Special polynomials as continuous dynamical systems*

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Abstract

The special polynomials play a central role in the applications of mathematics. Their number has notably increased during the years. Nevertheless, seen from the point of view of dynamical systems, they display an incredible number of common properties. At the core of this unifying approach is the Pascal matrix. Some examples will be discussed in this paper.

1 Introduction

The subject concerning special polynomials is perhaps the most studied of mathematics, considering their importance in Approximation Theory and then in most applications of mathematics. Nevertheless, they continue to display new properties according to the newest frameworks they are looked at.

In this paper we choose the framework of continuous dynamical systems which, as already mentioned in [2] and [3], gives a simplified way to obtain old results as well as new interesting results and new links between them. The even more fruitful point of view of discrete dynamical systems will be discussed in a forthcoming paper.

*Work developed within the project "Numerical methods and software for differential equations"

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The starting point to define the continuous dynamical system we are interested in is the *creation matrix* H defined as

$$H = \begin{pmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & 2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & n-1 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n},$$

with n a generic positive integer. The name *creation matrix* comes from its role in quantum mechanics where it [1] operates the creation of quantum states in an harmonic oscillator. Its role in the subject we are dealing with is not less important, as we shall see. To start with, we mention that it is strictly related to the shift matrix, which also plays an important role in discrete mathematics:

$$K = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

In fact, by defining $D_f = \mathrm{diag}(0!, 1!, 2!, \ldots, (n-1)!)$, it is easy to find that

$$K = D_f^{-1} H D_f. \tag{1}$$

Let now be

- $p_i(t), i = 0, 1, \ldots,$ a sequence of polynomials of degree i;
- $p(t) = (p_0(t), p_1(t), \ldots, p_{n-1}(t))^T$ [2];

[1] Actually, for normalization reasons, the quantum creation matrix has the elements which are the squares of those of H. With a little abuse of notation we shall denote it by $H^{1/2}$. Of course, $(H^T)^{1/2}$ is the annihilation matrix.

[2] Sometime, in order to avoid notational confusion, the vector $p(t)$ may not share the same letter as its entries.

the dynamical system is defined by

$$\frac{d}{dt}p(t) = H\,p(t), \tag{2}$$

where $p(t_0)$ is a set of n numbers to be defined later. The fundamental matrix of (2) is $P(t) = e^{tH}$ and then, when $t_0 = 0$, the solution of (2) is

$$p(t) = P(t)p(0). \tag{3}$$

Polynomials satisfying (2) are called *Appell polynomials.* Consequently, they may only differ by the initial condition $p(0)$. It turns out that by means of the matrix $P(t)$ we can obtain all the values of the entries of $p(t)$.

2 The Pascal matrix and a few of its properties.

The fame of the matrix H in the mathematical setting resides, so far, almost completely in the fact that $P \equiv P(1)$ is the Pascal matrix, i.e.,

$$P = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ \vdots & & & \ddots & \\ 1 & \binom{n-1}{1} & \binom{n-1}{2} & \dots & 1 \end{pmatrix} \in \mathbb{R}^{n\times n},$$

and then $P(t)$ is its t^{th} power.

The Pascal matrix is one of the most interesting number-patterns in the history of mathematics. It is named after the French mathematician Blaise Pascal (1623-1662) in much of the Western world, although it was invented by the Chinese mathematician Jia Xian around the first half of 11th century and in Europe it appeared already in the works of Niccolò Tartaglia (1499-1557). This matrix contains a lot of remarkable and deep information and in recent years has attracting a renewed attention of many experts not only in the field of pure mathematics but also in many different areas of applied mathematics such as Numerical Analysis, Computer-Aided Design and Combinatorics. New and surprising properties have been so derived (see, e.g., [2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13] to quote only few works).

We list here some of its properties which will be used subsequently (see [3] for more details).

- The entries of P^t are

$$(P^t)_{ij} = \begin{cases} t^{i-j} \begin{pmatrix} i \\ j \end{pmatrix} & \text{for} \quad i \geq j \\ \\ 0 & \text{otherwise} \end{cases} \qquad i, j = 0, 1, \ldots, n-1.$$

- The shifted Pascal matrix is defined by:

$$P_s := e^{H+sK}, \quad s \geq 0; \tag{4}$$

its entries are

$$(P_s)_{ij} = \begin{cases} \begin{pmatrix} i+s \\ j+s \end{pmatrix} & \text{for} \quad i \geq j \\ \\ 0 & \text{otherwise} \end{cases} \qquad i, j = 0, 1, \ldots, n-1.$$

In particular, $P_0 \equiv P$.

- P_s is diagonally similar to its powers:

$$P_s = D(t)^{-1} P_s^t D(t), \quad t \neq 0, \qquad D(t) = \mathrm{diag}(1, t, t^2, \ldots, t^{n-1}). \tag{5}$$

By means of the matrix D_s whose entries are

$$(D_s)_{ij} = \begin{cases} \begin{pmatrix} j+s \\ j \end{pmatrix} & \text{for} \quad i = j \\ \\ 0 & \text{otherwise} \end{cases} \qquad i, j = 0, 1, \ldots, n-1, \tag{6}$$

the relation between P_s^t and P^t is obtained:

$$D_s^{-1} P_s^t D_s = P^t. \tag{7}$$

In the papers [2, 4] we have considered the existing relations between the Pascal matrix and some classical Appell polynomials, like the Bernoulli and the Euler ones.

In this paper we will survey the results in this field and add some further considerations for polynomials which are not directly in the Appell class, but can be reduced to this class by means of more or less elementary transformations.

3 Polynomials in the Appell class

As already said many classical polynomials are in the Appell class, so their expression can be directly obtained by (3) once $p(0)$ is determined.

3.1 The monomial case

The easiest case is that of monomials, i.e., $p_i(t) = t^i$. Here $p(0) = e_0 \equiv (1, 0, \ldots, 0)^T$. Then, for example, $(1, 1, \ldots, 1)^T \equiv e = p(1) = Pe_0$ and $p(2) = (2^0, 2^1, \ldots, 2^{n-1}) = Pe$, which is the compact form of the well known combinatorial identity

$$\sum_{k=0}^{i} \binom{i}{k} = 2^i.$$

All combinatorial identities involving the matrix P can be derived in a similar way.

3.2 The Bernoulli polynomials

In order to treat this case, we need to define a new matrix, in the class of those generated by H, along with some of its properties. Let

$$L = \int_0^1 P^t dt = \int_0^1 e^{tH} dt,$$

from which one obtains $HL = P - I$, with I denoting the identity matrix. Equipped with the matrix L, the Bernoulli polynomials $B_i(t)$, which in the vector form will be labelled $b(t)$, are defined by (3) and $Lb(0) = e_0$ which is the vector form of the well known condition on such polynomials:

$$\int_0^1 B_i(t)\,dt = \begin{cases} 1 & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i > 0 \end{cases} \quad .$$

The entries of the vector $b(0)$ are the Bernoulli numbers, usually denoted by B_k. Since $b(0) = L^{-1}e_0$, such vector is the first column of the matrix L^{-1}. The complete matrix is

$$L^{-1} = \left(\int_0^1 P^t\,dt\right)^{-1} = \sum_{k=0}^{n-1} \frac{B_k}{k!} H^k.$$

Since it is a function of H, such matrix commutes with all functions of H and, in particular, with P^t. Thus,

$$b(t) = L^{-1}\,P^t e_0 \equiv L^{-1}\,\xi(t), \tag{8}$$

where

$$\xi(t) = (1, t, t^2, \ldots, t^{n-1})^T. \tag{9}$$

Relation (8) expresses the fact that the matrix L^{-1} transforms the monomials into the Bernoulli polynomials.

All the known identities concerning Bernoulli polynomials can be derived by the above vectorial relations. The interested reader may consult the cited papers.

3.3 The Euler polynomials

It is well known that the connection between the Euler polynomials $E_i(t)$ and the Euler numbers E_i is given by

$$E_i(1/2) = 2^{-i}E_i, \qquad i = 0, 1, \ldots, n-1.$$

By introducing the vectors $e(t) = (E_0(t), E_1(t), \ldots, E_{n-1}(t))^T$ and $E = (E_0, E_1, \ldots, E_{n-1})^T$, the previous relations can be written as (see (5))

$$e(1/2) = D(2)^{-1}E.$$

Then, considering that $e(t) = P^t e(0)$, one has $P^{\frac{1}{2}}e(0) = D(2)^{-1}E$, which implies

$$e(t) = P^{t-\frac{1}{2}} D(2)^{-1}E.$$

This express in compact form the known relations [1, p. 804]

$$E_i(t) = \sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} (t-1/2)^{i-k} \frac{E_k}{2^k}, \quad i = 0, 1, \dots, n-1.$$

From $E_i(t+1) + E_i(t) = 2t^i$, we deduce that (see (9))

$$e(t+1) + e(t) = 2\xi(t),$$

or, equivalently, $(P+I)e(t) = 2\xi(t)$. The matrix $\frac{1}{2}(P+I)$ allows us to express the Euler polynomials in terms of the monomials. Since it is invertible, we obtain

$$e(t) = 2(P+I)^{-1}\xi(t). \tag{10}$$

Moreover, by taking into account (8) and (10), the transformation between the Bernoulli and Euler polynomials can also be derived:

$$e(t) = 2(P+I)^{-1} L\, b(t). \tag{11}$$

Since $L^{-1}\,(P+I) = 2\,L^{-1} + H$, it is easy to find that

$$b(t) = \left(L^{-1} + \frac{1}{2}H\right)\, e(t).$$

3.4 The Genocchi polynomials

An other notable class of Appell polynomials is the set of the Genocchi polynomials $G_i(t)$. They are related to the Euler polynomials by the formula

$$G_i(t) = iE_{i-1}(t);$$

in particular, the quantities $G_i \equiv G_i(0)$ are called Genocchi numbers. By denoting with $g(t)$ the vector whose entries are the Genocchi polynomials, from the previous relation we obtain:

$$g(t) = He(t). \tag{12}$$

Moreover, recalling (11) and (8), it is easy to deduce that

$$g(t) = 2H(P+I)^{-1}\, L\, b(t), \qquad g(t) = 2H(P+I)^{-1}\, \xi(t).$$

4 Polynomials reducible in the Appell class

Many polynomials are not in the Appell class but, by means of more or less simple transformations, can be reducible to such class.

4.1 The Hermite polynomials

The simplest case is that of Hermite polynomials. As it is true for many special functions, in the literature there are some different types of Hermite polynomials depending on different normalizations and/or scalings. The two most commons are the one widely employed in physics, here denoted by $H_i(t)$, and the one used in probability derived by operating the change of variable $t = r/2$.

The vector $h(t) = (H_0(t), H_1(t), \ldots, H_{n-1}(t))^T$ verifies the differential equation

$$\frac{d}{dt} h(t) = 2\, H\, h(t),$$

whose solution is

$$h(t) = P^{2t} h(0).$$

In particular, this implies

$$h(t+1) = P\, h\left(t + \frac{1}{2}\right).$$

Consequently, the "probabilist's" Hermite polynomials satisfy (2) and then they are Appell polynomials.

4.2 The generalized Laguerre polynomials

The generalized Laguerre polynomials $L_i^{(\alpha)}(t)$ satisfy the following relations:

$$L_i^{(\alpha)}(t) = \sum_{k=0}^{i} (-1)^k \left(\begin{array}{c} i+\alpha \\ i-k \end{array} \right) \frac{t^k}{k!}, \qquad t \in [0, \infty), \;\; \alpha > -1.$$

Introducing $L^{(\alpha)}(t) = \left(L_0^{(\alpha)}(t), L_1^{(\alpha)}(t), \ldots, L_{n-1}^{(\alpha)}(t)\right)^T$ allows us to derive the vector form of the previous equation, i.e.,

$$L^{(\alpha)}(t) = P_\alpha D_f^{-1} D(-1)\, \xi(t), \tag{13}$$

where P_α is the shifted Pascal matrix defined in (4). Since $L^{(\alpha)}(0) = P_\alpha\, e_0$, such vector is the first column of the matrix P_α. Moreover, in correspondence to the value $\alpha = 0$, the special case of polynomials known in the literature as the Laguerre polynomials is obtained:

$$L^{(0)}(t) = P D_f^{-1} D(-1)\, \xi(t). \tag{14}$$

Such polynomials satisfy the differential equation

$$\frac{d}{dt} L^{(0)}(t) = -PKP^{-1}\, L^{(0)}(t).$$

In fact, by differentiating both sides of the relation (14) and by taking into account (1), we obtain

$$\begin{aligned} \frac{d}{dt} L^{(0)}(t) &= P D_f^{-1} D(-1) H \xi(t) = P D_f^{-1} D(-1) H D(-1) D_f P^{-1} L^{(0)}(t) \\ &= P D(-1) K D(-1) P^{-1} L^{(0)}(t) = -PKP^{-1}\, L^{(0)}(t). \end{aligned}$$

Therefore, the Laguerre polynomials do not seem to be Appell polynomials. However, if we consider the polynomials

$$t^i\, L_i^{(0)}(1/t), \qquad t \neq 0,$$

in matrix form we have (see (5))

$$\begin{aligned} D(t) L^{(0)}(1/t) &= D(t) P D_f^{-1} D(-1)\, \xi(1/t) = D(t) P D_f^{-1} D(-1) D(t)^{-1} \xi(1) \\ &= P^t D_f^{-1} D(-1) \xi(1) = e^{Ht} D_f^{-1} D(-1) \xi(1). \end{aligned}$$

Consequently,

$$\frac{d}{dt} D(t) L^{(0)}(1/t) = H\, D(t) L^{(0)}(1/t),$$

which express the fact that the entries of the vector $D(t)L^{(0)}(1/t)$ form an Appell set.

More generally, we consider the transformation suggested by Carlson in [9]:

$$\frac{i!}{(i+\alpha)^{(i)}}\, t^i\, L_i^{(\alpha)}(1/t),$$

where $\alpha^{(0)} = 1$ and $(i+\alpha)^{(i)} = (i+\alpha)(i+\alpha-1)\cdots(1+\alpha)$. These relations written in compact form lead to the vector $D_\alpha^{-1} D(t)\, L^{(\alpha)}(1/t)$, with D_α defined according to (6). From (13) and by using (5) and (7) it follows that

$$\begin{aligned}
D_\alpha^{-1} D(t)\, \mathbf{L}^{(\alpha)}(1/t) &= D_\alpha^{-1}\, D(t)\, P_\alpha\, D_f^{-1}\, D(-1)\, \xi(1/t)\\
&= D_\alpha^{-1}\, D(t)\, P_\alpha\, D(t)^{-1}\, D_f^{-1}\, D(-1)\, \xi(1)\\
&= D_\alpha^{-1}\, P_\alpha^t\, D_f^{-1}\, D(-1)\, \xi(1) = P^t\, D_\alpha^{-1}\, D_f^{-1}\, D(-1)\, \xi(1).
\end{aligned}$$

Therefore,

$$\frac{d}{dt} D_\alpha^{-1} D(t) \mathbf{L}^{(\alpha)}(1/t) = H\, D_\alpha^{-1} D(t) \mathbf{L}^{(\alpha)}(1/t).$$

The entries of the vector $D_\alpha^{-1} D(t) \mathbf{L}^{(\alpha)}(1/t)$ are Appell polynomials.

4.3 The Legendre polynomials

In order to express the Legendre polynomials $\mathcal{L}_i(u), u \in [0,1]$, in a vector form, we need to introduce the lower triangular matrix $\mathcal{S}$ whose entries are:

$$(\mathcal{S})_{ij} = \begin{cases} (-1)^{i-j} \begin{pmatrix} i+j \\ i-j \end{pmatrix} & \text{for} \quad i \geq j \\ \\ 0 & \text{otherwise} \end{cases} \qquad i,j = 0,1,\ldots,n-1,$$

and the diagonal matrix

$$\mathcal{D} = \mathrm{diag}(c_0, 2c_1, \ldots, nc_{n-1}),$$

being $c_i, i = 0,1,\ldots,n-1$ the Catalan numbers. By posing

$$\mathcal{L}(u) = (\mathcal{L}_0(u), \mathcal{L}_1(u), \ldots, \mathcal{L}_{n-1}(u))^T,$$

it can be checked that

$$\mathcal{L}(u) = \mathcal{S}\mathcal{D}\, \xi(u);$$

obviously, such vector does not satisfy the differential equation (2). If we consider the transformation $u = (x+1)/2$, the corresponding Legendre polynomials on the interval $[-1,1]$,

$$\mathcal{L}(x) = \mathcal{S}\mathcal{D}\xi\left(\frac{x+1}{2}\right) = \mathcal{S}\mathcal{D}D(2)^{-1}P\xi(x) \equiv M\xi(x)$$

are not of Appell type, too. However, this is not the case when $x = z/\sqrt{z^2+1}$. In fact, as it is stated by Carlson in [9], the polynomials

$$\sqrt{(z^2+1)^i}\mathcal{L}_i\left(\frac{z}{\sqrt{z^2+1}}\right)$$

form an Appell set. By direct calculation it follows that (see (5))

$$\mathcal{L}(z) = D\left(\sqrt{z^2+1}\right)MD\left(\sqrt{z^2+1}\right)^{-1}\xi(z) \equiv J_0(H)\xi(z),$$

where

$$J_0(y) = \sum_{j=0}^{+\infty}(-1)^j\frac{y^{2j}}{2^{2j}\,(j!)^2}$$

denotes the Bessel function of the first kind of index 0. Since $J_0(H)$ is a function of H it commutes with H. Therefore, we can conclude that the vector $\mathcal{L}(z)$ satisfies (2) and its entries are Appell polynomials.

5 Conclusions

The Pascal matrix plays a central role in unifying the treatment of many special polynomials. Such matrix allows to group many diverse interesting results in a very simple and logical framework. In this paper its connections with some Appell polynomials have been examined.

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