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Numerical solution of Fredholm integral equations of the second kind with discontinuous kernels

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Abstract

We consider a linear Fredholm integral equation of the second kind

$$u(t) - \int_0^1 K(t,s)u(s)ds = f(t), \quad 0 \le t \le 1, \tag{1}$$

where the kernel $K(t,s)$ may have, in addition to a diagonal singularity (a singularity at $s = t$), additional boundary singularities (singularities at $s = 0$ and/or $s = 1$) and a singularity at $s = d$ for some fixed point $d \in (0,1)$. More precisely, we assume that $K(t,s) = g(t,s)\kappa(t,s)$, where, for an integer $m \ge 1$,

A1 $\kappa \in C^m\left(([0,1] \times (0,1)) \setminus \text{diag}\right)$, $\text{diag} := \{(t,s) \in \mathbb{R}^2 : t = s\}$ and there exist real numbers $\nu \in (0,1)$, $\lambda_0 \in [0, 1-\nu)$, $\lambda_1 \in [0, 1-\nu)$ such that the estimate

$$\left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \kappa(t,s) \right| \le c\, s^{-\lambda_0 - j}(1-s)^{-\lambda_1 - j}\, |t-s|^{-\nu - i} \tag{2}$$

holds with a positive constant c for all $(t,s) \in ([0,1] \times (0,1)) \setminus \text{diag}$ and for all non-negative integers i and j such that $i + j \le m$;

A2 the function $g(t,s)$ is m times continuously differentiable with respect to t and s for $t \in [0,1]$, $s \in [0,1]\backslash\{d\}$, $d \in (0,1)$, and g itself and all its derivatives up to order m are bounded in the regions $[0,1] \times [0,d)$ and $[0,1] \times (d,1]$.

The solutions $u(t)$ for these types of integral equations are typically non-smooth at the endpoints of the interval of integration $[0,1]$ and at point $t = d$. When constructing a high-order numerical method for the solution of (1) one must therefore take this non-smooth behaviour into account. In this contribution we devise a numerical solution to (1) with the help of a suitable grading of the interval of integration $[0,1]$ and by utilizing a piecewise polynomial collocation approach. We prove the validity of the method by deriving a global convergence estimate and a superconvergence result. Finally, to illustrate the reliability of the proposed method a numerical example is given.

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