Testing the stability of some geometric integrators for large stepsizes

Luigi Brugnano, Felice Iavernaro

Geometric Integration aims to devise numerical methods able to reproduce, in the discrete setting, a number of relevant geometric features of the continuous problem at hand. The case of Hamiltonian systems is among the most interesting due, on the one hand, to the numerous applications in different disciplines and, on the other hand, to the long standing problem of stability in celestial mechanics.

Symplectic methods have had a primary role during the past decades. The fact that each step of integration is arranged by a canonical (or symplectic) transformation has some relevant consequences in the numerical solution y_n, such as the volume preservation properties of the flow in the phase space and the conservation of all quadratic first integrals, if any, of the original differential problem. What about non quadratic constant of motions like, for nonlinear problems, the Hamiltonian function itself? Though a symplectic method cannot conserve the energy function precisely, it possesses a weaker stability property referred to as *long time energy conservation.* More precisely, under suitable assumptions, the error in the energy function is, for $h < h_0$,

$$H(y(t_n)) - H(y_n) = O(nhe^{-h_0/h}) \tag{1}$$

and therefore remains under control as long as the covered time interval nh, satisfies for example $nh \le e^{-h_0/2h}$. A similar result may be extended to all the other first integrals in a completely integrable Hamiltonian system.

Looking at (1), one notices that the extension of the integration interval over long times requires the reduction of the stepsize, and this evidently contrasts with the classical idea of stability in Numerical Analysis initiated by Dahlquist's studies and inherited from the rooted theories of Poincaré and Liapunov.

A question then naturally arises about whether the stability behaviour descending from symplecticity and epitomized by (1) may be extended for large stepsizes, outside of a domain where the asymptotic results in (1) are applicable. This question is reinforced by the present-day studies on alternative integration techniques that aim to provide precise conservation properties over infinite time intervals, independently of the size of h. These include the recently devised energy-preserving Runge-Kutta methods called HBVMs and their generalization composed by the class of Linear Integral Methods (LIMs), capable of conserving any number of first integrals.

We propose a few numerical examples involving symplectic as well as first integrals preserving methods (HBVMs and LIMs) and intended to show that the long time simulation of Hamiltonian system is a branch of research far from being over.