that of HC_3, and the explicit functional expressions reported in Table 3 can be used in this case. Concerning $Q_3(y)$, we can see from Tables 5 and 6 that for the same value of N, Q_3^{BS} is more accurate than such scheme. On the other hand, considering that Q_d doesn't use the derivative information, in order to perform an honest comparison, our error ϵ_d^{BS} related to N should be compared to ϵ_d related to $2N$. If we take this point into account, our results in Table 5 and in Table 6 show that Q_d (which can be specifically used for the uniform case) produces little better results. However, we would like to remark that there are important applications where the derivatives at the mesh points are available, while this is not true for the function information at refined meshes. Finally, Tables 5–7 show that a little better accuracy is obtained in this case by $H_3(y)$. These results are however encouraging because, when the degree of the spline increases the computation of $H_d(y)$ is more expensive, since the scheme is global. Tables 8–10 relate to the case $d = 4$ and analogous comments can be done. Finally, Tables 11–13 relate to the case $d = 6$ and they show that, if d is suitably increased, $Q_d^{(BS)}$ can produce smooth function approximations with a very high accuracy, even with a relatively low number of knots. In Tables 11–13 we report also the errors for the approximation of the first derivative (denoted by ϵ' with the corresponding sub and superscripts) and the numerically computed approximation order (denoted as r' with the corresponding sub and superscripts). We note that the convergence order is respected and in many cases, even if the error for Q_6^{BS} is higher than the error for HI_6, the approximation of the derivative is more accurate. Another interesting behavior is that, with high N, the HI scheme is unstable for the geometric mesh distribution; this is mainly due to the global nature of the scheme. On the other hand, Q_6^{BS} has no problem because of its locality.

6 Conclusion

We have presented a new class of spline quasi-interpolants which can be easily used also in case of nonuniform knot distribution. Such class is of (first order) differential type and is based on the BS linear multistep collocation methods for the numerical solution of ordinary differential equations. Results concerning the convergence of the schemes have been proved and some numerical experiments related to function approximation have been presented. These formulas are particularly interesting when dealing with the numerical solution of differential problems because in this case approximations of both the solution and of its derivative at the mesh points are available. For applications of this quasi-interpolant to the numerical solution of Ordinary Differential Equations, see [9, 12, 13].

Acknowledgements We would like to thank the referee for his/her useful comments and suggestions which have allowed us to improve the quality of the paper.

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