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High-order time stepping schemes for parabolic PDEs with nonsmooth initial conditions

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Abstract

Nonsmooth payoff functions are common in financial contracts and pose difficulties in obtaining high-order solutions of the contract prices. In this work, we consider parabolic PDEs with initial conditions involving various types of nonsmoothness, such as Dirac delta, Heaviside, ramp and quadratic ramp. We apply a fourth-order finite difference (FD) discretization on a uniform grid in space, and BDF4 time stepping initialized with two steps of an explicit third-order Runge-Kutta (RK3) method and one step of BDF3. Using Fourier analysis on the discrete system, we prove that the low-order errors generated by RK3 for nonsmooth data in the high-frequency domain get damped away by BDF steps, which prevents spurious oscillations in the solution; while low-order errors in the low-frequency domain come from the placement of the discontinuity on the discrete grid used for the initial conditions, and form what is known as quantization error. To achieve globally fourth-order convergence, we apply fourth-order smoothing to the initial conditions, and provide explicit formulae of the discretization. By combining the initial condition smoothing with the proposed time-stepping scheme, we mathematically prove and numerically verify that fourth-order convergence is obtained. The analysis can be generalized to higher order methods of the BDF family. Furthermore, we extend the smoothings to nonuniform grids and to general unsmooth, but piecewise smooth, initial conditions.

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