Some known results and open problems related to the covariogram

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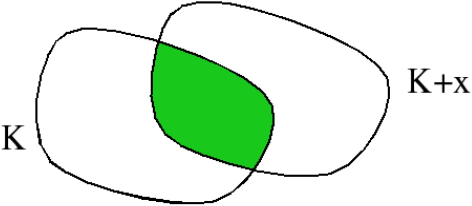
(with some update made in January 2020)

Definition of covariogram

$K \subset \mathbb{R}^n$ compact set, with $K = \mathrm{cl}\,(\mathrm{int}\, K)$
covariogram (or set covariance) of K = function $g_K : \mathbb{R}^n \to \mathbb{R}$ defined as

$$g_K(x) := \mathrm{vol}\,(K \cap (K + x))\,, \quad x \in \mathbb{R}^n$$



- g_K is invariant with respect to translations and reflections of K;
- the covariogram is the autocorrelation of 1_K,

$$g_K(x) = 1_K * 1_{(-K)}$$

that is, passing to Fourier transforms,

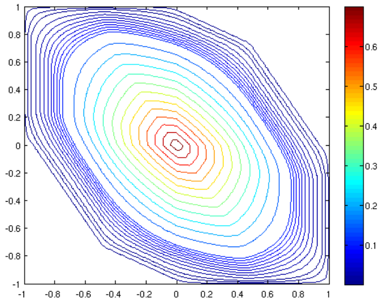
$$\widehat{g_K}(x) = |\widehat{1_K}(x)|^2 \quad \forall x \in \mathbb{R}^n.$$

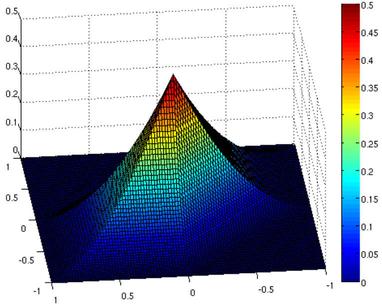
Structure of talk

1. Does g_K determine the set K? ("Covariogram problem")
2. Which functions are covariograms?
3. Which geometric properties of K can be explicitly read in g_K?
4. A view at Question 1 from the Fourier transform side, in complex variables.
   - Zero sets of Fourier transforms of 1_K

Properties of covariogram: general

- support of g_K is $K + (-K)$;
  - $\Longrightarrow$ covariogram gives the width of K in every direction;
- $g_K(0) = \mathrm{vol}\,(K)$;
- g_K is even;
- when K is convex:
  - $(g_K)^{1/n}$ is concave (Brunn-Minkowski inequality).





Properties of covariogram: regularity and derivatives

- If K has finite perimeter $\Longleftrightarrow$ g_K is Lipschitz;

  B. Galerne, Image Anal. and Stereol., 2011.

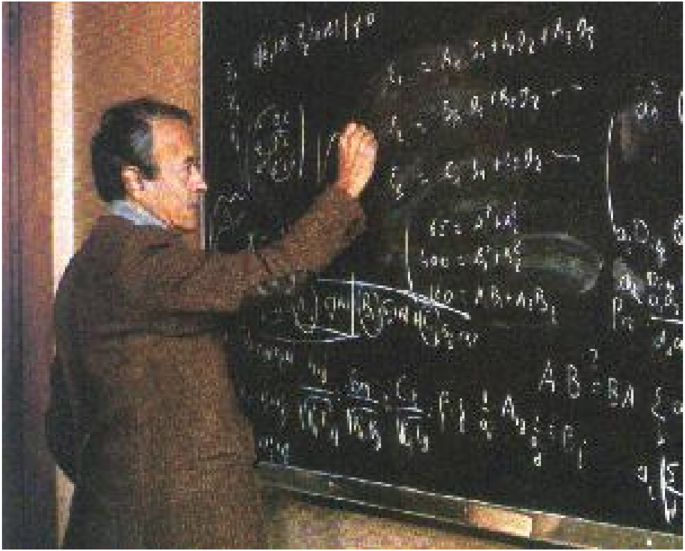
- when K is convex:
  - covariogram gives brightness:

$$-\frac{\partial g_K}{\partial u}(0) = \mathrm{vol}\,(K|u^\perp) \qquad \forall u \in S^{n-1}$$

  - If K is strictly convex $\Longrightarrow$ g_K is C^1 in $\mathrm{int}\,(\mathrm{supp}\, g_K) \setminus \{0\}$;
  - If K is C^1 $\Longrightarrow$ g_K is C^2 in $\mathrm{int}\,(\mathrm{supp}\, g_K) \setminus \{0\}$;
  - explicit formulae for ∇g_K and $D^2 g_K$

  Meyer, Reisner and Schmuckenslager, Mathematika (1993).

G. Matheron and mathematical morphology



G. Matheron and mathematical morphology

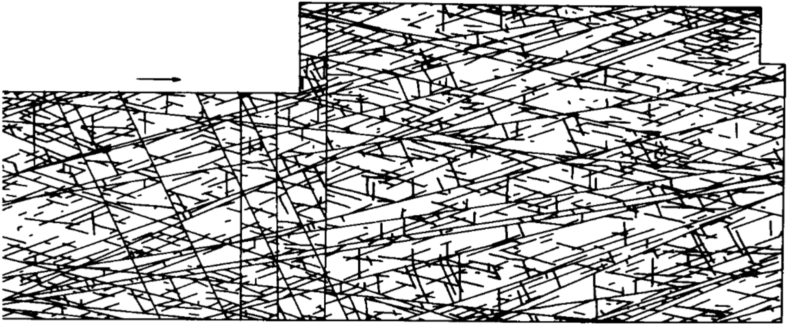


Figure IX.4. Simulation of geological faults.

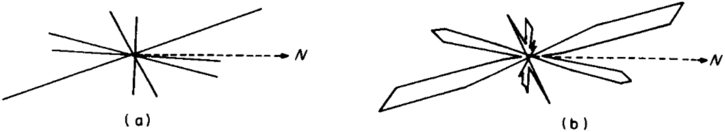


Figure IX.5. Rose of directions of the simulation on Fig. IX-4(a), and its estimation (b).

Section 1

Covariogram problem

Does g_K determine the set K?

- G. Matheron, Le covariogramme géometrique des compacts convexes de $\mathbb{R}^2$ (1986).
- R. Adler and R. J. Pyke, Inst. Math. Statistics Bull. (1991).

Results for Covariogram Problem I

Non-convex sets

In general NO

- Rosenblatt and Seymour (1982); Gardner, Gronchi and Zhong (2005).

Convex sets in $\mathbb{R}^2$

YES

- G. Averkov and G. Bianchi, J. Eur. Math. Soc. (2009).
- Other contributions from W. Nagel, G. Bianchi, F. Segala, A. Volčič.

Results for Covariogram Problem II

Convex Polytopes in any dimension

- YES for any generic polytope in $\mathbb{R}^n$, for any n

  P. Goodey, R. Schneider and W. Weil, Bull. London Math. Soc. (1997).

- NO when $n \geq 4$: In any dimension $n \geq 4$ there are different convex polytopes with equal covariogram

  G. Bianchi, J. London Math. Soc. (2005).

- YES in $\mathbb{R}^3$

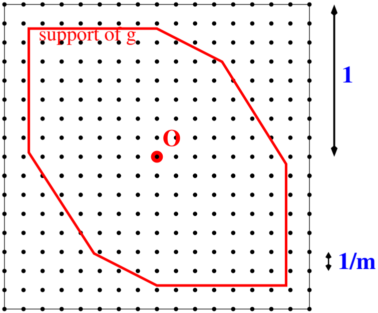
  G. Bianchi, Adv. Math. (2009).

Covariogram problem: Algorithm for reconstruction

G. Bianchi, R. J. Gardner and M. Kiderlen, Journal of the American Mathematical Society (2011).

INPUT knowledge, affected by a random error, of g_K at all points of a grid containing $\operatorname{supp} g_K$

OUTPUT a convex polytope P_m that approximates K



Theorem

Assume that $K \subset \mathbb{R}^n$ is a convex body determined by its covariogram. Then, as $m \to \infty$ (i.e. the grid gets finer and finer), almost surely,

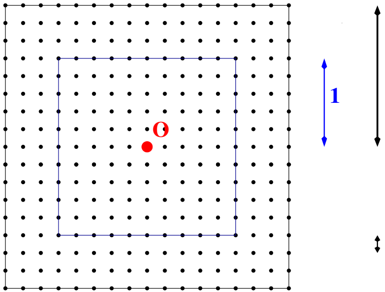
$$P_m \to K \quad \text{or} \quad P_m \to -K.$$

Covariogram problem: Algorithm for reconstruction

G. Bianchi, R. J. Gardner and M. Kiderlen, Journal of the American Mathematical Society (2011).

INPUT knowledge, affected by a random error, of $\widehat{g_K}$ at all points of a grid

OUTPUT a convex polytope P_m that approximates K



Theorem

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Some open problems:

- Polytopes: fill the gap (other counterexamples and positive results)
- Explicit stability estimates of the map $g_K \to K$ (when it is injective)
  - estimates needed to obtain converge rates in the previous algorithm
- C^2_+ convex bodies: open even in dimension 3

Case of C^2_+ convex bodies in $\mathbb{R}^n$, $n \geq 3$

For each $u \in S^{n-1}$ let us consider the "Hessian matrix" of h_K:

$$A_K(u) := \nabla^2 h_K(u) + h_K(u) I.$$

The covariogram gives for each u:

- width: $h_K(u) + h_K(-u)$
- brightness: $\mathrm{vol}\,(K|u^\perp)$
- asymptotic behav. of g_K near point of $\mathrm{supp}\, g_K$ with outer normal u:

Therefore we know:

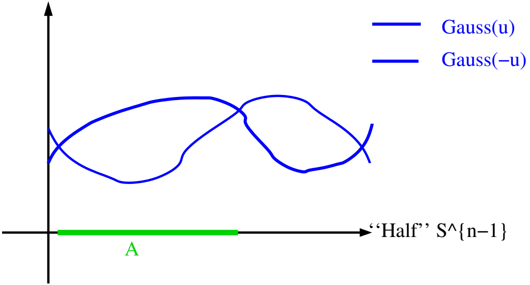
1. $\Longrightarrow A_K(u) + A_K(-u)$
2. $\Longrightarrow \det A_K(u) + \det A_K(-u)$
3. $\Longrightarrow \det \left(A_K(u)^{-1} + A_K(-u)^{-1}\right)$

C^2_+ convex: information easily readable

The information that is easily readable in g_K is thus:

1. $A_K(u) + A_K(-u)$ (i.e. the even part of h_K)
2. the non-ordered set $\{\mathrm{Gauss}_K(u), \mathrm{Gauss}_K(-u)\}$

Ambiguity: Which is $\mathrm{Gauss}_K(u)$ and which is $\mathrm{Gauss}_K(-u)$?



C^2_+ convex bodies: difficulties

- In $\mathbb{R}^3$ knowledge of Gauss_K on a subset of S^2 does not determine a portion of ∂K;
- How to use the extra information given by the knowledge of the matrix $A_K(u) + A_K(-u)$?
- Not enough information to fully determine both matrices $A(u)$ and $A(-u)$ (In $\mathbb{R}^3$ we have "6 unknowns and only 5 conditions");
- For particular subclasses of C^2_+ convex bodies this difficulty can be overcome (bodies with analytic boundaries; bodies of revolution)

Section 2

Which functions are covariograms

Necessary conditions for being a covariogram

There are many necessary conditions that a function h, defined in $\mathbb{R}^n$, must satisfy for being a covariogram but no "necessary and sufficient" one.

Some necessary conditions:

- for being covariogram of a general set:
  - h has to be even, ≥ 0, compactly supported, have maximum only at o;
  - h has to be positive definite, i.e. its Fourier transform $\widehat{h}$ has to be non-negative (indeed $\widehat{g_K} = |\widehat{1_K}|^2 \geq 0$);
- for being covariogram of a convex set:
  - $h^{1/n}$ has to be concave on its support

Problem

Which radial functions are covariograms?

- A. Fish, D. Ryabogin and A. Zvavitch

Which radial functions are covariograms

Problem

Let K be a regular compact set. If g_K is radial, is also K radial?
YES. Moreover K is determined by g_K, up to translations.

- Under the a-priori assumption that K is convex it was known for:
  - $n = 2$ (positive answer to the covariogram problem);
  - $n = 3$ (Howard's result for Nakajima problem);
  - any $n \geq 2$ under the further a-priori assumption that K is centrally symmetric (Meyer, Reisner and Schmuckenslager, 1993);
  - any $n \geq 2$ and $K \in C^2_+$ (it is a consequence of some formula presented in the previous slides:

    $$\text{the set} \quad \{\mathrm{Gauss}_K(u), \mathrm{Gauss}_K(-u)\} \quad \text{does not depend on } u \in S^{n-1}$$

    and continuity implies that Gauss_K is constant).

Proof of “radial g_K implies radial K”

Theorem

Assume $f \in L^2(\mathbb{R}^n)$, $n \geq 2$, real valued and with compact support. Assume that

$$|\widehat{f}|(x) \quad \text{is a radial function for } x \in \mathbb{R}^n.$$

Then f is a translation of a radial function.
Moreover f is determined in $L^2(\mathbb{R}^n)$ by $|\widehat{f}|$, up to translations.

W. Lawton, J. Opt. Soc. Am. (1981)

- Lawton uses techniques from th. of funct. of several complex var.
- Let $\widehat{f}(z)$ be Fourier transform defined for $z \in \mathbb{C}^n$. Then

$$\widehat{f(z)} = c\left(\sum_{j=1}^{n} z_j^2\right)^m e^{2\pi i\ x_0 \cdot z} \prod_{i=1}^{\infty}\left(1 - \frac{\sum_{j=1}^{n} z_j^2}{\lambda_i^2}\right)\left(1 - \frac{\sum_{j=1}^{n} z_j^2}{\overline{\lambda_i}^2}\right)$$

where $m \in \mathbb{N}$, $c \in \mathbb{C}$, $x_0 \in \mathbb{R}^n$, (λ_i) is a sequence in $\mathbb{C}$.

- theorem false if f is complex valued

Which radial functions are covariograms?

Consequences:

- a radial function h is the covariogram of a convex set if and only if

$$h(x) = R^n g_{B(o,1)}\left(\frac{x}{R}\right) \quad \text{for } R = \frac{1}{2}\left(\frac{\mathrm{vol}\,(\mathrm{supp}\, h)}{\mathrm{vol}\,(B(o,1))}\right)^{\frac{1}{n}}.$$

- reduction of the corresponding problem for general set to a one-dimensional problem

Section 3

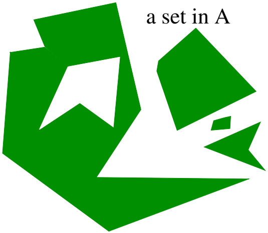
Which geometric properties of K can be explicitly read in g_K?

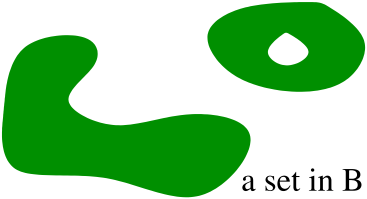
Determine whether the set is convex or not

Define two classes of sets:

$\mathcal{A} =$ {planar regular compact sets whose boundary consists of a finite number of closed disjoint simple polygonal curves.}

$\mathcal{B} =$ {planar regular compact sets whose interior has at most two components}





Theorem

Assume a-priori that g_K is the covariogram of a set in $\mathcal{A} \cup \mathcal{B}$.
Then there are explicitly computable properties of g_K that determine whether K is convex or not.

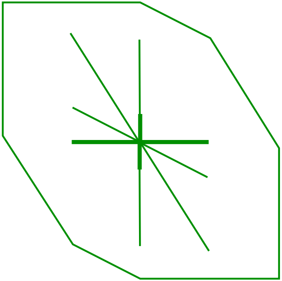
Benassi, Bianchi and D'Ercole, Mathematika (2010).

Determine whether the set is convex or not: test

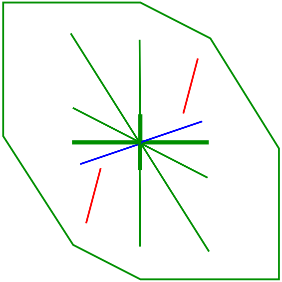
First test (for class $\mathcal{B}$) : Check whether, for each $u \in S^1$, it is

$$-\frac{\partial}{\partial u} g_K(o) = \frac{1}{2}\mathrm{width}(\mathrm{supp}\, g_K, \text{“}u + \pi/2\text{''})$$

Second test (for class $\mathcal{A}$) : Check certain properties of the set of discontinuities of ∇g_K



Convex



Non-convex

Recognize other properties

Theorem. Central symmetry

Let $K \subset \mathbb{R}^n$ be a regular compact set.
The set K is convex and centrally symmetric if and only if

$$g_K(o) = \frac{1}{2} \left(\mathrm{vol}\left(\mathrm{supp}\, g_K\right)\right)^{\frac{1}{n}}$$

When this holds $K = (1/2)\mathrm{supp}\, g_K$.

- A consequence of Brunn-Minkowski inequality and its equality cases.

Theorem. Homothetic level lines of g_K

Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. Assume that the level lines $\{x \ : \ g_K(x) = t\}$ are homothetic to each other for all $t > 0$ small.
Then K is an ellipsoid.

Meyer, Reisner and Schmuckenslager, Mathematika (1993).

- Connections with floating bodies.

Some open problems

Convexity or not It is known that when $K \subset \mathbb{R}^2$ is convex then $\sqrt{g_K}$ is concave on its support.
Does this property characterize the convexity of K in the class of planar regular compact set (or in some subclass)?

Central symmetry Assume K non-convex. Is it possible to read in g_K whether K is centrally symmetric or not?

Homothetic level lines (MRS result) Is the central symmetry of K necessary?
Is it sufficient to assume that only two level lines are homothetic?

Section 4

A view at the covariogram problem from the Fourier transform side, in complex variables.

- J. L. C. Sanz and T. S. Huang, J. Math. Anal. Appl. (1984).
- N. E. Hurt, Phase retrieval and zero crossing (1989).
- T. Kobayashi, J. Fac. Sci. Univ. Tokyo Sect. IA Math. (1989).

Phase Retrieval?

Phase Retrieval in complex variables

Fourier transform in $\mathbb{C}^n$

Let $f \in L^2(\mathbb{R}^n)$ with compact support. Define, for $z \in \mathbb{C}^n$,

$$\widehat{f}(z) = \int_{\mathbb{R}^n} e^{ix\cdot z} f(x)\, dx.$$

FACT: $\widehat{f}$ is an holomorphic function *of exponential type*.

- *"of exponential type"* means that $|\widehat{f}(z)| \leq a e^{b\|z\|}$ for some $a, b \in \mathbb{C}$ and all $z \in \mathbb{C}^n$

Phase Retrieval in $\mathbb{C}^n$: role of factorization

Theorem. J. Sanz and T. Huang, J. Math. Anal. Appl. (1984)

Let $f \in L^2(\mathbb{R}^n)$ with compact support. If $\widehat{f}$ is irreducible in $\mathbb{C}^n$ then f is determined by the knowledge of $|\widehat{f}(x)|$ for $x \in \mathbb{R}^n$.
($\widehat{f}$ is irreducible in $\mathbb{C}^n$ if it cannot be written as $\widehat{f} = f_1 \, f_2$, with f_1 and f_2 holomorphic entire functions, both with non-empty zero sets.)

Theorem. R. Barakat and G. Newsam, J. Math. Phys. (1984)

Let $f, g \in L^2(\mathbb{R}^2)$ with compact support. If $f \neq g$ and $|\widehat{f}(x)| = |\widehat{g}(x)|$ for each $x \in \mathbb{R}^2$, then, for all $z \in \mathbb{C}^2$,

$$\widehat{f}(z) = f_1(z)\, f_2(z) \quad \text{and} \quad \widehat{g}(z) = e^{ic\cdot z} f_1(z)\, \overline{f_2(\overline{z})},$$

for suitable holomorphic functions f_1 and f_2, both with non-empty zero set.

- The only known example of different convex sets with equal covariogram are $K_1 \times K_2$ and $K_1 \times (-K_2) \subset \mathbb{R}^n \times \mathbb{R}^m$. In Fourier space this corresponds to the phenomenon described in the Theorem.

Phase Retrieval in $\mathbb{C}^n$: zero set of $\widehat{1_K}(z)$

Theorem

Any $f \in L^2(\mathbb{R}^n)$ with compact support is determined, up to a translation, by the knowledge of

1) *the modulus of its FT, $|\widehat{f}(x)|$, for $x \in \mathbb{R}^n$;*
2) *the zero set of its FT, $\{z \in \mathbb{C}^n : \widehat{f}(z) = 0\}$.*

Question

Regarding the covariogram problem, what do we know about

$$\{z \in \mathbb{C}^n : \widehat{1_K}(z) = 0\}?$$

- The identity $g_K = 1_K * 1_{-K}$ seen in $\mathbb{C}^n$ becomes

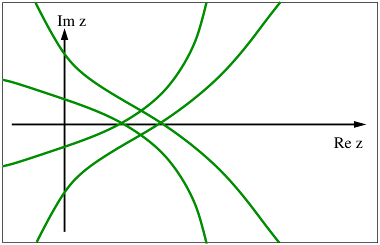
$$\widehat{g_K}(z) = \widehat{1_K}(z)\,\overline{\left(\widehat{1_K}(\overline{z})\right)} \qquad \forall z \in \mathbb{C}^n. \tag{1}$$

Thus we have:

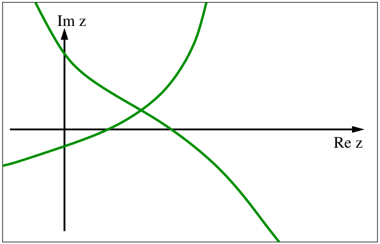
$$\text{zero set of } \widehat{g_K} = (\text{zero set of } \widehat{1_K}) \bigcup \overline{(\text{zero set of } \widehat{1_K})}$$

summarizing...

$$\text{zero set of } \widehat{g_K} = (\text{zero set of } \widehat{1_K}) \bigcup \overline{(\text{zero set of } \widehat{1_K})}$$



zero set of $\widehat{g_K}$



zero set of $\widehat{1_K}$

To determine 1_K it suffices:

- modulus of $\widehat{1_K}(x)$ for x real
- zero set of $\widehat{1_K}$ in $\mathbb{C}^n$

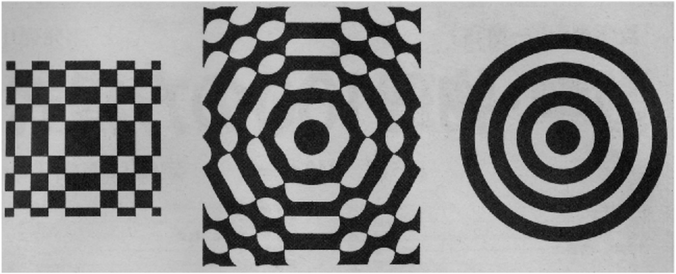
The knowledge of g_K gives us

- modulus of $\widehat{1_K}(x)$ for x real
- (zero set of $\widehat{1_K}$ in $\mathbb{C}^n$) $\bigcup$ (its conjugate)

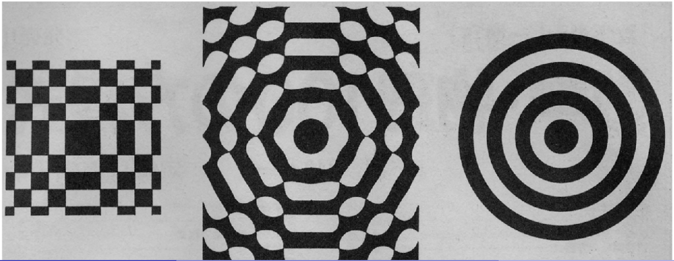
- Passing from $K_1 \times K_2$ to $K_1 \times (-K_2) \Longrightarrow$ substituting zeros of $\widehat{1_{K_2}}$ with their conjugates and leaving others zeros unchanged.

Kobayashi study of the zero set of $\widehat{1_K}$, for K convex

T. Kobayashi, J. Fac. Sci. Univ. Tokyo Sect. IA Math. (1989).



イ) Ω は正方形　　ロ) Ω は正六角形　　ハ) Ω は円板



zero set of $\widehat{1_K}$, for K convex, smooth, with $\mathrm{Gauss}_K > 0$

Kobayashi proves that when K is convex, smooth enough and with $\mathrm{Gauss}_K > 0$ the asymptotic behaviour is similar to that of a ball.

Define a subset S of $\mathbb{C}^n$:

$$S = \text{“}S^{n-1} \times \mathbb{C}\text{”} = \{z \in \mathbb{C}^n : z = \lambda u + i\mu u, \text{ with } \lambda, \mu \in \mathbb{R}, u \in S^{n-1}\}$$

Theorem

$$\exists m_0 \in \mathbb{N} : \qquad (\textit{zero set of } \widehat{1_K}) \cap S = (\textit{compact set}) \bigcup \left(\bigcup_{j=m_0}^{\infty} \mathcal{Z}_j \right)$$

and each $\mathcal{Z}_j$ is a regular submanifold in S ($\subset \mathbb{C}^n$) diffeomorphic to S^{n-1}. Moreover

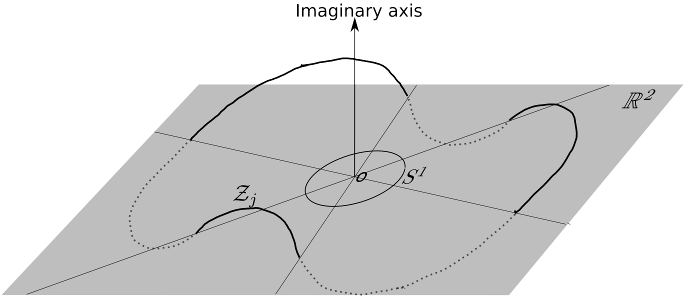
$$\mathcal{Z}_j = \{F_j(u)u \;:\; u \in S^{n-1}\}$$

with, as $j \to \infty$,

$$F_j(u) = \frac{\pi(2j + n - 1)}{\mathrm{width}_K(u)} + \mathbf{i} \,\log\left(\frac{\mathrm{Gauss}_K(u)}{\mathrm{Gauss}_K(-u)}\right) + \mathrm{O}\left(\frac{1}{j}\right).$$

shape of the components at ∞ of zero set of $\widehat{1_K}$

$$\mathcal{Z}_j : u \in S^{n-1} \to \left(\frac{\pi(2j+n-1)}{\mathrm{width}_K(u)} + \mathbf{i}\,\log\left(\frac{\mathrm{Gauss}_K(u)}{\mathrm{Gauss}_K(-u)}\right) + \mathrm{O}\left(\frac{1}{j}\right)\right) u.$$



- ∞ similar copies; each of them carries the same information;
- $\mathcal{Z}_j$ intersects $\mathbb{R}^n$ (i.e. $\mathrm{Im}\, z = 0$) for u such that $\mathrm{Gauss}_K(u) = \mathrm{Gauss}_K(-u)$;
- g_K gives us $\mathcal{Z}_j \bigcup \overline{\mathcal{Z}_j}$, and we need from this to determine $\mathcal{Z}_j$
  - the ambiguity is where $\mathcal{Z}_j$ and $\overline{\mathcal{Z}_j}$ meet: that is in $\mathcal{Z}_j \cap \mathbb{R}^n$ (again u such that $\mathrm{Gauss}_K(u) = \mathrm{Gauss}_K(-u)$!!!!!)

g_K determines a smooth enough K in any dimension

The key ingredient in resolving this ambiguity, when the body is smooth enough, is the fact that certain maps appearing in the description of the asymptotic behavior of the zero set are analytic.

Theorem

Let $n \geq 2$ and define $r(n) = 8$ when $n = 2, 4, 6$, $r(n) = 9$ when $n = 3, 5, 7$ and $r(n) = [(n-1)/2] + 5$ when $n \geq 8$. Let H and K be convex bodies in $\mathbb{R}^n$ of class $C_+^{r(n)}$. Then $g_H = g_K$ implies that H and K coincide, up to translations and reflections.

G. Bianchi, The covariogram and Fourier-Laplace transform in $\mathbb{C}^n$, Proc. London Math. Soc. (2016).

Does the zero set alone determine K?

- The paper by Kobayashi tries to understand whether the (asymptotics of the) zero set alone does determine K.
- His formula proves that this zero set determines, for each $u \in S^{n-1}$,

$$\mathrm{width}_K(u) \quad \text{and} \quad \frac{\mathrm{Gauss}_K(u)}{\mathrm{Gauss}_K(-u)}.$$

Theorem (Kobayashi)

Assume K planar "smooth enough" convex body with $\mathrm{Gauss}_K > 0$. The asymptotic behaviour of the set

$$(\textit{zero set of } \widehat{1_K}) \cap S$$

determines K, up to translations.

Problem

What about dimension $n \geq 3$?