REDUCED ORDER OF THE LOCAL ERROR OF SPLITTING FOR PARABOLIC PROBLEMS

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We study the local error of splitting methods applied to parabolic initial-boundary value problems under homogeneous Dirichlet or Neumann boundary conditions. For the Lie–Trotter splitting, we provide a theoretical local error analysis that rigorously explains the order reduction observed in numerical experiments. Thus, on $\Omega \subset \mathbb{R}^d$ we consider the numerical solution of problems of the form

$$\begin{aligned} &\partial_t\, u(x,t) = \Delta u(x,t) + F(x,u(x,t)), \quad (x,t) \in \Omega \times (0,T), \\ &u(x,t) = 0, \quad \text{or} \quad \partial_n\, u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ &u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \end{aligned}$$

by exponential splitting methods. We show that the following convergence orders hold for the local error (we denote $A = \Delta$ with boundary conditions):

Theorem: Under suitable assumptions, the following local error estimates hold for the Lie–Trotter splitting method:

(1) If $u \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, the Lie–Trotter approximation satisfies the local error bound

$$\|\mathcal{L}(t,u)\| \le \mathcal{C}\, t^2,$$

where $\mathcal{C}$ depends on $\|u\|_{H^2}$ for both Dirichlet and Neumann boundary conditions.

(2) If $u \in H^2(\Omega)$, the Lie–Trotter approximation satisfies the following local error bounds for any $\varepsilon > 0$ under homogeneous Dirichlet or Neumann boundary conditions,

$$\|\mathcal{L}(t,u)\| \le \begin{cases} \mathcal{C}\, t^{\frac{3}{2}-\frac{d}{4}-\varepsilon} & \text{(Dirichlet)} \\ \mathcal{C}\, t^{\frac{9}{4}-\frac{d}{4}-\varepsilon} & \text{(Neumann)} \end{cases}$$

Here $\mathcal{C}$ depends on $\|u\|_{H^2}$ for both boundary conditions.

References

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