A generalization of Numerov's method using the BVM approach for Sturm-Liouville eigenvalue estimates

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Abstract

The approximation of the eigenvalues of a regular Sturm-Liouville problem (SLP)

$$-y'' + q(x)y = \lambda y, \qquad y(0) = y(\pi) = 0 \tag{1}$$

is a classical problem that has led to the development of many numerical techniques. Among them, the matrix methods are based on the application of difference schemes for reducing the SLP to a generalized matrix eigenvalue problem. In this context, the most popular methods are the three-point finite difference scheme and the Numerov's method. In particular, the latter one can be considered as a 2-step Boundary Value Method (BVM) of order four designed for solving second order ODEs in their original formulation. The schemes that we propose are then obtained by increasing the number of steps of the formula and by using the BVM approach for the additional conditions that such formula requires. This allows to derive methods having arbitrarily high-order p of accuracy that, when applied to the SLP (1) for $p > 4$, provide an approximation of its k-th eigenvalue with an error that behaves as $O(k^{p+1}h^{p-\frac{1}{2}}) + O(k^{p+2}h^p)$. Here $h = \pi/(N+1)$, $k = 1, 2, \ldots, N$, and $N + 2$ is the number of points composing the discretization of the interval of integration. Some numerical results showing the possible advantages that may arise from the use of the new schemes will be presented.