Quadratic Spline Collocation and Parareal Deferred Correction Method for Parabolic PDEs

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Abstract

Parareal is a kind of time parallel numerical methods for time dependent systems. In this paper, we consider the following linear parabolic PDE,

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + f(x,t), \quad x \in \Omega, \quad t \in [0,T] \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in [0,T], \\ u(x,0) = g(x), \quad x \in \Omega. \end{cases}$$

We use optimal quadratic spline collocation(QSC) methods for the space discretization to find its approximation in the space of piecewise quadratic polynomials, and the coefficients of the basis polynomials satisfy a system of ordinary differential equations (ODEs). We apply the parareal technique on the time domain of the ODEs. Meanwhile, deferred correction technique is also used to improve the accuracy during the iterations.

We can deduce that the iteration error of parareal can be estimated by

$$\|u(T_n) - U_n^k\| \leq \frac{C_3}{C_1}\frac{(C_1 T_n \Delta T^p)^{k+1}}{(k+1)!}e^{C_2(T_n - T_{k+1})} + \sum_{i=0}^{k-1}\frac{C_1^i \Delta T^{pi}(T_n)^i}{(i+1)!}T_n e^{C_2(T_n - T_{i+1})}C_T h^{k-i},$$

where C_i are constants, p is the convergence order of the coarse propagator. It means that the method converges superlinearly in a rough meaning. We also analyze the stability numerically. There are two separate unstable regions belonging to the coarse propagator and the fine propagator respectively. As the iteration number increases, both of the two unstable regions enlarge. We can see that, if the two propagators are created by implicit numerical methods, the parareal deferred correction will be A-stable; if implicit coarse propagator and explicit fine propagator are employed, the stability will be bounded after a few iterations.

In fact, there are two kinds of iterations, parareal and deferred correction, and they usually converge at different rates. We can take the steps of the fine propagator on each time sub-interval as a parameter, and choose a proper value for it to balance the two iterations for optimized performance. Moreover, the speedup and parallel efficiency are presented. Numerical experiments are carried out on a computer with at least 41 CPUs, and the results show that the hybrid algorithm can achieve a desired accuracy within much less running time than some reference algorithms.

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